DIRICHLET PROBLEM FOR HERMITIAN-EINSTEIN EQUATIONS OVER BI-HERMITIAN MANIFOLDS

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ABSTRACT. In this paper, we solve the Dirichlet problem for α-Hermitian-Einstein equations on $I_{\pm}$-holomorphic bundles over bi-Hermitian manifolds. As a corollary, we obtain an analogue result about generalized holomorphic bundles on generalized Kähler manifolds.

1. INTRODUCTION

A bi-Hermitian structure on a $2n$-dimensional manifold M consists of a triple (g, I_+, I_-), where g is a Riemannian metric on M and $I_{\pm}$ are integrable complex structures on M that are both orthogonal with respect to g. Let (M, g, I_+, I_-) be a bi-Hermitian manifold. Let E be a holomorphic vector bundle on M endowed with two holomorphic structures $\bar{\partial}_+$ and $\bar{\partial}_-$ with respect to the complex structures I_+ and I_-, respectively. Suppose H ia a Hermitian metric on E. Let $F_{\pm}^H$ be the curvatures of the Chern connections $\nabla_{\pm}^H$ on E associated to the Hermitian metric H and the holomorphic structures $\bar{\partial}_{\pm}$. Motivated by Hitchin [14], Hu *et al.* [16] introduced the following α-Hermitian-Einstein equation, where $\alpha \in (0,1)$ and $\lambda \in \mathbb{R}$:

$$\sqrt{-1}(\alpha F_+^H \wedge \omega_+^{n-1} + (1-\alpha) F_-^H \wedge \omega_-^{n-1}) = (n-1)!\lambda \cdot \mathrm{Id}_E \cdot \mathrm{vol}_g, \tag{1.1}$$

where $\omega_{\pm}(\cdot,\cdot) = g(I_{\pm}\cdot,\cdot)$ are the fundamental 2-forms of g. Once $I_+ = I_-$, (1.1) reduces to the Hermitian-Einstein equation. A Hermitian metric H on E is called α-Hermitian-Einstein if it satisfies (1.1).

Recently, the existence of Hermitian-Einstein metrics on holomorphic vector bundles has attracted a lot of attention. The celebrated Donaldson-Uhlernbeck-Yau theorem states that holomorphic vector bundles over compact Kähler manifolds admit Hermitian-Einstein metrics if they are polystable. It was proved by Narasimhan and Seshadri [26] for compact Riemann surface, by Donaldson [8] for algebraic manifolds and by Uhlenbeck and Yau [34] for general compact Kähler manifolds. There are many interesting generalized Donaldson-Uhlernbeck-Yau theorem (see the References [1, 2, 3, 4, 13, 15, 16, 17, 18, 19, 20, 21, 25, 27, 36], etc.). It is natural to hope that geometric results dealing with closed manifolds will extend to yield interesting information for manifolds with boundary. In [9], Donaldson solved the Dirichlet problem for Hermitian-Einstein equations over compact Kähler manifolds with non-empty boundary. Zhang [37] generalized Donaldson's result to the general Hermitian manifolds. Later, Li and Zhang [23] solved the Dirichlet problem for a class of vortex equations, which generalize the well-known Hermitian-Einstein

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equations. At the same time, Zhang [38] also solved the the Dirichlet problem for Hermitian Yang-Mills-Higgs equations for holomorphic vector bundles on compact Kähler manifolds.

Just very recently, Hu *et al.* [16] proved that the $I_\pm$-holomorphic vector bundles admit α-Hermitian-Einstein metrics iff they are α-polystable, for any $\alpha \in (0,1)$. In this paper, we want to consider the Dirichlet boundary value problem for α-Hermitian-Einstein equations. We obtain the following theorem.

Theorem 1.1. *Let (M, g, I_+, I_-) be a compact bi-Hermitian manifold with non-empty boundary ∂M such that $\mathrm{vol}_g = \frac{\omega_\pm^n}{n!}$. Suppose $(E, \bar\partial_+, \bar\partial_-)$ is an $I_\pm$-holomorphic bundle on M. Then for any Hermitian metric φ on the restriction of E to ∂M, there is a unique α-Hermitian-Einstein metric H on E such that $H = \varphi$ on ∂M.*

Remark 1.2. In Theorem 1.1, we assume the bi-Hermitian manifold (M, g, I_+, I_-) satisfying $\mathrm{vol}_g = \frac{\omega_\pm^n}{n!}$. The existence of such manifold can be found in Remark 6.14 in [10]. In this case, one can rewrite (1.1) as

$$\alpha\sqrt{-1}\Lambda_+ F_+^H + (1-\alpha)\sqrt{-1}\Lambda_- F_-^H - \lambda \cdot \mathrm{Id}_E = 0,$$

where $\Lambda_\pm$ are the contraction operators associated to $\omega_\pm$, respectively.

Our motivation for studying such bundles also comes from generalized complex geometry. In [11], Gualtieri introduced generalized holomorphic bundles, which are analogues of holomorphic vector bundles on complex manifolds. For instance, on a complex manifold M, generalized holomorphic bundles correspond to co-Higgs bundles, which is a holomorphic vector bundle E on M together with a holomorphic map $\phi : E \to E \otimes T_M$ for which $\phi \wedge \phi = 0$. Some of the general properties of co-Higgs bundles were studied by Hitchin in [14] and moduli spaces of stable co-Higgs bundles were studied in [28, 29, 30, 35], etc. Given the relationship between the generalized complex geometry and the bi-Hermitian geometry, one can study generalized holomorphic bundles in terms of $I_\pm$-holomorphic bundles. Recall that any $\mathbb{J}$-holomorphic bundle over generalized Kähler manifold $(M, \mathbb{J}, \mathbb{J}')$ induces an $I_\pm$-holomorphic bundle on (M, g, I_+, I_-) (see [16, Proposition 2.11]). We will not list the definitions on generalized complex geometry (see [11, 16] for more details). Therefore, combining Theorem 1.1, we have the following result.

Corollary 1.3. Let $(M, \mathbb{J}, \mathbb{J}')$ be a compact generalized Kähler manifold with non-empty boundary ∂M whose associated bi-Hermitian structure (g, I_+, I_-) is such that $\mathrm{vol}_g = \frac{\omega_\pm^n}{n!}$. Moreover, suppose $(E, \bar\partial_+, \bar\partial_-)$ is a $\mathbb{J}$-holomorphic bundle on M. Then for any Hermitian metric φ on the restriction of E to ∂M, there is a unique α-Hermitian-Einstein metric H on E such that $H = \varphi$ on ∂M.

This paper is organized as follows. In Section 2, we will introduce the α-Hermitian-Einstein flow on bi-Hermitian manifolds. And some elementary calculations will be presented. In Section 3, we prove the long-time existence of the α-Hermitian-Einstein flow over a compact bi-Hermitian manifold. At last, we deal with convergence of the α-Hermitian-Einstein flow over a compact bi-Hermitian manifold with boundary, that is we complete the proof of Theorem 1.1.

2. Preliminary results

Suppose $(E, \bar{\partial}_+, \bar{\partial}_+)$ is an $I_\pm$-holomorphic bundle on a bi-Hermitian manifold (M, g, I_+, I_-). Let us fix the $I_\pm$-holomorphic structures $\bar{\partial}_\pm$ and a Hermitian metric H_0 on $(E, \bar{\partial}_+, \bar{\partial}_+)$. For any positive-definite Hermitian endomorphism $h \in \mathrm{Herm}^+(E, H_0)$, let $H := H_0 h$ be the Hermitian metric defined by

$$\langle s, t\rangle_H := \langle hs, t\rangle_{H_0},$$

for $s, t \in C^\infty(E)$. Let $\nabla^H_\pm = \bar{\partial}_\pm + \partial^H_\pm$ be the corresponding Chern connections. The relation between $\partial^H_\pm$ and $\partial^{H_0}_\pm$ is given by

$$\partial^H_\pm = \partial^{H_0}_\pm + h^{-1}\partial^{H_0}_\pm h. \tag{2.1}$$

Then the curvatures with respect to $\nabla^H_\pm$ and $\nabla^{H_0}_\pm$ satisfy

$$F^H_\pm = F^{H_0}_\pm + \bar{\partial}_\pm(h^{-1}\partial^{H_0}_\pm h). \tag{2.2}$$

We turn to a family of Hermitian metrics $H(t)$ on E with an initial metric $H(0) = H_0$. We will follow the classical heat flow method to deduce the existence of α-Hermitian-Einstein metric. Actually, we introduce the following α-Hermitian-Einstein flow

$$H^{-1}\frac{\partial}{\partial t}H = -\left(\alpha\sqrt{-1}\Lambda_+F^H_+ + (1-\alpha)\sqrt{-1}\Lambda_-F^H_- - \lambda\cdot \mathrm{Id}_E\right). \tag{2.3}$$

If $I_+ = I_-$, (2.3) is the Hermitian-Einstein flow considered in [8, 9]. By taking a local holomorphic basis e_α $(1 \le \alpha \le r)$ on bundle E and local complex coordinates $\{z^i\}_{i=1}^m$ on M, the α-Hermitian-Einstein flow (2.3) can be written as following:

$$\begin{aligned}\frac{\partial H}{\partial t} =& -\alpha\sqrt{-1}\Lambda_+\bar{\partial}_+\partial_+H + \alpha\sqrt{-1}\Lambda_+\bar{\partial}_+HH^{-1}\partial_+H\\ &-(1-\alpha)\sqrt{-1}\Lambda_-\bar{\partial}_-\partial_-H + (1-\alpha)\sqrt{-1}\Lambda_-\bar{\partial}_-HH^{-1}\partial_-H\\ &+\lambda\cdot H,\end{aligned} \tag{2.4}$$

where H denote the Hermitian matrix $(H_{\alpha\bar{\beta}})_{1\le\alpha,\beta\le r}$ and $\partial_\pm$ denote the (1,0)-parts of the exterior differential d with respect to the complex structures $I_\pm$, respectively. From the above equation, we see that the α-Hermite-Einstein evolution equation is a non-linear strictly parabolic equation.

We define

$$\Delta_{\bar{\partial}_\pm} := -\sqrt{-1}\Lambda_\pm\bar{\partial}_\pm\partial_\pm$$

and

$$\Delta_{\bar{\partial},\alpha} := \alpha\Delta_{\bar{\partial}_+} + (1-\alpha)\Delta_{\bar{\partial}_-}.$$

We will see later the following proposition plays an important role in our discussion.

Proposition 2.1. Let $H(t)$ be a solution of the flow (2.3), then

$$(\Delta_{\bar{\partial},\alpha} - \frac{\partial}{\partial t})|\alpha\sqrt{-1}\Lambda_+F^H_+ + (1-\alpha)\sqrt{-1}\Lambda_-F^H_- - \lambda\cdot\mathrm{Id}_E|^2_H \ge 0.$$

Proof. For simplicity, set

$$\eta = \alpha\sqrt{-1}\Lambda_+F^H_+ + (1-\alpha)\sqrt{-1}\Lambda_-F^H_- - \lambda\cdot\mathrm{Id}_E.$$

Then from (2.1) and (2.2), we have

$$\Delta_{\bar{\partial}_\pm}|\eta|^2_H = -\sqrt{-1}\Lambda_\pm\bar{\partial}_\pm\partial_\pm\mathrm{tr}\{\eta H^{-1}\bar{\eta}^T H\}$$

$$= -\sqrt{-1}\Lambda_\pm\bar\partial_\pm \mathrm{tr}\{\partial_\pm\eta H^{-1}\bar\eta^T H - \eta H^{-1}\partial_\pm H H^{-1}\bar\eta^T H$$
$$+ \eta H^{-1}\overline{\bar\partial_\pm\eta}^T H + \eta H^{-1}\bar\eta^T H H^{-1}\partial_\pm H\}$$
$$= 2\mathrm{Re}\langle -\sqrt{-1}\Lambda_\pm\bar\partial_\pm\partial_\pm^H\eta, \eta\rangle_H + |\partial_\pm^H\eta|_H^2 + |\bar\partial_\pm\eta|_H^2,$$

and

$$\begin{aligned}\frac{\partial}{\partial t}(\sqrt{-1}\Lambda_\pm F_\pm^H) &= \frac{\partial}{\partial t}(\sqrt{-1}\Lambda_\pm\bar\partial_\pm(h^{-1}\partial_\pm^{H_0}h))\\ &= \sqrt{-1}\Lambda_\pm\bar\partial_\pm\frac{\partial}{\partial t}(h^{-1}\partial_\pm h + h^{-1}H_0^{-1}\partial_\pm H_0 h)\\ &= \sqrt{-1}\Lambda_\pm\bar\partial_\pm\Big(\partial_\pm(h^{-1}\frac{\partial h}{\partial t}) - h^{-1}\frac{\partial h}{\partial t}H^{-1}\partial_\pm H + H^{-1}\partial_\pm H h^{-1}\frac{\partial h}{\partial t}\Big)\\ &= \sqrt{-1}\Lambda_\pm\bar\partial_\pm(\partial_\pm^H(h^{-1}\frac{\partial h}{\partial t}))\\ &= -\sqrt{-1}\Lambda_\pm\bar\partial_\pm\partial_\pm^H\eta.\end{aligned}$$

Hence

$$\begin{aligned}(\Delta_{\bar\partial,\alpha} - \frac{\partial}{\partial t})|\eta|_H^2 &= \Delta_{\bar\partial,\alpha}|\eta|_H^2 - 2\mathrm{Re}\langle\frac{\partial}{\partial t}\eta,\eta\rangle_H\\ &= \alpha(|\partial_+^H\eta|_H^2 + |\bar\partial_+\eta|_H^2) + (1-\alpha)(|\partial_-^H\eta|_H^2 + |\bar\partial_-\eta|_H^2)\\ &\geq 0.\end{aligned}$$

□

Now we recall the Donaldson's distance on the space of Hermitian metrics as follows.

Definition 2.2. For any two Hermitian metrics H and K on the bundle E, we define

$$\sigma(H,K) = \mathrm{tr}(H^{-1}K) + \mathrm{tr}(K^{-1}H) - 2r,$$

where $r = \mathrm{rank(E)}$.

If we choose a local frame to diagonalize $H^{-1}K$ to be $\mathrm{diag}(\lambda_1,\dots,\lambda_r)$, then

$$\sigma(H,K) = \sum_{i=1}^r(\lambda_i + \lambda_i^{-1} - 2),$$

from which we can see that $\sigma \geq 0$, with equality holds if and only if $H = K$. Let d be the Riemannian distance function on the metric space, then

$$f_1(d) \leq \sigma \leq f_2(d)$$

holds for some monotone functions f_1 and f_2. So we can conclude from this inequality that a sequence of metrics H_i converge to some H in the usual C^0-topology if and only if $\sup_M \sigma(H_i, H) \to 0$.

Proposition 2.3. Let H,K be two α-Hermitian-Einstein metrics, then

$$\Delta_{\bar\partial,\alpha}\sigma(H,K) \geq 0.$$

Proof. Let $h = K^{-1}H$, from (2.2) we have

$$\mathrm{tr}\{\sqrt{-1}h(\Lambda_\pm F_\pm^H - \Lambda_\pm F_\pm^K)\} = -\Delta_{\bar\partial_\pm}\mathrm{tr}h + \mathrm{tr}(-\sqrt{-1}\Lambda_\pm\bar\partial_\pm h h^{-1}\partial_\pm^K h),$$

and

$$\mathrm{tr}\{\sqrt{-1}h^{-1}(\Lambda_\pm F_\pm^K - \Lambda_\pm F_\pm^H)\} = -\Delta_{\bar\partial_\pm}\mathrm{tr}h^{-1} + \mathrm{tr}(-\sqrt{-1}\Lambda_\pm\bar\partial_\pm h^{-1} h\partial_\pm^H h^{-1}).$$

On the other hand, by doing calculation locally([8]), it is easy to check that

$$\mathrm{tr}(-\sqrt{-1}\Lambda_{\pm}\bar{\partial}_{\pm}hh^{-1}\partial^{K}_{\pm}h) \geq 0$$

and

$$\mathrm{tr}(-\sqrt{-1}\Lambda_{\pm}\bar{\partial}_{\pm}h^{-1}h\partial^{H}_{\pm}h^{-1}) \geq 0.$$

Hence we complete the proof. □

Next, instead of considering H and K to be α-Hermitian-Einstein metrics, we assume $H = H(t)$, $K = K(t)$ to be two solutions of the α-Hermitian-Einstein flow (2.3) with the same initial value H_0. Similar to Proposition 2.3, we can prove the following.

Proposition 2.4.

$$(\Delta_{\bar{\partial},\alpha} - \frac{\partial}{\partial t})\sigma(H(t), K(t)) \geq 0.$$

Proof. Set $h(t) = K(t)^{-1}H(t)$. Notice that

$$\frac{\partial}{\partial t}\mathrm{tr}h = \mathrm{tr}(K^{-1}HH^{-1}\frac{\partial}{\partial t}H - K^{-1}\frac{\partial}{\partial t}KK^{-1}H),$$

$$\frac{\partial}{\partial t}\mathrm{tr}h^{-1} = \mathrm{tr}(-H^{-1}\frac{\partial}{\partial t}HH^{-1}K + H^{-1}KK^{-1}\frac{\partial}{\partial t}K).$$

These two identities together with Proposition 2.3 show that

$$(\Delta_{\bar{\partial},\alpha} - \frac{\partial}{\partial t})\left(\mathrm{tr}h + \mathrm{tr}h^{-1}\right) \geq 0.$$

□

3. α-Hermitian-Einstein flow on compact bi-Hermitian manifold

In this section our primary purpose is to prove the long-time existence of the α-Hermitian-Einstein flow over a compact bi-Hermitian manifold. When the base manifold M is closed, we consider the following problem:

$$\begin{cases} H^{-1}\frac{\partial}{\partial t}H = -\left(\alpha\sqrt{-1}\Lambda_{+}F^{H}_{+} + (1-\alpha)\sqrt{-1}\Lambda_{-}F^{H}_{-} - \lambda \cdot \mathrm{Id}_E\right), \\ H(0) = H_0. \end{cases} \tag{3.1}$$

And when M is a compact manifold with a non-empty smooth boundary ∂M, for any given initial metric φ over ∂M we instead consider the following boundary value problem:

$$\begin{cases} H^{-1}\frac{\partial}{\partial t}H = -\left(\alpha\sqrt{-1}\Lambda_{+}F^{H}_{+} + (1-\alpha)\sqrt{-1}\Lambda_{-}F^{H}_{-} - \lambda \cdot \mathrm{Id}_E\right), \\ H(0) = H_0, \\ H|_{\partial M} = \varphi. \end{cases} \tag{3.2}$$

Since (2.3) is non-linear strictly parabolic. So we get the short-time existence from the standed parabolic PDE theory [12].

Theorem 3.1. *For sufficiently small $\varepsilon > 0$, the problem (3.1) and (3.2) have a smooth solution defined for $0 \leq t < \varepsilon$.*

Next, following a standard argument, we can show the long-time existence of (3.1) and (3.2).

Lemma 3.2. *Suppose that a smooth solution H_t to (3.1) or (3.2) is defined for $0 \leq t < T$. Then H_t converges in C^0 topology to some continuous non-degenerate metric H_T as $t \to T$.*

Proof. In order to prove the convergence, it suffices to show that, given any $\varepsilon > 0$ we can find $\delta > 0$ such that

$$\sup_M \sigma(H_t, H_{t'}) < \varepsilon, \text{ for all } t, t' > T - \delta.$$

And this can be easily seen from the continuity at $t = 0$ combining with Proposition 2.4 and the maximum principle.

So, it remains to show H_T is non-degenerate. By Proposition 2.1 we know that

$$\sup_{M\times[0,T)} |\alpha\sqrt{-1}\Lambda_+ F_+^H + (1-\alpha)\sqrt{-1}\Lambda_- F_-^H - \lambda \cdot \mathrm{Id}_E|_H^2 < C,$$

where $C = C(H_0)$ is a uniform constant. By a direct calculation we have

$$\begin{aligned}\left|\frac{\partial}{\partial t}(\ln \mathrm{tr} h)\right| &= \left|\mathrm{tr}\left\{h\left(\alpha\sqrt{-1}\Lambda_+ F_+^H + (1-\alpha)\sqrt{-1}\Lambda_- F_-^H - \lambda \cdot \mathrm{Id}_E\right)\right\}\frac{1}{\mathrm{tr} h}\right|_H \\ &\leq \left|\alpha\sqrt{-1}\Lambda_+ F_+^H + (1-\alpha)\sqrt{-1}\Lambda_- F_-^H - \lambda \cdot \mathrm{Id}_E\right|_H.\end{aligned}$$

And similarly

$$, \left|\frac{\partial}{\partial t}(\ln \mathrm{tr} h^{-1})\right| \leq \left|\alpha\sqrt{-1}\Lambda_+ F_+^H + (1-\alpha)\sqrt{-1}\Lambda_- F_-^H - \lambda \cdot \mathrm{Id}_E\right|_H.$$

Then we can conclude that $\sigma(H, H_0)$ are uniformly bounded on $M \times [0, T)$, which implies that H_T is non-degenerate. □

For further consideration, we prove the following lemma in the same way as [8, Lemma 19] and [31, Lemma 6.4] (also see [37, Lemma 3.3]).

Lemma 3.3. *Suppose (M, g, I_+, I_-) is a closed bi-Hermitian manifold without boundary (compact with non-empty boundary). Let $H(t)$, for $0 \leq t < T$, be a one-parameter family of Hermitian metrics on $(E, \bar\partial_+, \bar\partial_-)$ over M (satisfying the Dirichlet boundary condition), such that*

(i) *$H(t)$ converges in C^0 topology to some continuous metric H_T as $t \to T$,*
(ii) *$\sup_M |\alpha\sqrt{-1}\Lambda_+ F_+^H + (1-\alpha)\sqrt{-1}\Lambda_- F_-^H|_{H_0}$ is uniformly bounded for $t < T$.*

Then $H(t)$ is bounded in C^1, and also bounded in L_2^p (for any $1 < p < \infty$) uniformly in t.

Proof. Let us first follow Donaldson's arguments [8, Lemma 19]. Let $h(t) = H_0^{-1} H(t)$. We claim that $h(t)$ are bounded uniformly in C^1. If this not true, then for some subsequence t_j there are points $x_j \in M$ with $\sup |\mathrm{d} h_j| = l_j$ achieved at x_j, and $l_j \to \infty$, here we denoted by $h_j = h(t_j)$.

(a) When M is a closed manifold. We can suppose that the x_j converges to a point x in M after taking a subsequence. Once choosing local coordinates $\{z_\alpha\}_{\alpha=1}^n$ around x_j and translating the coordinates slightly, we can suppose that

$$\sup |\mathrm{d} h_j| = l_j$$

is attained at $z = 0$. Rescale $\{z_\alpha\}$ to new coordinates $\{w_\alpha\}$ by $w_\alpha = l_i z_\alpha$; that is, via the maps $\{|w_\alpha| < 1\} \to \{|w_\alpha| < l_j^{-1}\}$, pull back the matrices h_j to matrices $\widetilde{h}_j$

defined for $|w_\alpha| < 1$. With respect to the rescaled coordinates,

$$\sup_{|w_\alpha|<1} |\mathrm{d}\widetilde{h}_j| = 1$$

is attained at the origin point. For convenience, we set $\partial^0_\pm := \partial^{H_0}_\pm, \widetilde{F}^j_\pm := \widetilde{F}^{H(t_j)}_\pm$. Under the assumption of the lemma, we have

$$\begin{aligned}
&|\alpha\sqrt{-1}\Lambda_+\widetilde{F}^j_+ + (1-\alpha)\sqrt{-1}\Lambda_-\widetilde{F}^j_- - \alpha\sqrt{-1}\Lambda_+\widetilde{F}^0_+ - (1-\alpha)\sqrt{-1}\Lambda_-\widetilde{F}^0_-| \\
&= |\alpha\widetilde{h}_j^{-1}(\Lambda_+\bar{\partial}_+\partial^0_+\widetilde{h}_j - \Lambda_+\bar{\partial}_+\widetilde{h}_j\widetilde{h}_j^{-1}\partial^0_+\widetilde{h}_j) \\
&+ (1-\alpha)\widetilde{h}_j^{-1}(\Lambda_-\bar{\partial}_-\partial^0_-\widetilde{h}_j - \Lambda_-\bar{\partial}_-\widetilde{h}_j\widetilde{h}_j^{-1}\partial^0_-\widetilde{h}_j)|
\end{aligned} \tag{3.3}$$

is bounded in $\{|w_\alpha| < 1\}$. Since $\widetilde{h}_j$ and $\mathrm{d}\widetilde{h}_j$ are bounded, $|\Lambda_\pm\bar{\partial}_\pm\partial^0_\pm\widetilde{h}_j|$ are bounded independent of j, then $|\Delta_{\bar{\partial},\alpha}\widetilde{h}_j|$ is also bounded independent of j. By the properties of the elliptic operator $\Delta_{\bar{\partial},\alpha}$ on L^p spaces, $\widetilde{h}_j$ are uniformly bounded in L^p_2 on a small ball. After taking $p > 2n$, $L^p_2 \to C^1$ is compact, then we deduce that some subsequence of the $\widetilde{h}_j$ converge strongly in C^1 to $\widetilde{h}_\infty$. But on the other hand the variation of $\widetilde{h}_\infty$ is zero, since the original metrics approached a C^0 limit, which contradicts the fact

$$|\mathrm{d}\widetilde{h}_\infty|_{z=0} = \lim_{j\to\infty} |\mathrm{d}\widetilde{h}_j|_{z=0} = 1.$$

(b) When M is a compact manifold with non-empty boundary ∂M. We will adapt Simpson's arguments [31, Lemma 6.4] to our settings. Let d_j denote the distance from x_j to the boundary ∂M, then there are two cases.

(b1) If $\limsup d_j l_j > 0$, then we can choose balls of radius $\leq d_j$ around x_j and rescale by a factor of $\frac{l_j}{\epsilon}$ to a ball of radius 1 (where $\epsilon < \limsup d_j l_j$), and pull back the matrixes h_j to matrixes $\widetilde{h}_j$ defined on $\{|w_\alpha| < 1\}$. With respect to the rescaled coordinates,

$$\sup |\mathrm{d}\widetilde{h}_j| = \epsilon$$

is attained at the origin. By condition of the lemma, and discussing like that in (a), we will deduce contradiction.

(b2) On the other hand, if $\limsup d_j l_j = 0$, we can assume that x_j approach a point y on the boundary. Then let $\check{x}_j \in \partial M$ so that $\mathrm{dist}(\check{x}_j, x_j) = d_j$, also $\check{x}_j$ approach y. Choose half-balls of radius $\frac{1}{l_j}$ around $\check{x}_j$ and rescale by l_j to the unit half-balls. In the rescaled picture, x_j approach $z = 0$. In the rescaled coordinates, $|\Lambda_\pm\bar{\partial}_\pm\partial^0_\pm\widetilde{h}_j|$ are still bounded, $\widetilde{h}_j$ are uniformly bounded, and $\sup|\mathrm{d}\widetilde{h}_j| = 1$. Since $\widetilde{h}_j$ satisfy boundary condition along the face of the half-ball, then using elliptic estimates with boundary, and discussing like that in (a), we can also deduce contradiction.

From the above discussion, $h(t)$ are uniformly bounded in C^1. Using (3.3) together with the bounds on $h(t)$, $|\alpha\sqrt{-1}\Lambda_+F^H_+ + (1-\alpha)\sqrt{-1}\Lambda_-F^H_-|$, and $\mathrm{d}h$ imply that $\Lambda_\pm\bar{\partial}_\pm\partial^0_\pm h$ are uniformly bounded. The by the elliptic estimates with boundary conditions, $h(t)$ are uniformly bounded in L^p_2 for any $1 < p < \infty$.

□

Theorem 3.4. *(3.1) and (3.2) have a unique solution $H(t)$ which exists for $0 \leq t < \infty$.*

Proof. By Theorem 3.1, we suppose that there is a solution $H(t)$ existing for $0 \leq t < T$. By Lemma 3.2, $H(t)$ converges in C^0 topology to a continuous non-degenerate metric H_T. This together with the fact that $\sup_M |\alpha\sqrt{-1}\Lambda_+ F_+^H + (1-\alpha)\sqrt{-1}\Lambda_- F_-^H|_{H_0}$ is bounded uniformly in t implies that $H(t)$ are bounded in C^1, and also bounded in L_2^p (for any $1 < p < \infty$) uniformly in t. Since (3.1) and (3.2) are quadratic in the first derivative of H, we can apply Hamilton's method [12] to deduce that $H(t) \to H_T$ in C^∞, and the solution can be extended past T. Hence we have showed the long-time existence of problem (3.1) and (3.2). As for the uniqueness, one can easily achieve it from Proposition 2.4 and the maximum principle. □

4. Proof of Theorem 1.1

Since we have proved the long-time existence of (3.2), it remains for us to show that the solution $H(t)$ converges to a metric H_∞ as the time t approaches to the infinity, and that the limit H_∞ is α-Hermitian-Einstein.

Suppose $H(t)$ is a solution to (3.2) for $0 \leq t < \infty$. As in the previous section we still set

$$\eta = \alpha\sqrt{-1}\Lambda_+ F_+^H + (1-\alpha)\sqrt{-1}\Lambda_- F_-^H - \lambda \cdot \mathrm{Id}_E.$$

From Proposition 2.1 and the fact that $|\mathrm{d}\,|\theta|_H|_H^2 \leq |\mathrm{d}\theta|_H^2$ holds for any section θ of $\mathrm{End}(E)$, we have

$$(\Delta_{\bar{\partial},\alpha} - \frac{\partial}{\partial t})\,|\eta|_H \geq 0. \tag{4.1}$$

Next, according to the Proposition 1.8 of Chapter 5 in [33], the following Dirichlet problem is solvable:

$$\begin{cases} \Delta_{\bar{\partial},\alpha} v = -\left|\alpha\sqrt{-1}\Lambda_+ F_+^{H_0} + (1-\alpha)\sqrt{-1}\Lambda_- F_-^{H_0} - \lambda \cdot \mathrm{Id}_E\right|_{H_0}, \\ v|_{\partial M} = 0. \end{cases} \tag{4.2}$$

Then set $w(x,t) = \int_0^t |\eta|_H (x,s)\mathrm{d}s - v(x)$, where $v(x)$ is a solution to the problem above. From (4.1), (4.2) and the boundary condition satisfied by H we can see that for $t > 0$, $|\eta|_H$ vanishes over the boundary ∂M. Then one can easily check that $w(x,t)$ satisfies

$$\begin{cases} (\Delta_{\bar{\partial},\alpha} - \frac{\partial}{\partial t})w(x,t) \geq 0, \\ w(x,0) = -v(x), \\ w(x,t)|_{\partial M} = 0. \end{cases} \tag{4.3}$$

Therefore the maximum principle implies that

$$\int_0^t |\eta|_H (x,s)\mathrm{d}s \leq \sup_{y\in M} v(y), \tag{4.4}$$

for any $y \in M$ and $0 \leq t < \infty$.

Let $0 \leq t_1 \leq t < \infty$, $\bar{h} = H^{-1}(x,t_1)H(x,t)$. Obviously $\bar{h}$ satisfies

$$\bar{h}^{-1}\frac{\partial}{\partial t}\bar{h} = -\left(\alpha\sqrt{-1}\Lambda_+ F_+^H + (1-\alpha)\sqrt{-1}\Lambda_- F_-^H - \lambda \cdot \mathrm{Id}_E\right) = -\eta.$$

Then we have

$$\frac{\partial}{\partial t}\ln \mathrm{tr}\bar{h} \leq 2\,|\eta|_H\,.$$

Integrating it over $[t_1, t]$ gives

$$\mathrm{tr}\bar{h} = \mathrm{tr}\left(H^{-1}(x,t_1)H(x,t)\right) \leq r\exp\left(2\int_{t_1}^{t} |\eta|_H \,\mathrm{d}s\right).$$

Treating $\bar{h}^{-1}$ in the same way one can get a similar estimate for it. Combining them together we can conclude that

$$\sigma\left(H(x,t), H(x,t_1)\right) \leq 2r\left(\exp(2\int_{t_1}^{t} |\eta|_H \,\mathrm{d}s) - 1\right). \tag{4.5}$$

From (4.4) and (4.5), we have that $H(t)$ converges in the C^0 topology to some continuous metric H_∞ as $t \longrightarrow +\infty$. Hence using Lemma 3.3 again we know that $H(t)$ has uniform C^1 and L_2^p bounds. This together with the fact that $|H^{-1}\frac{\partial}{\partial t}H|$ is uniformly bounded and the standard elliptic regularity arguments shows that, by passing to a subsequence if necessary, $H(t) \to H_\infty$ in C^∞ topology. And from (4.4) we have

$$\alpha\sqrt{-1}\Lambda_+ F_+^\infty + (1-\alpha)\sqrt{-1}\Lambda_- F_-^\infty - \lambda\cdot \mathrm{Id}_E = 0,$$

i.e. H_∞ is the desired α-Hermitian-Einstein metric satisfying the Dirichlet boundary condition. The uniqueness of the solution comes from Proposition 2.3 and the maximum principle. Hence we complete the proof of Theorem 1.1.

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