# Geometrical formulation of physics.

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**Abstract**

We define a unique geometrical poisson bracket for classical physics and construct geometrical quantum operators.

## 1 Introduction.

Consider a particle moving in a complex bundle $\mathcal{E}$ over spacetime $\mathcal{M}$, which locally trivializes as as $\mathbb{C}^n \times \mathcal{V}$ where $\mathcal{V}$ is an open set in $\mathbb{R}$, and consider the vector valued sections $(v^i(x))_{i:1\ldots n}$ which take values in the flat fiber which is endowed with the sesquilinear Cartan metric and $\mathbb{C}^n$ carries an irreducible representation of a compact gauge $\mathcal{G}$ with generators $\tau_a$. The vector sections are prone to local gauge transformations which require the introduction of a gauge connection $A^a_\mu(x)$. We shall do two things in this short paper: (a) we choose our kinematical variabls as such that everyghing is poored into manifestly quantum mechanical form with the standard Heisenberg commutation relations apart from a factor of $\hbar$ (b) the equations of motion are defined in a manifestly covariant fashion without recourse to any slicing or simplectic geomtry whatsoever. Everything is expressed in terms of evolution of the worldline with as physical momentum, the four momentum of the particl itself. The "Hamitonian" is of first order in the momenta and of rather trivial nature. Technically, we shall not need a single worldline but a slight "thickening" thereof meaning that we consider worldvolumes $\gamma : (t, \vec{s}) \to \mathcal{M}$ where, say, $\vec{s} \in (-\epsilon, +\epsilon)^3$ and of course we are only interested in teh equation at $s = 0$. Nothing depends upon that thickening but it is mandatory to make the math well defined. We shall be interested here with the time evolution of bundle vector sections. Before we proceed, let us make the following basic observations: as mentioned alreqdy, you should regard the wordline as an immersion $\gamma : \mathbb{R} \times (-\epsilon, +\epsilon)^3 \to \mathcal{M}$ and the momentum as the push forward of $\partial_t$ which we denote by $(\partial_t)_\star$. Given that we shall work with functions $f, g : \mathcal{M} \to \mathbb{C}$ we can define the linear operator $\gamma_f$ by

$$[(\gamma_f)(g)](t, \vec{s}) := f(\gamma(t, \vec{s}))g(\gamma(t, \vec{s}))$$

and

$$[p_\gamma g](t, \vec{s}) := i\frac{d}{dt}g(\gamma(t, \vec{s})).$$

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We have moreover,

$$\gamma_f(gh) = \gamma_f(g)\gamma_f(h)$$

and

$$i[(\partial_t)(\gamma_f g)](t,\vec{s}) := i[(\partial_t)_\star f](t,\vec{s})g(\gamma(t,\vec{s})) + if(\gamma(t,\vec{s}))[(\partial_t)_\star(g)](t,\vec{s}).$$

This suggests to extend the definition of the momentum in this way to functions $\mathbb{R}\times(-\epsilon,+\epsilon)^3 \to \mathbb{R}\times(-\epsilon,+\epsilon)^3$. The same comment holds for $\gamma_f$. In this vein,

$$[\gamma_g\gamma_f h](t,\vec{s}) = g(\gamma(t,\vec{s}))f(\gamma(t,\vec{s}))h(\gamma(t,\vec{s}))$$

$$[p_\gamma\gamma_f h](t,\vec{s}) := i\partial_t(f(\gamma(t,\vec{s}))h(\gamma(t,\vec{s})))$$

as well as

$$[\gamma_f p_\gamma h](t,\vec{s}) := if(\gamma(t,\vec{s}))\partial_t h(\gamma(t,\vec{s})).$$

Finally,

$$[p_\gamma p_\gamma h](t,\vec{s}) = -(\partial_t)^2 h(\gamma(t,\vec{s}))$$

which induces a complex algebra generated by

$$\gamma_g, p_\gamma$$

where $\gamma$ varies over all immersions. This algebra is represented by means of linear operators on the function algebra

$$\mathcal{B} := C^\infty(\mathbb{R}\times(-\epsilon,+\epsilon)^3)\otimes C^\infty(\mathcal{M})$$

which may be given the structure of an Hilbert algebra in the usual $L^2$ sense. Concretely

$$[\gamma_f,\gamma_h](g) = 0 = [p_\gamma,p_\gamma](g), [p_\gamma,\gamma_f](g) = p_\gamma(f)\gamma_\star(g) = \gamma_{p_\gamma(f)}(g)$$

where $\gamma_\star$ is the pull back defined by the immersion $\gamma$. Here, the commutation relations employ the full $\mathcal{B}$ action but are understood to apply on $f,g,h \in C^\infty(\mathcal{E})$ and result in an element of $C^\infty(\mathbb{R})$. So, here we have the standard Heisenberg commutation relations without $\hbar$ of course.

## 2 Classical dynamics.

Covariant dynamics requires dynamics without "potential energy" terms; therefore, any force has to be implemented in the momentum what explains the bundle $\mathcal{E}$. Moreover, according to Einstein, the gravitational field can be gauged away in some point so that physically every particle is a free one meaning that, in absence of other force fields, the correct equation is the geodesic equation. Therefore, in that case, the covariant "Hamiltonian" becomes trivially the momentum $p_\gamma$; indeed

$$[\frac{D}{dt}\gamma_f](g) := [\frac{D}{dt}\triangle\gamma_f](g) = [p_\gamma,\gamma_f](g) = \gamma_{p_\gamma(f)}(g)$$

and

$$[\frac{D}{dt}(\partial_t)_\star](g) := [\frac{D}{dt}\triangle p_\gamma](g) = [p_\gamma,p_\gamma](g) = 0$$

where

$$[\frac{D}{dt}\triangle\zeta](g) = [\frac{D}{dt}, \zeta](g).$$

So, in this view the geodesic equation $\frac{D}{dt}(\partial_t)_\star = 0$ is implemented and the correct Hamiltonian is $p_\gamma$. There is nothing more to say really apart from the constraint $g(p_\gamma, p_\gamma) = m^2$ which is the mass energy relation. In case you consider gauge fields, the picture becomes slightly more complicated. Here, we are interested in the action of the Lie algebra on vector sections $v(x)$. We propose as Hamiltonian

$$H = i(\partial_t)_\star - \frac{q}{m}\gamma_{\tau_a A^a_\mu(\gamma(t,\vec{s}))\dot{\gamma}^\mu(t,\vec{s})}$$

which is nothing but the gauge covariant derivative along the worline and we have extend our notion of $\gamma$ to the action of matrices on vector sections given by

$$\gamma_{\tau_a A^a_\mu(\gamma(t,\vec{s}))\dot{\gamma}^\mu(t,\vec{s})} v := \tau_a A^a_\mu(\gamma(t,\vec{s}))\dot{\gamma}^\mu(t,\vec{s}) v(\gamma(t,\vec{s})).$$

Now, the momenta we shall ve interested in are gauge covariant derivatives in commuting "space" directions $V(\gamma(t,\vec{s}))$ such that $[V, (\partial_t)_\star] = 0$. Hence, we define

$$p_{\gamma,V} := iV - \frac{q}{m}\gamma_{\tau_a A^a_\mu(\gamma(t,\vec{s}))V^\mu(t,\vec{s})}.$$

Hence, we propose as equations of motion

$$g\left(\frac{D}{dt}p_\gamma, V\right) = w(\gamma(t,\vec{s}))^\dagger \left[H, p_{\gamma,V}\right] w(\gamma(t,\vec{s}))$$

and

$$Hw(\gamma(t,\vec{s})) = 0$$

where $w$ is again a vector bundle section over $\mathcal{M}$. A small and elementary computation yields

$$g\left(\frac{D}{dt}p_\gamma, V\right) = w(\gamma(t,\vec{s}))^\dagger \left(-\frac{q}{m}\tau_a(\nabla A)^a_{[\nu\alpha]}(\gamma(t,\vec{s}))\dot{\gamma}^\nu(t,\vec{s})\right) V^\alpha(\gamma(t,\vec{s}))w(\gamma(t,\vec{s}))$$

$$+w(\gamma(t,\vec{s}))^\dagger \left(\frac{q^2}{m^2}f^{abc}\tau_c A_{a,\nu}(\gamma(t,\vec{s}))A_{b,\alpha}(\gamma(t,\vec{s}))\dot{\gamma}^\nu(t,\vec{s})\right) V^\alpha(\gamma(t,\vec{s}))w(\gamma(t,\vec{s}))$$

which is equivalent to the standard classical non abelian Yang Mills equations, see [1] given that one can safely drop $V^\alpha$ from all considerations. Here, I mention that

$$[\tau_a, \tau_b] = i f_{abc}\tau_c$$

and the Cartan metric is just $\delta_{ab}$. Finally, we must insist upon

$$g((\partial_t)_\star, (\partial_t)_\star) = 1$$

where $g$ is the Lorentzian metric of signature $+ - --$. This is all what is allowed in classical physics of point particles really and we now proceed to the quantum theory. Notice that the dynamical content is completely implied by the commutator algebra which is precisely the same as in quantum mechanics.

# References

[1] A.P. Balachandran et al, Classical description of a particle interacting with a non abelian gauge field, Phys Rev D, volume 15, number 6, 1977