Identity Featuring Gamma Function: Ramanujan's Integration

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(1) $$\int_{-k}^{\infty} \frac{a^x}{\Gamma(x+1)}\,dx + \int_0^{\infty} \frac{e^{-ax}\,x^{k-1}}{\pi^2+\log^2 x}\left(\cos\pi k - \frac{1}{\pi}\sin\pi k\log x\right) dx = e^a$$

<u>Proof</u>:- Let LHS of above be $f(a,k)$ and differentiating this w.r.t 'k' we get,

$$\frac{\partial}{\partial k} f(a,k) = \frac{a^{-k}}{\Gamma(1-k)} + \int_0^{\infty} \frac{e^{-ax}\,x^{k-1}\log x}{\pi^2+\log^2 x}\left(\cos\pi k - \frac{1}{\pi}\sin\pi k \log x\right) dx$$

$$+ \int_0^{\infty} \frac{e^{-ax}\,x^{k-1}}{\pi^2+\log^2 x}\left(-\pi\sin\pi k - \cos\pi k\log x\right) dx$$

$$= \frac{a^{-k}}{\Gamma(1-k)} - \frac{\sin\pi k}{\pi}\underbrace{\int_0^{\infty} e^{-ax}\,x^{k-1}\,dx}$$

$$\Rightarrow \frac{a^{-k}}{\Gamma(1-k)} - \frac{\sin \pi k}{\pi}\, \Gamma(k)\, a^{-k}$$

$$\Rightarrow \frac{a^{-k}}{\pi}\, \Gamma(\cancel{k}) \sin \pi k - \cancel{\frac{\sin \pi k}{\pi}}\, \Gamma(k)\, a^{-k} = 0$$

Hence $f(a,k)$ is constant w.r.t 'k'. - - - - - - - - (I)

Now diffrentiating w.r.t 'a', we get.

$$\frac{\partial}{\partial a} f(a,x) = \int_{-k}^{\infty} \frac{x a^{x-1}}{\Gamma(x+1)}\, dx - \int_{0}^{\infty} \frac{e^{-ax}\, x^{k}}{\pi^2 + \log^2 x} \left(\cos \pi k - \frac{1}{\pi} \sin \pi k \log x \right) dx$$

$$\Rightarrow \int_{-k-1}^{\infty} \frac{a^{x}}{\Gamma(x+1)}\, dx + \int_{0}^{\infty} \frac{e^{-ax}\, x^{k}}{\pi^2 + \log^2 x} \left(\cos \pi(k+1) - \frac{1}{\pi} \sin(k+1) \log x \right) dx$$

$\Rightarrow f(a, k+1)$ - - - - - - - - (II) from (I) & (II), we get,

$$\frac{\partial}{\partial a} f(a,k) = f(a,k)$$

$$\boxed{f(a,k) = c e^{a}}$$

for calculating value of 'c', we can put $a=0, k=0$ in the definition of $f(a,k)$

$$f(0,0) = \int_0^{\infty} \frac{dx}{x(\pi^2+\log^2 x)} = \int_{-\infty}^{+\infty} \frac{du}{\pi^2+u^2} = 1$$

Finally, we conclude that,

$$\boxed{f(a,k) = e^{a}}$$

Q.E.D