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Identity Featuring Gamma Function: Ramanujan's Integration

Method · October 2018

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$$\textcircled{1} \int_{-k}^{\infty} \frac{a^x}{\Gamma(x+1)} dx + \int_0^{\infty} \frac{e^{-ax} x^{k-1}}{\pi^2 + \log^2 x} \left(\cos \pi k - \frac{1}{\pi} \sin \pi k \log x \right) dx = e^a$$

Proof: - Let LHS of above be $f(a, k)$ and differentiate w.r.t. k we get,

$$\begin{aligned} \frac{\partial}{\partial k} f(a, k) &= \frac{a^{-k}}{\Gamma(1-k)} + \int_0^{\infty} \frac{e^{-ax} x^{k-1} \log x}{\pi^2 + \log^2 x} \left(\cos \pi k - \frac{1}{\pi} \sin \pi k \log x \right) dx \\ &\quad + \int_0^{\infty} \frac{e^{-ax} x^{k-1}}{\pi^2 + \log^2 x} \left(-\pi \sin \pi k - \cos \pi k \log x \right) dx \\ &= \frac{a^{-k}}{\Gamma(1-k)} - \frac{\sin \pi k}{\pi} \underbrace{\int_0^{\infty} e^{-ax} x^{k-1} dx}_{= e^{-a}} \end{aligned}$$

$$\Rightarrow \frac{a^{-k}}{\Gamma(1-k)} - \frac{\sin \pi k}{\pi} \Gamma(k) a^{-k}$$

$$\Rightarrow \frac{a^{-k}}{\pi} \cancel{\Gamma(k)} \sin \pi k - \frac{\sin \pi k}{\pi} \cancel{\Gamma(k)} a^k = 0$$

Hence $f(a, k)$ is constant w.r.t 'k'. ----- (1)

Now differentiate w.r.t 'a', we get,

$$\frac{\partial f(a, k)}{\partial a} = \int_{-k}^{\infty} \frac{x a^{x-1}}{\Gamma(x+1)} dx - \int_0^{\infty} \frac{e^{-ax} x^k}{\pi^2 + \log^2 x} \left(\cos \pi k - \frac{1}{\pi} \sin \pi k \log x \right) dx$$

$$\Rightarrow \int_{-k-1}^{\infty} \frac{a^x}{\Gamma(x+1)} dx + \int_0^{\infty} \frac{e^{-ax} x^k}{\pi^2 + \log^2 x} \left(\cos \pi(k+1) - \frac{1}{\pi} \sin \pi(k+1) \log x \right) dx$$

$$f(a, k+1) \text{ ----- (11)}$$

from ① & ②, we get,

$$\frac{\partial}{\partial a} f(a, k) = f(a, k)$$

$$\boxed{f(a, k) = ce^a}$$

for calculating value of 'c', we can put $a=0, k=0$ in the definition of $f(a, k)$ to

$$f(0,0) = \int_0^{\infty} \frac{dx}{x(\pi^2 + \log^2 x)} = \int_{-\infty}^{\infty} \frac{du}{\pi^2 + u^2} = 1$$

Finally, we conclude that,

$$\boxed{f(a, k) = e^a}$$

Q.E.D