An introduction to non-commutative (pseudo) metric geometry.

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Abstract

We introduce the reader to the problematic aspects of formulating in concreto a suitable notion of geometry. Here, we take the canonical approach and give some examples.

1 Introduction.

Non-commutative geometry is much less rigid as commutative geometry is in the sense that it has a very poor group of symmetry transformation. This allows for many different types of differential operators to exist depending upon subtle intricacies of the algebraic relationships existing between differen point-operators. There is of course still a universal definition in a way to do geometry; alas it has many more inequivalent representations than one would desire for - a problem of choice between blondé and brown woman. Certainly, it is not desirable to try to mimic local differential geometry by means of "coordinate" operators satisfying some ad-hoc algebra. If anything goes, then coordinate transformations do not exist anymore but diffeomorphisms do; the former coordinate dependent approach has already been investigated by this author to a sufficient degree of despair.

The approach taken in this paper is simple and direct and shows that the real problem is the representation one. Often, many things will not exist such as the human as a computer.

What is a point? Democritos thought that points correspond to entities which cannot be further devided; in an operational setting this means that if you try to disturb a point it shows itself or not. One thing is for sure, the point is there and you know it, but then, how could you if you do not measure it? With particles, we assume they exist when going through a *second* detector, because something akin passed previously the first one and we assume a previous one not to have stopped at the particle beach lounge. Points clearly do not have such status because we cannot manipulate them; then, the only option is to measure as many times as required, an infinite number to be precise before a click occurs.

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Therefore, we have two preliminary options, either to assume that points exist and resort to an approach where the outcome is definite (classical geometry) or to assume our apparatus is perfect but that something else changes. This last option does not make much sense a priori given that we cannot *control* points. Moreover, our apparatus, whatever it may be must have "relationships" with the point which is usually expressed through is support in classical thinking. Point gheists in a way carry the material gheists of the particle making up the apparatus and I have never known someone who could interrogate its gardian angel. So, we must conclude points do not exist as measurable atomic entities from the human or elementary particle point of view. This takes away all the charm of spacetime and we must resort to "relations" between conscious perceptions of point gheists which are precluded from our perspective. Noncommutative geometry is the most stupid of all such enterprises; here, the point gheists can only ask yes or no questions (in the framework of a divinely preconceived space of an infinite number of allowed questions) and spacetime does therefore not a priori exist but corresponds to an intelligent process correlating outcomes of the "do I exist?" point-questions. Given, therefore, a preconceived space of questions, we ask for the proximity between the existential question of point gheists delineating a kind of similarity of temporarily intelligence.

As an example, take a classical manifold $\mathcal{M}$ and a cover by means of open sets $\mathcal{O}_i$, let $P_i = \chi_{\mathcal{O}_i}$ be the projection on the open set $\mathcal{O}_i$; then,

$$P_i \wedge P_j = P_{\mathcal{O}_i \cap \mathcal{O}_j}$$

and the P_i all commute. The $\mathcal{O}_i$ also define a sub-Hilbertspace $\mathcal{H}_i$ of $L^2(\mathcal{M}, \mu)$ where μ is some volume measure with $\int_{\mathcal{M}} d\mu = 1$. Taking as state the unit function, assuming that $\mathcal{M}$ is compact, we have that $P_i.1 = P_i$ where P_i is to be seen as the identity 1_i in $\mathcal{H}_i = L^2(\mathcal{O}_i, \frac{d\mu}{\int_{\mathcal{O}_i} d\mu})$. Hence, 1, in a way perfectly correlates the similtanious existence of all point regions P_i and is a sign of emergent classicality. Therefore, the generalized Heisenberg dynamics must ensure that all point projection operators commute and that there exists a isomorphism ψ between state-vectors and a subalgebra of operators such that $P_i = \psi(v_i)$ and there exist a vector 1 such that

$$P_i.\psi(1) = P_i = \psi(P_i(1)).$$

In other words, we *must* depart from the idea of an Heisenberg xor Schroedinger picture, otherwise we are screwed in one way or another in explaining classicality.

2 Mathematical formulation.

Let $\mathcal{H}$ be a Hilbert involutive *algebra*, a so called Hilbert $C^\star$ algebra, and $\mathcal{A}$ the weak dual of the $C^\star$ algebra of bounded operators on $\mathcal{H}$ whose weak pre-dual is given by the compact operators on $\mathcal{H}$. $\mathcal{A}$ is given by the trace class operators B,

$$\mathbf{Tr}(B^\dagger B) < \infty.$$

Let P_i be some Hermitian projection operators on $\mathcal{H}$ where $i \in I$, an index set which is finite, countable or real numbered. The space of Hermitian projection operators has the structure of an atomistic, unimodular lattice meaning it has a maximal 1 and minimal element 0 as well as minimal nonzero elements such that the intersection $\wedge$, union $\vee$ satisfying the de Morgan rule and are well defined within the context of a partial order.

More concretely, let $\mathcal{P}$ be the algebraic variety of Hermitian, trace-class, projection operators on $\mathcal{H}$ and $\mathcal{S}^+$ the cone of positive, trace-class operators, then a metric geometry is characterized by a bifunction

$$d : \mathcal{P} \times \mathcal{P} \rightarrow \mathcal{S}^+$$

satisfying

$$d(P,P) = 0,\ d(P,Q) = d(Q,P) > 0,\ \tilde{d}(P,Q) + \tilde{d}(Q,R) \geq \tilde{d}(P,R)$$

where $\tilde{d}$ is a classical metric associated to d in a way explained below. It is clear that the bijective linear operators commuting with the multiplication on $\mathcal{H}$, the so-called algebra automorphisms, determine a diffeomorphism on the underlying space is $\mathcal{H} = L^2(\mathcal{M}, \mu)$ for some compact manifold $\mathcal{M}$ and measure μ with the complex conjugation as involution. The proof is simple; note that an automorphism maps characteristic functions to characteristic functions preserving the entire algebra. Therefore, it induces a mapping on the points which must be bijective given that the automorphism is. In case the automorphism is unitary as a linear mapping, then it corresponds to an isometry of the measure. If, moreover, the distance function is preserved, then we recover the Killing fields. So, classically, the atomistic (Hermitian) projective elements χ of $\mathcal{H}$ which are of measure zero in the sense

$$||\chi|| = 0$$

correspond to points. Here, atomistic means that χ cannot be written as a sum of alike elements and we have put Hermitian between brackets because in the classical situation projective elements are automatically Hermitian. This point of view is not really exact and we better speak about a sequence of decreasing Hermitian projective elements $(\chi_k)_{k \in \mathbb{N}}$ such that $\chi_k < \chi_l$ for $k > l$ and

$$\lim_{k \rightarrow \infty} ||\chi_k|| = 0.$$

Generalized δ functions can be constructed with regard to a dense, unital, subalgebra $\mathcal{K}$ by demanding that there exists an increasing sequence of positive numbers a_k such that for any $\psi \in \mathcal{K}$

$$\lim_{k \rightarrow \infty} a_k \langle \chi_k | \psi \rangle := \widehat{\psi}_\chi$$

where the latter defines an automorphism on $\mathcal{K}$. The question now is how to suitably relax this structure; clearly if we keep the field of the complex numbers in the definition of the Hilbert algebra, as well as the standard commutatative, associative and distributive rules for the sum and product, then nothing is gained. We are stuck to classical spaces. On the other hand, if we throw away the Hilbert space character of $\mathcal{H}$, then there is very little structure in the operator algebra over it. Connes prefers to throw away the notion of multiplication on $\mathcal{H}$ (but keeps the Hilbert space character); unfortunately, that leaves us with no points or atoms. Hence, the most general setting which appears to be satisfying is one of a quaternionic Hilbert algebra with the standard abelian sum and scalar multiplication rules but with a multiplication between vectors which is generically non-abelian but satisfies the distributive law. We baptise these geometries to be quaternionic.

For example, on flat Minkowski compactified on a n-dimensional torus from $-L$ to L in every orthonormal direction, points are determined by distributional states $\delta(x-z) := |x\rangle$ and the unit operator is given by

$$1 = \int dx |x\rangle\langle x|.$$

Notice that

$$0 \leq \pm(T(h) \pm T(-h))^2 = \pm(T(h)^2 + T(-h)^2 \pm 2)$$

and therefore

$$-2 \leq T(h)^2 + T(-h)^2 \leq 2$$

where T constitues a representation of the n dimensionial translation group which is nothing but the maximal abelian group of the isometry group of $\mathbb{R}^n$. Also, we know that

$$\int_{T_n^{[0,L]}} dr \int_{T_n^{[0,L]}} ds\, r.s\, T_{-r}T_s = \int_{T_n^{[0,L]}} dr \int_{T_n^{[-L,L]}} ds\, (||r||^2 - \frac{||s||^2}{4})\, T_s =$$
$$\left(\int_{T_n^{[0,L]}} dr ||r||^2\right)\left(\int_{T_n^{[-L,L]}} ds\, T_s\right) - \frac{1}{4}\mathrm{Vol}(T_n^{[0,L]}) \int_{T_n^{[-L,L]}} ds\, ||s||^2\, T_s$$

is a positive definite matrix. Moreover,

$$\int_{T_n^{[-L,L]}} ds\, T_s$$

usurpates (in the limit to maximal L) the action of T_t and therefore must equal the (distributional if $L = \infty$) state

$$|1\rangle\langle 1|$$

in the functional representation. Hence,

$$\left(\int_{T_n^{[0,L]}} dr ||r||^2\right) |1\rangle\langle 1| - \frac{1}{4}\mathrm{Vol}(T_n^{[0,L]}) \int_{T_n^{[-L,L]}} ds ||s||^2 T_s$$

is the expression of our concern and the reader notices there is no way to regularize the operator in the limit for the compactification scale L to infinity. Therefore, a good definition of an operator valued distance d is determined by the "positive scalar product" operator

$$\langle A|B\rangle_{\mathrm{op}} = \int_{T_n^{[0,L]}} dr \int_{T_n^{[0,L]}} ds\, r.s\, A^{\dagger} T_{-r} T_s B.$$

Given that the quantity

$$A(|x\rangle\langle x|, |y\rangle\langle y|) = \int_{T_n^{[-L,L]}} dh ||h||^2 |x\rangle\langle x|T(h)|y\rangle\langle y|$$

is given by

$$A(|x\rangle\langle x|, |y\rangle\langle y|) = d(x,y)^2 \, |x\rangle\langle y|$$

where d is the distance on the n torus. Therefore, $|x\rangle\langle x|$ is a distributional operator of norm squared $\frac{nL^{n+2}}{3}|x\rangle\langle x|$. The norm is then given by its square root which does not exist; it is however possible to construct quasi-roots be considering the operators

$$B^{\epsilon}(x) := \frac{1}{\epsilon^{\frac{5n}{2}}}\sqrt{\frac{nL^{n+2}}{3}} \int_{\times_n[-\frac{\epsilon}{2},\frac{\epsilon}{2}]} dh T_h |x\rangle\langle x| \int_{\times_n[-\frac{\epsilon}{2},\frac{\epsilon}{2}]} dz T_z$$

then

$$(B^{\epsilon}(x))^2 = \frac{nL^{n+2}}{3}\frac{1}{\epsilon^{2n}} \int_{\times_n[-\frac{\epsilon}{2},\frac{\epsilon}{2}]} dh T_h |x\rangle\langle x| \int_{\times_n[-\frac{\epsilon}{2},\frac{\epsilon}{2}]} dz T_z$$

which agrees in the limit for ϵ to zero with

$$\frac{nL^{n+2}}{3}|x\rangle\langle x|.$$

By definition, the distance formula equals

$$d(|x\rangle\langle x|, |y\rangle\langle y|)^2 = \int_{T_n^{[0,L]}} dr \int_{T_n^{[0,L]}} ds \, r.s \, (|y\rangle\langle y| - |x\rangle\langle x|) \, T_{r-s} \, (|y\rangle\langle y| - |x\rangle\langle x|) =$$

$$\frac{nL^{n+2}}{3}\left(|x\rangle\langle x| + |y\rangle\langle y| - |x\rangle\langle y| - |y\rangle\langle x|\right) + \frac{1}{4}L^n \, d(x,y)^2 \, \left(|x\rangle\langle y| + |y\rangle\langle x|\right).$$

Hence, we obtain

$$\frac{nL^{n+2}}{3}\left(|x\rangle\langle x| + |y\rangle\langle y|\right) - \left(\frac{nL^{n+2}}{3} - \frac{L^n}{4}d(x,y)^2\right)\left(|x\rangle\langle y| + |y\rangle\langle x|\right).$$

We may again look for quasi-roots of the operator

$$\widehat{d}(x,y)^2 := a\left(|x\rangle\langle x| + |y\rangle\langle y|\right) - b(x,y)\left(|x\rangle\langle y| + |y\rangle\langle x|\right)$$

where $a = \frac{nL^{n+2}}{3}, b(x,y) = \left(\frac{nL^{n+2}}{3} - \frac{L^n}{4}d(x,y)^2\right)$ satisfying $0 < b(x,y) < a$. They are all characterized by matrices of the type

$$B(x,y) := \begin{pmatrix} 1 & \epsilon d(x,y) - 1 \\ \epsilon d(x,y) - 1 & 1 \end{pmatrix}$$

with ϵ a very small number. Regarding a quantum triangle inequality

$$B(x,y) + B(y,z) - B(x,z) \sim \begin{pmatrix} 0 & \epsilon d(x,y) - 1 & 1 - \epsilon d(x,z) \\ \epsilon d(x,y) - 1 & 2 & \epsilon d(y,z) - 1 \\ \epsilon 1 - d(x,z) & \epsilon d(y,z) - 1 & 0 \end{pmatrix}$$

a Hermitian matrix with two positive and one negative eigenvalue due to the triangle inequality

$$d(x,y) + d(y,z) \geq d(x,z).$$

A reverse inequality, such as is the case in Lorentzian geometry, results in two negative eigenvalues and one positive one. It is therefore clear that no triangle inequality is satisfied at the level of $\widehat{d}(x, y)$ given that those operators do not commute and therefore Cauchy-Schwartz does not apply. These regard "quantum fluctuations" of metric geometry which are not present classicaly where the distance is positive real valued. We shall come back to this in the next chapter, but it must be clear that the quantity

$$\widetilde{d}(x, y) = \sqrt{-(\langle x| - \langle y|)\frac{1}{4}\mathrm{Vol}(T_n^{[0,L]}) \int_{T_n^{[-L,L]}} ds\, ||s||^2\, T_s(|x\rangle - |y\rangle)}$$

reproduces the classical distance and metric geometry. Indeed, notice that

$$(\langle x| - \langle y|)\, |1\rangle = 0$$

and, moreover,

$$d(|x\rangle, |y\rangle)^2 := -(\langle x| - \langle y|)\frac{1}{4}\mathrm{Vol}(T_n^{[0,L]}) \int_{T_n^{[-L,L]}} ds\, ||s||^2\, T_s(|x\rangle - |y\rangle) = \frac{L^n}{2} d(x, y)^2$$

which is clearly a satisfying formula allowing for regularisation in the limit for L towards infinity. Clearly,

$$d(|x\rangle, |y\rangle)$$

restricted to those "atomic" states satisfies the full triangle inequality given that d does. For more general states;

$$d(|\psi\rangle, |\phi\rangle)^2 := \frac{nL^{n+2}}{3} \left| \int_{T_n^{[-L,L]}} dh\, (\psi(h) - \phi(h)) \right|^2 -$$

$$\frac{L^n}{4} \int_{T_n^{[-L,L]}} dx \int_{T_n^{[-L,L]}} dy\, d(x, y)^2 \overline{(\psi(x) - \phi(x))}(\psi(y) - \phi(y)).$$

We show this quantity is indeed positive; suppose ζ has only support in a region for which $d(x, y)^2 = (x - y)^2$, then

$$\int_{T_n^{[-L,L]}} dx \int_{T_n^{[-L,L]}} dy\, (x^2 - 2x.y + y^2)\overline{\zeta(x)}\zeta(y) =$$

$$\overline{\int_{T_n^{[-L,L]}} dx\, \zeta(x)} \int_{T_n^{[-L,L]}} dy\, y^2 \zeta(y) + cc - 2|| \int_{T_n^{[-L,L]}} dy\, y\zeta(y)||^2$$

which allows for explicit verification of positivity of

$$d(|\zeta\rangle, 0)^2.$$

Hence, the triangle inequality is satisfied and coincides with the usual one on distributional atomistic elements. It is clear that Lorentzian geometry and non-abelian generalizations thereof can be treated in an entirely similar manner.

3 Differential geometry.

In the previous section we has cast flat, compactified, Euclidean and Minkowskian geometry into a new jacket and the only task is to study the limit L to infinity in a very succint way which necessitates giving up on the concept of a function space hereby introducing the concept of "infinitesimal vectors" and operators by means of a Cauchy procedure. Quantum geometry obviously necessitates such thing given that "points", given by Hermitian projection operators, are atomistic in a much weaker sense than it is for classical vectors in the Hilbert algebra. This weaker notion is holistic given that it has non-zero measure whereas the classical one is limiting to the zero measure case or distributional if appropriate rescalings are applied for. The reader must have noticed by now that

$$d(x,y) \sim d(|x\rangle,|y\rangle) \sim (\langle x|+\langle y|) \int_{\times_n\left[-\frac{\epsilon}{2},\frac{\epsilon}{2}\right]} dh\, T_h\, d(|x\rangle\langle x|,|y\rangle\langle y|) \int_{\times_n\left[-\frac{\epsilon}{2},\frac{\epsilon}{2}\right]} dh\, T_h\, (|x\rangle+|y\rangle).$$

Therefore, regarding

$$\triangle(x,y;z) := d(|x\rangle\langle x|,|y\rangle\langle y|) + d(|y\rangle\langle y|,|z\rangle\langle z|) - d(|x\rangle\langle x|,|z\rangle\langle z|)$$

one obtains that

$$\lim_{\epsilon\to 0}(\langle x|+\langle y|+\langle z|) \int_{\times_n\left[-\frac{\epsilon}{2},\frac{\epsilon}{2}\right]} dh\, T_h\, \triangle(x,y;z) \int_{\times_n\left[-\frac{\epsilon}{2},\frac{\epsilon}{2}\right]} dh\, T_h\, (|x\rangle+|y\rangle+|z\rangle) \geq 0$$

due to the classical triangle inequality. So, this is our classical-quantum correspondance: from a democratic state over all points, such as is the state associated to the barycenter of the triangle, the triangle inequality is satisfied on a average. The reader may compute the second moment of a smearing of the operator $\triangle(x,y;z)$ over the points x,y,z by means of the translation operators in order to study "quantum" fluctuations.

The reader must correctly understand that underlying the quantum geometry is a fixed classical one just as is the case in this author's work on quantum gravity. We now generalize this work to a curved classical background by means of the exponential map which is after all immediately determined by the geodesic equation and vierbein and generalizes the idea of a translation group towards non-abelian bi-groups. That is, locally, we may write

$$T_{[T_x(v)]}(w) = T_x((w\oplus v)_x)$$

where $w\oplus v$ is uniquely given if we demand that geodesics do not leave a certain open region $\mathcal{O}$ around x and $T_x(v) = \exp_x(v)$. On the other hand $T_x(v)$ may be thought of as representing a translation on the tangent space at x in which case the usual law

$$T_x(w)T_x(v) = T_x(v+w)$$

holds. We shall be interested in the first representation which is isomorphic to the second in flat Minkowski with respect to a global inertial frame so that there, the x dependency can be dropped in T_x as well as $\oplus_x$. Specifically, the global action T is

$$(T(v)f)(x) := f(T_x(v(x)))$$

where $v(x)$ is a vectorfield on $\mathcal{M}$. The element $v(x)$, seen as an ultralocal vector, may also serve as $T_{v(x)}$ on the flat geometry modelled at x. It is the exponential map which connects both representations as we shall see soon. One also has

$$[T(w)(T(v)f)](x) := [T(v\oplus w)f](x) = f(T_x(v\oplus w)_x) = f(T_{T_x(w(x))}(v(T_x(w(x))))).$$

Therefore, the right framework for curved geometry is the one of the induced non-abelian sum on the vectorfields. This calls for an extension of our previous setting; one could work with the Hilbert-algebra $\mathcal{H}$ of functions on $\mathcal{M}$ where $\mathcal{M}$ is compact, equipped with the real *Leibniz* topological dual $\mathcal{H}^{\star,L}$ on it defined by the continuous, real linear functionals D satisfying

$$D(fg) = D(f)\tilde{g} + \tilde{f}D(g)$$

where

$$\tilde{f}(x) = \lim_{\epsilon \to 0} \frac{1}{\mathrm{Vol}(B_\epsilon(x))} \int_{B_\epsilon(x)} f(y) dy \sqrt{h_y}.$$

The Leibniz rule is there to ensure the locality aspect and enables one to define $D(x)$ which is what we need; notice that the previous definition of $\mathcal{H}^{\star,L}$ does not depend upon the choice of H whereas quantum mechanically it might. Given that $\mathcal{H}^{\star,L}$ is infinite dimensional, we cannot integrate over it; however, we restrict to *constant* elements D which are those satisfying some equation of constancy. Note that we have something as a pull back defined by

$$f_\star D$$

where $[(f_\star D)(g)](x) = [D(g \circ f^{-1})](f(x))$ for $f \in \mathrm{Diff}(\mathcal{M})$ which is an automorphism of $\mathcal{H}$. Formulated more algebraically, every automorphism χ of $\mathcal{H}$ induces a mapping $\chi_\star : \mathcal{H}^{\star,L} \to \mathcal{H}^{\star,L}$ by means of

$$[\chi_\star D](f) = \chi[D(\chi^{-1}(f))].$$

Indeed, one checks that

$$[\chi_\star D](fg) = \chi[D(\chi^{-1}[fg])] = \chi[D(\chi^{-1}(f))]g + f\chi[D(\chi^{-1}[g])] = [[\chi_\star D](f)]g + f[[\chi_\star D](g)]$$

which shows its sanity. In general, constancy of an element requires metric information to have a benchmark. Here, our translations might come in handy:

$$T(D)^2 = T(2D)$$

as an equality between automorphisms on $\mathcal{H}$. This restricts the field to be geodesic; however, that leaves plenty of freedom. It is better to fix a point x and drag $D(x)$ along the geodesics emanating from it. Concretely, we look for a mapping

$$(\exp_x)_\star : \mathbb{R}^n \to \mathcal{H}^{\star,L} : D(x) \to D$$

where

$$D(f)(\exp_x(v)) := \lim_{\epsilon \to 0} \frac{[T_{\epsilon D(x)}(f \circ T_x)(v) - (f \circ T_x)(v)]}{\epsilon}$$

and the reader verifies that the Leibniz rule is satisfied. To implement this idea in abstracto, we need to make use of the fact that the T map really connects the Leibniz dual $\mathcal{H}^{\star,L}$ with the Hilbert algebra $\mathcal{H}$ given that the specialization to a "stalk" of the Leibniz dual at a point provides for a local automorphism between the respective local Hilbert algebra in $\mathbb{R}^n$ and a part of the Hilbert algebra $\mathcal{H}$ by means of the associated local diffeomorphism T_x. Here, the local Hilbert algebra $\mathcal{H}^{x,\mathcal{O}}_{\mathrm{loc}}$ is canonically defined by

$$(T_x)_\star \chi_{\mathcal{O}}$$

where χ_A is the standard characteristic function on $A \subset \mathbb{R}^n$. In vector language,

$$\langle (T_x)_\star \chi_{\mathcal{O}} | y \rangle := \chi_{\mathcal{O}}((T_x)^{-1}(y))$$

which determines the mapping completely given that we assume disjunct atoms to ca. Therefore, we have to take into account that $(\exp_x)_\star$ is only defined on a neighborhood of the origin of $\mathbb{R}^n$ given that one meets serious problems *globally*. Here, locality is hiding in the classical distance on the natural Hilbert-algebra $\mathcal{H}^{\text{flat}}(x)$ associated to the localized Leibniz topological dual $\mathcal{H}^{\star,L}$ at x. The formula for Df then reads

$$\lim_{\epsilon \to 0} \frac{f(T_x(v + \epsilon D(x))) - f(T_x(v))}{\epsilon}$$

and is to be understood in the usual way. $\mathcal{H}^{\text{flat}}(x)$ is defined by noticing that

$$(D(f\chi_{\mathcal{O}}))(x) = (Df)(x)$$

since

$$[D(\chi_{\mathcal{O}})](x) = [D(\chi^2_{\mathcal{O}})](x) = 2[D(\chi_{\mathcal{O}})](x) = 0$$

and therefore D depends at x only on $f\chi_{\mathcal{O}}$ and not the entire f. Since $\mathcal{O}$ was arbitrary, the limit to zero size can be taken what justifies the notation $D(x)$. This requires an a priori input of a topological class of idempotent elements $\zeta^2 = \zeta$. More in particular, we demand the existance of a Boolean isomorphism from

$$\psi : I(\mathcal{H}) \to \tau(A(\mathcal{H}))$$

where $A(\mathcal{H})$ is the set of all atoms and τ is a topology on it locally homeomorphic with a $\mathbb{R}^n$ metric topology. ψ is defined by resorting to a notion of inclusion which is the restriction of the partial order $\prec$ on $I(\mathcal{H})$ where

$$\alpha \prec \beta \leftrightarrow \alpha\beta = \alpha.$$

Moreover, an idempotent ζ is an open neighborhood $\mathcal{O}$ of an atom x if and only if for any equivalence class $[\xi_n]_n$ of x, there exists an n such that for any $m \geq n$ holds that $\xi_m \prec \zeta$. $\psi(\zeta) = \chi_{\mathcal{O}}$ meaning that all other information about ζ is redundant with regard to the scalar product on $\mathcal{H}$. It is clear that ψ induces a mapping between atoms and points which allows one to speak about differentiable structure. We assume moreover a C^∞ atlas to exist on $A(\mathcal{H})$ equipped with its local topology homeomorphic to $\mathbb{R}^n$.

This prepares the setting for a generalization of the geometry defined in the previous section. The crucial part is to use the standard spectral theorem on $\mathcal{H}$ to know that every element can be written as a sum of complex multiples of Hermitian idempotents which in their turn can be written as an integral of distributional atomistic idempotents (a Hilbert algebra is a commutative $C^\star$ algebra as well as a Hilbert space, where the $C^\star$ algebra is represented on itself). Therefore, the position "basis" of atoms always is a basis of orthogonal elements in the general distributional sense. A classical metric is defined in the following way: pick a point x and a scalar product $h_x(v(x), w(x))$ on $\mathcal{H}^{\star,L}(x)$ which we assume to be isomorphic, as a vector space, to $\mathbb{R}^n$. The pull back of h_x is defined as

$$(\chi_\star h)_{\chi^{-1}(x)}((\chi_\star v)_{\chi^{-1}(x)}, (\chi_\star w)_{\chi^{-1}(x)}) := h_x(v(x), w(x)).$$

If one were to define the h field by means of

$$[(T_x^{-1})_\star h]_{T_x(v)} = h_x$$

or

$$h_{T_x(v)} = [(T_x)_\star h_x](v)$$

where $h_x(v) = h_x$ then that definition would be x dependent and result in a flat geometry. To rectify this, note that T defines the full connection and therefore the parallel transporter which we denote with $\widehat{T}$. T and $\widehat{T}$ satisfy

$$T_{T_x(v)}(-((T_v)_\star(v))(x)) = x$$

for all x and $v \in \mathcal{H}^{\star,L}(x)$. Moreover, locally,

$$(\epsilon v) \oplus (\epsilon w) = \epsilon(v + w) + O(\epsilon^3).$$

As is well known from differential geometry, this issue *does* depend upon the choice of h_x if the latter is nondegenerate and symmetric and of fixed signature. Indeed, take a matrix field $O(x)$, then the connection associated to $O(x)g(x)O^T(x)$ is given by

$$O(x)\gamma(x)O^T(x) \otimes O^T(x) + \frac{1}{2}(O^T)^{-1}g^{-1}O^{-1}(\text{first derivatives of O}).$$

There are in general $\frac{n^2(n+1)}{2}$ equations and n^2 variables so that inconsistencies arise. This issue is pretty easily solved by demanding that

$$\lim_{\epsilon \to 0} \frac{\widehat{T}_{\epsilon v} h - h}{\epsilon} = 0$$

for the appropriate metric h and any field $v \in \mathcal{H}^{\star,L}$. Consistency then implies that

$$\lim_{\epsilon \to 0} \frac{\widehat{T}_{\epsilon v}\widehat{T}_{\epsilon w} h - \widehat{T}_{\epsilon(v+w)} h}{\epsilon^2} = 0$$

for any fields v, w and the two conditions on T which define one parameter subgroups and restrict the coincidental behaviour of $\oplus$, together with the fact that $\widehat{T}$ must define an infinitesimal isometry of the metric field, fix the classical geometry entirely.

We now proceed towards the end of this short introduction which is by no means complete. Delta *densities* are defined by

$$\int_{A(\mathcal{H})} dx\, \delta(x, z) f(x) := f(z)$$

where the integral has been constructed by making use of ψ and the local charts at $A(\mathcal{H})$ and the vector f maps to a continuous function on atomic space by means of a Hilbert-space limiting procedure. This gives meaning to

$$\langle z|f\rangle := f(z)$$

and

$$f = \int_{A(\mathcal{H})} dx\, f(x)\sqrt{h_x}|x\rangle$$

with
$$\langle z|w\rangle = \frac{\delta(z,w)}{\sqrt{h_z}}.$$
To ensure that it is really $\sqrt{h_z}$ showing up, we demand that the $T(\epsilon v)$ are unitary in the limit ϵ to zero up to second order in ϵ for conformal vectorfields v satisfying
$$\lim_{\epsilon\to 0} \frac{(T(\epsilon v))_\star\sqrt{h} - \sqrt{h}}{\epsilon} = 0.$$
More in particular, for those Leibniz dual elements, we have that
$$\lim_{\epsilon\to 0} \frac{\langle T(\epsilon v)\alpha|T(\epsilon v)\beta\rangle - \langle\alpha|\beta\rangle}{\epsilon} = 0.$$
An alternative route consists in taking a point $z = [\chi_n]$ as a generalized vector satisfying
$$\langle z|f\rangle = \widehat{f}(z)$$
where $\widehat{f}(z)$ is an algebra homomorphism from $\mathcal{H}$ to $\mathbb{C}$. The reader should proof that the a_n accomplishing this are given by $\frac{1}{||\chi_n||^2}$. The vector sub-algebra spoken about before is then simply defined by demanding that this expression exists and is independent of the equivalence. $|z\rangle$ is then not a generalized density but a generalized function which is the better way to follow. Define the "formal" operator B with as prescription
$$B|f\rangle := \int dz f(z)\sqrt{h}_z \int_{\mathcal{O}_z} dh\, h\, |T_z(h)\rangle$$
where the integral in $\mathcal{H}^{\star,L}(z)$ is executed with respect to inertial coordinates associated to an orthonormal basis of h_z. The reader then sees that we are really interested in the expression
$$d(|\alpha\rangle,|\beta\rangle)^2 := \langle B(|\alpha\rangle - |\beta\rangle)|B(|\alpha\rangle - |\beta\rangle)\rangle$$
which reproduces, at least locally, the correct classical distance obeying the triangle inequality. Quantum distances are then constructed by means of the scalar product
$$\langle A|C\rangle_{\mathrm{op}} := A^\dagger B^\dagger BC$$
for trace class operators A, C on $\mathcal{H}$.

4 Afterword.

In a previous paper of mine, I have generalized classical geometry to general path metric spaces. Careful elaboration on the setting introduced in this paper could pave the way for an alike suitable generalization. However, we are not there yet and it remains to be seen how far the scheme in this paper could lead towards a fruitful theory.