Complex Programming

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Abstract

We introduce and suggest to research a special class of optimization problems, wherein an objective function is a real-valued complex variables function and constraints comprising complex-valued complex variables functions.

Keywords: optimization, target function, constraints, minimizers, complex plane, real-valued complex variables target.

1. Introduction

Its well-known that an optimization problem can be represented in the following way:

Given: a function f: $\mathbf{G} \to \mathbf{R}$ from some set **G** to the real numbers
Sought: an element $x_0 \in \mathbf{G}$ such that $f(x_0) \leq f(x)$ for all $x \in \mathbf{G}$ ("minimization") or such that $f(x_0) \geq f(x)$ for all $x \in \mathbf{G}$ ("maximization").

Typically, **G** is some subset of the Euclidean space $\mathbf{R}^n$, specified by a set of constraints and the function f is called an objective function, target function.

The case, when **G** is some subset of two-dimensional complex plane and target function f: $\mathbf{C} \to \mathbf{R}$ is real-valued complex variable function is not investigated yet. Accordingly, **G** is supposed to be specified by constraints, cocomprising complex variable functions. In more general models, function f: $\mathbf{C}^n \to \mathbf{R}$ is supposed to be defined on the multi-dimensional complex space.

The purpose of this paper is to introduce and describe such optimization problems of f: $\mathbf{C} \to \mathbf{R}$ and f: $\mathbf{C}^n \to \mathbf{R}$ target functions over subsets of C and $\mathbf{C}^n$.

2. Overview

Thus, the history of mathematical optimization or mathematical programming is the history of optimization of functions that do not depend on complex arguments, under constraints that do not include functions that depend on complex arguments as well.

In fact, mathematical optimization or mathematical programming problem has the form:

$$\text{minimize } f(\mathbf{x}),$$

$$\text{subject to } g_i(\mathbf{x}) \le 0,\ i = 1, \dots, m,$$

$$\mathbf{x} \in \mathbf{R}^n,\ f\colon \mathbf{R}^n \to \mathbf{R},\ g_i\colon \mathbf{R}^n \to \mathbf{R},\ n \in \mathbf{N},\ m \in \mathbf{N}.$$

The following are the major subfields of the Optimization Theory:

convex programming, non-linear programming, mixed-integer programming quadratic programming, combinatorial optimization, stochastic optimization, etc (see, e.g., Boyd and Vandenberghe [1], Hemmecke et al. [2]).

A convex optimization problem is one in which the objective and constraint functions are convex.

Linear programming, where both: objective and constraints are linear is the important example of convex optimization:

$$\text{minimize } \mathbf{c}^T\mathbf{x} \text{ subject to}$$

$$A\mathbf{x} \le \mathbf{b},\ \mathbf{x} \ge 0,$$

$$\mathbf{x} = (x_j),\ x_j \in \mathbf{R},\ A = (a_{ij}),\ a_{ij} \in \mathbf{R},$$

$$\mathbf{b} = (b_i),\ b_i \in \mathbf{R},\ 1 \le i \le m,$$

$1 \leq j \leq n, m \in \mathbf{N}, n \in \mathbf{N}$.

Non-linear optimization or non-linear programming is the term to describe an optimization problem when the objective or constraint functions aren't linear but not known to be convex.

Quadratic programming allows the objective function to have quadratic terms, while constraints must be specified by linear equalities or inequalities:

minimize $0.5\mathbf{x}^T Q\mathbf{x} + \mathbf{c}^T\mathbf{x}$ subject to

$A\mathbf{x} \leq \mathbf{b}$,

$\mathbf{x} = (x_j)$, $1 \leq i \leq n$, $x_j \in \mathbf{R}$,

$Q = (q_{ij})$, $1 \leq i \leq n$, $1 \leq j \leq n$, $q_{ij} \in \mathbf{R}$,

$A = (a_{ij})$, $a_{ij} \in \mathbf{R}$,

$\mathbf{b} = (b_i)$, $b_i \in \mathbf{R}$, $1 \leq i \leq m$,

$1 \leq j \leq n, m \in \mathbf{N}, n \in \mathbf{N}$.

A general model of mixed-integer programming could be written as:

minimize $f(\mathbf{x})$,

subject to $g_i(\mathbf{x}) \leq 0$, $i = 1, ..., m$,

$\mathbf{x} \in \mathbf{R}^k \times \mathbf{Z}^s$, $f: \mathbf{R}^k \times \mathbf{Z}^s \to \mathbf{R}$, $g_i: \mathbf{R}^k \times \mathbf{Z}^s \to \mathbf{R}$,

where f and g_i are arbitrary non-linear functions.

Ones again, all existing mathematical programming models do not comprise any complex numbers or functions.

3. Complex Programming

Let $|z|$ be the absolute value of a complex number $z = \mathrm{Re}(z) + \mathrm{Im}(z)\mathbf{i} = a + \mathbf{i}b$, $a \in \mathbf{R}$, $b \in \mathbf{R}$, $\mathbf{i}^2 = -1$ and arg(z) the argument of z(the principal value). (See, e.g., Scheidemann [3], Shaw [4]).

Let us introduce and demonstrate optimization problems, defined in terms of complex numbers and functions.

Example 1.

cp1 = { minimize $|z|$ subject to $|z| \geq 1$ }, argmin(cp1) = {z: $|z| = 1$}.

Example 2.

cp2 = { minimize $-\mathrm{Im}(z)$ subject to $|z| \leq 1$ }, argmin(cp2) = $\mathbf{i}$.

Example 3.

cp3 = { minimize $\mathrm{Re}(z)$ subject to $|z| \leq 1$ }, argmin(cp3) = -1.

Example 4.

cp4 = { maximize $|z|$ subject to $0 \leq \mathrm{Re}(z) \leq 1$, to $0 \leq \mathrm{Im}(z) \leq 1$ }, argmax(cp4) = $1 + \mathbf{i}$.

Example 5.

cp5 = { maximize $\mathrm{Re}(z) + \mathrm{Im}(z)$ subject to $0 \leq \mathrm{Re}(z) \leq 1$, $0 \leq \mathrm{Im}(z) \leq 1$ }, argmax(cp5) = $1 + \mathbf{i}$.

Example 6.

cp6 = { maximize $\mathrm{Re}(z) + \mathrm{Im}(z)$ subject to $0 \leq \mathrm{Re}(z) \leq 1$, $0 \leq \mathrm{Im}(z) \leq 1$, $\arg(z) = 0$ }, argmax(cp6) = 1.

Example 7.

cp7 = { maximize Im(z) subject to $0 \leq \mathrm{Re}(z) \leq 1$, $0 \leq \mathrm{Im}(z) \leq 1$, $\mathrm{Im}(z) \leq \mathrm{Re}(z)$ }, argmax(cp7) = 1 + **i**.

Example 8.

cp8 = { maximize | z | subject to $\mathrm{Im}(z) \geq \mathrm{Re}^2(z)$, $\mathrm{Re}(z) \geq \mathrm{Im}^2(z)$ }, argmax(cp8) = 1 + **i**.

Example 9. "Polynomial" Complex Programming.

cp9 = { maximize $| c_n z^n + \ldots + c_1 z |$ subject to

$$| a_{1n}z^n + \ldots + a_{11}z + a_{10} | \leq b_1,$$
$$\ldots \quad \ldots \quad \ldots$$
$$| a_{mn}z^n + \ldots + a_{m1}z + a_{m0} | \leq b_m,\ z \in \mathbf{C},\ a_{ij} \in \mathbf{C},\ b_i \in \mathbf{R},\ c_j \in \mathbf{R},$$

$$1 \leq i \leq m,\ 0 \leq j \leq n,\ n \in \mathbf{N},\ m \in \mathbf{N} \}.$$

(More sophisticated examples would contain rational meromorphic complex functions).

Example 10. Several complex variables.

cp10 = { maximize $| z_1 + z_2 |$ subject to $| z_1 | \leq 1$, $| z_2 | \leq 1$ }.

Example 11.

cp11 = { maximize $| z_1 + z_2 |$ subject to $| z_1 | \leq 1$, $| z_2 | \leq 1$, $\arg(z_1 z_2) \leq \pi/4$ }.

Example 12. "Linear" Complex Programming.

cp12 = { maximize $| c_1 z_1 + \ldots + c_n z_n |$ subject to

$$| a_{11}z_1 + \ldots + a_{1n}z_n | \leq b_1,$$
$$\ldots \quad \ldots \quad \ldots$$
$$| a_{m1}z_1 + \ldots + a_{mn}z_n | \leq b_m,\ z_j \in \mathbf{C},\ a_{ij} \in \mathbf{C},\ b_i \in \mathbf{R},\ c_j \in \mathbf{R},$$

$1 \leq i \leq m, 1 \leq j \leq n, n \in \mathbf{N}, m \in \mathbf{N} \}$.

Example 13.

cp13 = **{** maximize $| z_1 + ... + z_n |$ subject to

$$\mathrm{Re}(a_{11}z_1 + ... + a_{1n}z_n) \leq b_1,$$
$$... \quad ... \quad ...$$
$$\mathrm{Re}(a_{m1}z_1 + ... + a_{mn}z_n) \leq b_m,$$

$$\mathrm{Im}(a_{11}z_1 + ... + a_{1n}z_n) \leq c_1,$$
$$... \quad ... \quad ...$$
$$\mathrm{Im}(a_{m1}z_1 + ... + a_{mn}z_n) \leq c_m,$$

$z_j \in \mathbf{C}$, $a_{ij} \in \mathbf{C}$, $b_i \in \mathbf{R}$, $c_i \in \mathbf{R}$, $1 \leq i \leq m$, $1 \leq j \leq n$,

$n \in \mathbf{N}$, $m \in \mathbf{N}$ }.

Example 14.

cp14 = { maximize $| z_1 + ... + z_n |$ subject to

$$\arg(a_{11}z_1 + ... + a_{1n}z_n) \leq b_1,$$
$$... \quad ... \quad ...$$
$$\arg(a_{m1}z_1 + ... + a_{mn}z_n) \leq b_m,$$

$$\mathrm{Im}(a_{11}z_1 + ... + a_{1n}z_n) \leq c_1,$$
$$... \quad ... \quad ...$$
$$\mathrm{Im}(a_{m1}z_1 + ... + a_{mn}z_n) \leq c_m,$$

$z_j \in \mathbf{C}$, $a_{ij} \in \mathbf{C}$, $b_i \in \mathbf{R}$, $c_i \in \mathbf{R}$, $1 \leq i \leq m$, $1 \leq j \leq n$,

$n \in \mathbf{N}$, $m \in \mathbf{N}$ }.

Example 15.

cp15 = **{** maximize $\arg(z_1 ... z_n)$ subject to

$\mathrm{Re}(a_{11}z_1 + ... + a_{1n}z_n) \leq b_1,$

... ...

$\mathrm{Re}(a_{m1}z_1 + ... + a_{mn}z_n) \leq b_m,$

$\mathrm{Im}(a_{11}z_1 + ... + a_{1n}z_n) \leq c_1,$

... ...

$\mathrm{Im}(a_{m1}z_1 + ... + a_{mn}z_n) \leq c_m,$

$\arg(z_j) \leq d_j,$

$z_j \in \mathbf{C}$, $a_{ij} \in \mathbf{C}$, $b_i \in \mathbf{R}$, $c_i \in \mathbf{R}$, $d_j \in \mathbf{R}$, $1 \leq i \leq m$, $1 \leq j \leq n$,

$n \in \mathbf{N}$, $m \in \mathbf{N}$ }.

Example 16. "Quadratic" Complex Programming.

cp16 = **{** maximize $| z_1^2 + ... + z_n^2 - \mathbf{i}z_1z_2 |$ subject to

$| a_{11}z_1 + ... + a_{1n}z_n | \leq b_1,$

... ...

$| a_{m1}z_1 + ... + a_{mn}z_n | \leq b_m,$

$z_j \in \mathbf{C}$, $a_{ij} \in \mathbf{C}$, $b_i \in \mathbf{R}$,

$1 \leq i \leq m$, $1 \leq j \leq n$, $n \in \mathbf{N}$, $m \in \mathbf{N}$ }.

Example 17. "Non-Linear" Complex Programming.

cp17 = **{** maximize $| e^z + \sin(\pi z) |$ subject to

$| \cos(\pi z) | \leq a$, $0 \leq \mathrm{Re}(z) \leq 1$, $0 \leq \mathrm{Im}(z) \leq 1$,

$z \in \mathbf{C}$, $a \in \mathbf{R}$ }.

Example 18. "Integer" Complex Programming(Over Gaussian Integers).

cp18 = **{** maximize $| z_1^4 + ... + z_n^4 |$ subject to

$$| a_{11}z_1 + \dots + a_{1n}z_n | \le b_1,$$
$$\dots \quad \dots \quad \dots$$
$$| a_{m1}z_1 + \dots + a_{mn}z_n | \le b_m,$$

$$z_j \in \mathbf{C} \cap \mathbf{Z}^2, \; a_{ij} \in \mathbf{C}, \; b_i \in \mathbf{R},$$

$$1 \le i \le m, \; 1 \le j \le n, \; n \in \mathbf{N}, \; m \in \mathbf{N} \}.$$

(Similarly for "Mixed-Integer" Complex Programming).

4. Open Problems

Despite such optimization problems actually could be translated and considered in terms of optimization problems over the Euclidean space, it may be not always so "easy" task(complexity problems, etc).

That is why, it would be preferable to develop specific, "direct" methods for complex programming problems using complex analysis.

The corresponding complexity evaluations for the Complex Programming problems would be developed as well: for example in binary encoded length of the coefficients.

5. Conclusions

We unveiled a new class of optimization problems that contain complex numbers and complex functions in their targets and constraints.

References

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