On the Convergence Speed of Tetration

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Abstract: In 2011, in his book "La strana coda della serie n^n^...^n", M. Ripà analyzed some properties involving the rightmost figures of integer tetration, the iterated exponentiation ${}^{b}a$, characterized by an increasing number of stable digits for any base $a > 1$. A few conjectures arose about how many new stable digits are generated by any unitary increment of the hyperexponent b, and Ripà indicated this value as $V(a)$ or "convergence speed" of a. In fact, when b is large enough, $V(a)$ seems to not depend from b, taking on a (strictly positive) unique value, and many observations supported this claim. Moreover, we claim that $V(a) = 1$ for any $a(mod\ 25) \in \{2, 3, 4, 6, 8, 9, 11, 12, 13, 14, 16, 17, 19, 21, 22, 23\}$ and $V(a) \geq 2$ otherwise.

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1 Introduction

In the present paper, we introduce some conjectures involving the rightmost digits of the Tetration ${}^{b}a = a^{a^{a^{\cdots}}}$($b$-times) [1], observing that, when the hyperexponent b is sufficiently large and $a \pmod{25} \not\equiv \{0, 1, 5, 7, 10, 15, 18, 20, 24\}$, the amount of new stable digits generated by any unitary increment of b is unitary as well: it depends only on the congruence modulo 25 of the base a [2].
This new result, if formally proved, would contribute to improve big numbers rightmost digits calculations, opening new scenarios in cryptography/cryptanalysis too [3-4].

2 Convergence Speed and Congruence (mod 25)

It is well known that, for any arbitrarily large n, ${}^{b}a$ originates a string of n stable figures, thus we can say that ${}^{b}a$ is well-defined modulo 10^n, for any $b \geq b'(n, a)$ [2-5-6].

We can easily prove also that, $\forall k \in \mathbb{N}$, $a^{20 \cdot k+1} \equiv a (mod\ 25)$. In fact, $\forall n \in \mathbb{N} \backslash \{0\}, \lambda(n) \leq \varphi(n)$. Thus, $\lambda(25) = \varphi(25) = 20$.
Let a be such that $\gcd(a, 25) = 1 \Leftrightarrow \gcd(a, 5) = 1$, $a^{\lambda(25)} \equiv 1 (mod\ 25) \Rightarrow a^{20+1} \equiv a (mod\ 25)$. Hence $a^{20 \cdot k+1} \equiv a\ (mod\ 25)$.
For any a such that $a \equiv 5 (mod\ 10)$, $\forall m \in \mathbb{N} \backslash \{0, 1\}$, $a^m \equiv 0 (mod\ 25) \Rightarrow a^m \equiv a^{m+1} (mod\ 25)$.
Therefore, $a^2 (mod\ 25) \equiv a^3\ (mod\ 25) \equiv \cdots \equiv a^{20+1}\ (mod\ 25) \equiv \cdots \equiv a^{20 \cdot k+1}\ (mod\ 25)$. $\square$

Let we now introduce the definition of "convergence speed" as it was originally presented by Ripà in his book about the righmost digits of ${}^{b}a$ [1] .

Defintion 1: Let $a \in \mathbb{N} \backslash \{1\}$ be an arbitrary base which is not a multiple of 10 and let $b \in \mathbb{N} \backslash \{0, 1\}$ be such that ${}^{(b-1)}a \equiv {}^{b}a\ (mod\ 10^d) \wedge {}^{(b-1)}a \not\equiv {}^{b}a \left(mod\ 10^{(d+1)}\right)$, where $d \in \mathbb{N}$, we consider $V(a) \ni' {}^{b}a \equiv {}^{(b+1)}a \left(mod\ 10^{((d+V(a)))}\right) \wedge {}^{b}a \not\equiv {}^{(b+1)}a \left(mod\ 10^{(d+V(a)+1)}\right)$.

For simplicity, from here on out, we refer to V(a) as the "convergence speed" of the natural base $1 < a \not\equiv 0$ (mod 10) of the tetration ${}^{(b \geq b')}a$.

3 The Conjectures about V(a)

In this section we present the conjectures and a few remarks to point out their main implications.

Conjecture 1: $\forall a \in \mathbb{N} \backslash \{1, 2\}$ such that $a \not\equiv 0$ (mod 10), $\exists b' < a \in \mathbb{N} \backslash \{0\} \ni', \forall b \geq b'$, $V(a) \in \mathbb{N} \backslash \{0\}$ is constant (see A317905 of the OEIS - ruling out the first term of the sequence [2]).

Conjecture 2: Assume $b \in \mathbb{N} \backslash \{0, 1, 2\}, \forall a \in \mathbb{N} \backslash \{1\}$ such that $a \not\equiv 0$ (mod 10), ${}^{b}a \equiv {}^{(b+1)}a \left(mod\ 10^{((b-2) \cdot V(a))}\right)$.

Remark: If Conjecture 2 holds, it follows that $(b - 2) \cdot V(a) \leq d + V(a)$, hence $\forall b > 3$, $V(a) \leq \frac{d}{b-3}$ (e.g., if $a = 143^{625}$ and $b \geq 5$, $4 = V(a) \leq \frac{0+6+6+5+\sum_{i=5}^{b} 4}{b-3} = \frac{17+(b-4) \cdot 4}{b-3}$ is true).

<u>**Ripà's hypothesis**</u>: $\forall a \in \mathbb{N} \backslash \{1, 2\}$ such that $a \not\equiv 0$ (mod 10), $\exists b' < a \in \mathbb{N} \backslash \{0\} \ni', \forall b \geq b'$,

$$\begin{cases} V(a) = 1 \Leftrightarrow a (mod\ 25) \in \mathbb{C}^{\complement} = \{2, 3, 4, 6, 8, 9, 11, 12, 13, 14, 16, 17, 19, 21, 22, 23\} \\ V(a) \geq 2 \Leftrightarrow a (mod\ 25) \in \mathbb{C} = \{0, 1, 5, 7, 15, 18, 24\} \end{cases}$$

Remark: It is very important to notice that, given $a(mod\ 25) \in \mathbb{C}^{\complement}$ (or equivalently $V(a) = 1$), it follows that $V(a^m) \geq 2,\ \forall m = 5 \cdot n \in \mathbb{N}\backslash\{0\}$, and $V(a^m) = 1$ otherwise (for any m such that $m\ (mod\ 10)\ \equiv \{1, 2, 3, 4, 6, 7, 8, 9\}$).
On the contrary, for any base such that $a(mod\ 25) \in \mathbb{C}$, $V(a^n) \geq 2$, since $a^n(mod\ 25) \in \mathbb{C}$ too ($\forall n \in \mathbb{N}\backslash\{0\}$). We point out that $V(a) \geq 2 \Rightarrow a^{m+1}(mod\ 25) \equiv a(mod\ 25), \forall m = 4 \cdot n$.

Conjecture 3: $\forall v \in \mathbb{N}\backslash\{0\}$, $\exists a$, not a multiple of 10, such that $V(a) = v$.

Remark: In order to prove Conjecture 3, it is sufficient to verify that, for any *n*-digits long base $a \coloneqq a_n \dots a_2 a_1$, where $a_1 = a_2 = \cdots = a_n = 9$, $V(a = 9 \dots 9) = n$ ($\forall b$) (see [1], pp. 25-26).
From Ripà's hypothesis, it follows that $a(n = 1) \in \mathbb{C}^{\complement} \Rightarrow V(a) = 1$ and $a(n \geq 2) \in \mathbb{C} \Rightarrow V(a) \geq 2$.

Conjecture 4: Let $len\big(a(i)\big)$ denote the length of the i-th term any (strictly positive) integer sequence $a(n)$ constructed through the juxtaposition of integers, $\forall i \in \mathbb{N}$ such that $len\big(a(i)\big) \geq 2$,
${}^{a(i)}a(i) \equiv {}^{a(i+1)}a(i+1)(mod\ 10^{len(a(i))})$.

Remark: This conjecture was firstly introduced in 2011 [1] and two examples of this property are given by the sequences A317903 and A317824 of the OEIS [7-8].

4 Conclusion

It is not easy to provide a short proof of any of the conjectures introduced in Section 3 and this could be the subject of another paper that we hope to release in the near future.
We conclude with a very important question that we wish to answer:
"Let $V(a) = n$ be given (assume $b \geq a$ and $a \in \mathbb{N}\backslash\{1, 2\}$ such that $a \not\equiv 0$ (mod 10)), is it possible to identify, for any $n \in \mathbb{N}\backslash\{0\}$, a new function

$$\boldsymbol{\mathcal{R}\big(V(a)\big) := min_{R \in \mathbb{N}\backslash\{0\}} \mid \{V(a + k \cdot R) = V(a)\}\ \forall k \in \mathrm{N_0}}$$

(e.g., $\mathcal{R}(V(a) = 1) = \mathcal{R}(1) = 25$ by Ripà's hypothesis)?".
Any original contribute to help us to prove the aforementioned conjectures would be appreciated.

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