Generalization of the Bernstein-Vazirani algorithm beyond qubit systems

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First, we review the Bernstein-Vazirani algorithm for determining a bit string. Next, we discuss the generalized Bernstein-Vazirani algorithm for determining a natural number string. Finally, we discuss the generalized Bernstein-Vazirani algorithm for determining an integer string. All of the generalized algorithms presented here have the following structure. Given the set of real values $\{a_1, a_2, a_3, \ldots, a_N\}$ and a special function g, we determine N values of the function $g(a_1), g(a_2), g(a_3), \ldots, g(a_N)$ simultaneously. The speed of determining the strings is shown to outperform the best classical case by a factor of N in every case.

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I. INTRODUCTION

Quantum mechanics [1–6] provides exact and frequently remarkably accurate numerical predictions, as reported for over a century. There is a remarkable link in recent decades between quantum theory and information theory. It gives rise to the rich field of quantum information theory, which says novel proposals that outperform classical tasks or simply have no classical counterpart [6].

A fundamental research of quantum theory is the Leggett-type non-local variables theory [7], which is experimentally explored [8–10]. The experiments report that quantum theory does not accept a Leggett-type non-local variables interpretation, although some controversy remains around the conclusions and interpretations of the experimental outcomes [11–13].

Applications of quantum information theory also include the implementation of quantum algorithms. For example the implementation of the Deutsch's problem [14–16] is first experimentally realized on a nuclear magnetic resonance proof-of-principle quantum computer [17]. Implementation of the Deutsch-Jozsa algorithm on an ion-trap quantum computer is also achieved [18]. There are, as well, several other attempts to use single-photon two-qubit states for quantum computing. Oliveira *et al* implements the Deutsch's algorithm with polarization and transverse electromagnetic spatial modes as qubits [19]. Other achievements also include single-photon Bell states preparation and measurement [20], a decoherence-free implementation of Deutsch's algorithm using single-photon and using two logical qubits [21], and more recently a one-way quantum computing implementation of the algorithm[22].

The Deutsch-Jozsa algorithm is very well related to the so called Bernstein-Vazirani algorithm [23, 24], which can be considered as an extended version of the previous one. After these two algorithms, Simon's algorithm [25] is discovered, among others. There is an experimental implementation of a quantum algorithm that solves the Bernstein-Vazirani parity problem without entanglement [26]. Additionally, fiber-optics implementations of the Deutsch-Jozsa and Bernstein-Vazirani quantum algorithms with three qubits are realized [27]. Also, a variant of the algorithm for quantum learning being robust against noise is introduced [28], as well as a quantum algorithm for approximating the influences of Boolean functions and its applications [29]. The Bernstein-Vazirani algorithm is also versatile in quantum key distribution [30, 31] and transport implementation with ion qubits [32]. Quantum circuit by cne step method and similarity with neural network are discussed [33]. A generalization of the Bernstein-Vazirani algorithm to qudit systems is discussed [34].

The original Bernstein-Vazirani algorithm [23, 24] determines a bit string. Using a Boolean-valued function, it is extended to determining the values of the function [35, 36]. The values of the function are restricted in $\{0, 1\}$. By using the extension, we can consider quantum algorithm of calculating a multiplication [36]. Further, we can consider the root finding problem [37].

By extending the Bernstein-Vazirani algorithm more, we give an algorithm of determining the values of a function that are extended to the natural numbers $\mathbf{N}$. That is, the extended algorithm determines a natural number string instead of a bit string. So we have the generalized Bernstein-Vazirani algorithm for determining a natural number string. By using the extension, quantum algorithm for determining a homogeneous linear function is studied [38].

By extending the quantum algorithm more and more, we present an algorithm of determining the values of a function that are extended to the integers **Z**. That is, the extended algorithm determines an integer string instead of a natural number string.

In this article, first, we review the Bernstein-Vazirani algorithm for determining a bit string. Next, we discuss the generalized Bernstein-Vazirani algorithm for determining a natural number string. Finally, we discuss the generalized Bernstein-Vazirani algorithm for determining an integer string. All of the generalized algorithms presented here have the following structure. Given the set of real values $\{a_1, a_2, a_3, \ldots, a_N\}$ and a special function g, we determine N values of the function $g(a_1), g(a_2), g(a_3), \ldots, g(a_N)$ simultaneously. The speed of determining the strings is shown to outperform the best classical case by a factor of N in every case.

This article is organized as follows.

In Sec. II, we discuss the Bernstein-Vazirani algorithm for determining a bit string.

In Sec. III, we discuss the generalized Bernstein-Vazirani algorithm for determining a natural number string.

In Sec. IV, we discuss the generalized Bernstein-Vazirani algorithm for determining an integer string.

Section V concludes the article.

II. ALGORITHM FOR DETERMINING A BIT STRING

Let us suppose that the following sequence of real values is given

$$a_1, a_2, a_3, \ldots, a_N. \tag{1}$$

Let us now introduce a function

$$g : \mathbf{R} \rightarrow \{0, 1\}. \tag{2}$$

Our goal is of determining the following values (a bit string)

$$g(a_1), g(a_2), g(a_3), \ldots, g(a_N). \tag{3}$$

Recall that in the best classical case, we need N queries, that is, N separate evaluations of the function (2). In our quantum algorithm, we shall require a single query.

Throughout the discussion, we consider the problem in the modulo 2. Assume $g(a_j) \in \{0, 1\}$, and we define

$$g(a) = (g(a_1), g(a_2), g(a_3), \ldots, g(a_N)), \tag{4}$$

where each entry of $g(a)$ is a bit. Here $g(a) \in \{0, 1\}^N$. We define $f(x)$ as follows:

$$\begin{aligned} f(x) &= g(a) \cdot x \bmod 2 \\ &= g(a_1)x_1 + g(a_2)x_2 + \cdots + g(a_N)x_N \bmod 2, \end{aligned} \tag{5}$$

where $x = (x_1, ..., x_N) \in \{0, 1\}^N$. Let us follow the quantum states through the algorithm.

The input state is

$$|\psi_0\rangle = |0\rangle^{\otimes N}|1\rangle, \tag{6}$$

where $|0\rangle^{\otimes N}$ means $\overbrace{|0, 0, ..., 0\rangle}^{N}$. Here $|0\rangle$ is a 2-dimensional state and $|1\rangle$ is a 2-dimensional state. We discuss the Fourier transform of $|0\rangle$

$$|0\rangle \rightarrow \sum_{y=0}^{1} \frac{\omega^{y \cdot 0}|y\rangle}{\sqrt{2}} = \sum_{y=0}^{1} \frac{|y\rangle}{\sqrt{2}} = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \tag{7}$$

where $\omega = e^{\pi i} = -1$ and $\omega^0 = 1$.

Subsequently let us define the 2-dimensional state $|\phi\rangle$ as follows:

$$|\phi\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle). \tag{8}$$

In the following, we discuss the Fourier transform of $|1\rangle$

$$\begin{aligned} |1\rangle &\rightarrow \sum_{y=0}^{1} \frac{\omega^{y \cdot 1}|y\rangle}{\sqrt{2}} = \sum_{y=0}^{1} \frac{\omega^{y2-y}|y\rangle}{\sqrt{2}} \\ &= \sum_{y=0}^{1} \frac{\omega^{2-y}|y\rangle}{\sqrt{2}} = |\phi\rangle, \end{aligned} \tag{9}$$

where $\omega^{y2} = \omega^2 = 1$.

The Fourier transform of $|x_1 ... x_N\rangle$ is as follows:

$$\begin{aligned} &|x_1 ... x_N\rangle \\ &\rightarrow \sum_{z_1=0}^{1} \cdots \sum_{z_N=0}^{1} \frac{\omega^{z_1 x_1}|z_1\rangle}{\sqrt{2}} \cdots \frac{\omega^{z_N x_N}|z_N\rangle}{\sqrt{2}} \\ &= \sum_{z \in K} \frac{\omega^{z \cdot x}|z\rangle}{\sqrt{2^N}}, \end{aligned} \tag{10}$$

where $K = \{0, 1\}^N$ and z is $(z_1, z_2, ..., z_N)$. Hence, for completeness, $\sum_{z \in K}$ is a shorthand to the compound sum

$$\sum_{z_1 \in \{0,1\}} \cdots \sum_{z_N \in \{0,1\}}. \tag{11}$$

After the componentwise Fourier transforms of the first N 2-dimensional states and after the Fourier transform of the 2-dimensional state $|1\rangle$ in (6)

$$\overbrace{F|0\rangle \times F|0\rangle \times ... \times F|0\rangle}^{N} \times F|1\rangle, \tag{12}$$

we have

$$|\psi_1\rangle = \sum_{x \in K} \frac{|x\rangle}{\sqrt{2^N}}|\phi\rangle. \tag{13}$$

Here, the notation $F|0\rangle$ means the Fourier transform of $|0\rangle$ and the notation $F|1\rangle$ means the Fourier transform of $|1\rangle$.

We introduce $SUM_{f(x)}$ gate

$$|x\rangle|j\rangle \rightarrow |x\rangle|(f(x) + j) \bmod 2\rangle, \tag{14}$$

where

$$f(x) = g(a) \cdot x \bmod 2. \tag{15}$$

We have

$$SUM_{f(x)}|x\rangle|\phi\rangle = \omega^{f(x)}|x\rangle|\phi\rangle. \quad (16)$$

In what follows, we discuss the rationale behind of the above relation (16). Now consider applying the $SUM_{f(x)}$ gate to the state $|x\rangle|\phi\rangle$. Each term in $|\phi\rangle$ is of the form $\omega^{2-j}|j\rangle$. We see

$$SUM_{f(x)}\omega^{2-j}|x\rangle|j\rangle \rightarrow \omega^{2-j}|x\rangle|(j + f(x)) \bmod 2\rangle. \quad (17)$$

We introduce k such as $f(x)+j = k \Rightarrow 2-j = 2+f(x)-k$. Hence (17) becomes

$$SUM_{f(x)}\omega^{2-j}|x\rangle|j\rangle \rightarrow \omega^{f(x)}\omega^{2-k}|x\rangle|k \bmod 2\rangle. \quad (18)$$

When $k < 2$ we have $|k \bmod 2\rangle = |k\rangle$ and thus, the terms in $|\phi\rangle$ such that $k < 2$ are transformed as follows:

$$SUM_{f(x)}\omega^{2-j}|x\rangle|j\rangle \rightarrow \omega^{f(x)}\omega^{2-k}|x\rangle|k\rangle. \quad (19)$$

When $k = 2$, we have $|k \bmod 2\rangle = |0\rangle$ and

$$\omega^{2-k} = \omega^{2-2} = 1 = \omega^{2-0}. \quad (20)$$

So the above relation (19) holds. Therefore, the relation (16) holds.

We have $|\psi_2\rangle$, by operating $SUM_{f(x)}$ to $|\psi_1\rangle$,

$$SUM_{f(x)}|\psi_1\rangle = |\psi_2\rangle = \sum_{x\in K} \frac{\omega^{f(x)}|x\rangle}{\sqrt{2^N}}|\phi\rangle. \quad (21)$$

After the Fourier transform of $|x\rangle$, using the previous equations (10) and (21), we can now evaluate $|\psi_3\rangle$ as follows:

$$|\psi_3\rangle = \sum_{z\in K}\sum_{x\in K} \frac{(\omega)^{x\cdot z+f(x)}|z\rangle}{2^N}|\phi\rangle = \sum_{z\in K}\sum_{x\in K} \frac{(\omega)^{x\cdot z+g(a)\cdot x}|z\rangle}{2^N}|\phi\rangle. \quad (22)$$

Notice

$$\sum_{x\in K}(\omega)^{x\cdot(z+g(a))} = 2^N\delta_{z+g(a),\vec{2}} = 2^N\delta_{z,\vec{2}-g(a)}, \quad (23)$$

where $\vec{2} = \overbrace{(2, 2, ..., 2)}^{N}$. Therefore, the above summation is zero if $z \neq \vec{2} - g(a)$ and the above summation is 2^N if $z = \vec{2} - g(a)$. Thus we have

$$|\psi_3\rangle = \sum_{z\in K}\sum_{x\in K} \frac{(\omega)^{x\cdot z+g(a)\cdot x}|z\rangle}{2^N}|\phi\rangle = \sum_{z\in K} \frac{2^N\delta_{z,\vec{2}-g(a)}|z\rangle}{2^N}|\phi\rangle = |\vec{2} - (g(a_1), g(a_2), g(a_3), \ldots, g(a_N))\rangle|\phi\rangle \quad (24)$$

from which

$$|\vec{2} - (g(a_1), g(a_2), g(a_3), \ldots, g(a_N))\rangle \quad (25)$$

can be obtained. That is to say, if we measure the first N 2-dimensional states of the state $|\psi_3\rangle$, that is, $|\vec{2} - (g(a_1), g(a_2), g(a_3), \ldots, g(a_N))\rangle$, then we can retrieve the following values (a bit string)

$$g(a_1), g(a_2), g(a_3), \ldots, g(a_N) \quad (26)$$

using a single query.

III. ALGORITHM FOR DETERMINING A NATURAL NUMBER STRING

Let us suppose that the following sequence of real values is given

$$a_1, a_2, a_3, \ldots, a_N. \quad (27)$$

Let us now introduce a function

$$g : \mathbf{R} \rightarrow \mathbf{N}. \quad (28)$$

Our goal is of determining the following values (a natural number string)

$$g(a_1), g(a_2), g(a_3), \ldots, g(a_N). \quad (29)$$

Recall that in the best classical case, we need N queries, that is, N separate evaluations of the function (28). In our quantum algorithm, we shall require a single query.

We introduce a positive integer d. Throughout the discussion, we consider the problem in the modulo d. Assume the following

$$0 \leq \overbrace{g(a_1), g(a_2), g(a_3), \ldots, g(a_N)}^{N} \leq d-1, \quad (30)$$

where $g(a_j) \in \{0, 1, ..., d-1\}$, and we define

$$g(a) = (g(a_1), g(a_2), g(a_3), \ldots, g(a_N)), \quad (31)$$

where each entry of $g(a)$ is a natural number. Here $g(a) \in \{0, 1, ..., d-1\}^N$. We define $f(x)$ as follows:

$$f(x) = g(a)\cdot x \bmod d = g(a_1)x_1 + g(a_2)x_2 + \cdots + g(a_N)x_N \bmod d, \quad (32)$$

where $x = (x_1, ..., x_N) \in \{0, 1, ..., d-1\}^N$. Let us follow the quantum states through the algorithm.

The input state is

$$|\psi_0\rangle = |0\rangle^{\otimes N}|d-1\rangle, \quad (33)$$

where $|0\rangle^{\otimes N}$ means $\overbrace{|0, 0, ..., 0\rangle}^{N}$. Here $|0\rangle$ is a d-dimensional state and $|d-1\rangle$ is a d-dimensional state. We discuss the Fourier transform of $|0\rangle$

$$|0\rangle \rightarrow \sum_{y=0}^{d-1} \frac{\omega^{y\cdot 0}|y\rangle}{\sqrt{d}} = \sum_{y=0}^{d-1} \frac{|y\rangle}{\sqrt{d}}, \quad (34)$$

where $\omega^0 = 1$.

Subsequently let us define the d-dimensional state $|\phi\rangle$ as follows:

$$|\phi\rangle = \frac{1}{\sqrt{d}}(\omega^d|0\rangle + \omega^{d-1}|1\rangle + \cdots + \omega|d-1\rangle), \quad (35)$$

where $\omega = e^{2\pi i/d}$. In the following, we discuss the Fourier transform of $|d-1\rangle$

$$|d-1\rangle \to \sum_{y=0}^{d-1} \frac{\omega^{y\cdot(d-1)}|y\rangle}{\sqrt{d}} = \sum_{y=0}^{d-1} \frac{\omega^{yd-y}|y\rangle}{\sqrt{d}}$$
$$= \sum_{y=0}^{d-1} \frac{\omega^{d-y}|y\rangle}{\sqrt{d}} = |\phi\rangle, \quad (36)$$

where $\omega^{yd} = \omega^d = 1$.

The Fourier transform of $|x_1 ... x_N\rangle$ is as follows:

$$|x_1...x_N\rangle$$
$$\to \sum_{z_1=0}^{d-1} \cdots \sum_{z_N=0}^{d-1} \frac{\omega^{z_1 x_1}|z_1\rangle}{\sqrt{d}} \cdots \frac{\omega^{z_N x_N}|z_N\rangle}{\sqrt{d}}$$
$$= \sum_{z\in K} \frac{\omega^{z\cdot x}|z\rangle}{\sqrt{d^N}}, \quad (37)$$

where $K = \{0, 1, ..., d-1\}^N$ and z is $(z_1, z_2, ..., z_N)$. Hence, for completeness, $\sum_{z\in K}$ is a shorthand to the compound sum

$$\sum_{z_1\in\{0,1,\ldots,d-1\}} \cdots \sum_{z_N\in\{0,1,\ldots,d-1\}}. \quad (38)$$

After the componentwise Fourier transforms of the first N d-dimensional states and after the Fourier transform of the d-dimensional state $|d-1\rangle$ in (33)

$$\overbrace{F|0\rangle \times F|0\rangle \times ... \times F|0\rangle}^{N} \times F|d-1\rangle, \quad (39)$$

we have

$$|\psi_1\rangle = \sum_{x\in K} \frac{|x\rangle}{\sqrt{d^N}}|\phi\rangle. \quad (40)$$

Here, the notation $F|0\rangle$ means the Fourier transform of $|0\rangle$ and the notation $F|d-1\rangle$ means the Fourier transform of $|d-1\rangle$.

We introduce $SUM_{f(x)}$ gate

$$|x\rangle|j\rangle \to |x\rangle|(f(x) + j) \bmod d\rangle, \quad (41)$$

where

$$f(x) = g(a) \cdot x \bmod d. \quad (42)$$

We have

$$SUM_{f(x)}|x\rangle|\phi\rangle = \omega^{f(x)}|x\rangle|\phi\rangle. \quad (43)$$

In what follows, we discuss the rationale behind of the above relation (43). Now consider applying the $SUM_{f(x)}$ gate to the state $|x\rangle|\phi\rangle$. Each term in $|\phi\rangle$ is of the form $\omega^{d-j}|j\rangle$. We see

$$SUM_{f(x)}\omega^{d-j}|x\rangle|j\rangle$$
$$\to \omega^{d-j}|x\rangle|(j + f(x)) \bmod d\rangle. \quad (44)$$

We introduce k such as $f(x)+j = k \Rightarrow d-j = d+f(x)-k$. Hence (44) becomes

$$SUM_{f(x)}\omega^{d-j}|x\rangle|j\rangle$$
$$\to \omega^{f(x)}\omega^{d-k}|x\rangle|k \bmod d\rangle. \quad (45)$$

Now, when $k < d$ we have $|k \bmod d\rangle = |k\rangle$ and thus, the terms in $|\phi\rangle$ such that $k < d$ are transformed as follows:

$$SUM_{f(x)}\omega^{d-j}|x\rangle|j\rangle \to \omega^{f(x)}\omega^{d-k}|x\rangle|k\rangle. \quad (46)$$

Also, as $f(x)$ and j are bounded above by $d-1$, k is strictly less than $2d$. Hence, when $d \le k < 2d$ we have $|k \bmod d\rangle = |k-d\rangle$. Now, we introduce m such that $k - d = m$ then we have

$$\omega^{f(x)}\omega^{d-k}|x\rangle|k \bmod d\rangle = \omega^{f(x)}\omega^{-m}|x\rangle|m\rangle$$
$$= \omega^{f(x)}\omega^{d-m}|x\rangle|m\rangle. \quad (47)$$

Hence the terms in $|\phi\rangle$ such that $k \ge d$ are transformed as follows:

$$SUM_{f(x)}\omega^{d-j}|x\rangle|j\rangle \to \omega^{f(x)}\omega^{d-m}|x\rangle|m\rangle. \quad (48)$$

Hence from (46) and (48) we have

$$SUM_{f(x)}|x\rangle|\phi\rangle = \omega^{f(x)}|x\rangle|\phi\rangle. \quad (49)$$

Therefore, the relation (43) holds.

We have $|\psi_2\rangle$, by operating $SUM_{f(x)}$ to $|\psi_1\rangle$,

$$SUM_{f(x)}|\psi_1\rangle = |\psi_2\rangle = \sum_{x\in K} \frac{\omega^{f(x)}|x\rangle}{\sqrt{d^N}}|\phi\rangle. \quad (50)$$

After the Fourier transform of $|x\rangle$, using the previous equations (37) and (50), we can now evaluate $|\psi_3\rangle$ as follows:

$$|\psi_3\rangle = \sum_{z\in K}\sum_{x\in K} \frac{(\omega)^{x\cdot z+f(x)}|z\rangle}{d^N}|\phi\rangle$$
$$= \sum_{z\in K}\sum_{x\in K} \frac{(\omega)^{x\cdot z+g(a)\cdot x}|z\rangle}{d^N}|\phi\rangle. \quad (51)$$

Notice

$$\sum_{x\in K}(\omega)^{x\cdot(z+g(a))} = d^N \delta_{z+g(a),\vec{d}}$$
$$= d^N \delta_{z,\vec{d}-g(a)}, \quad (52)$$

where $\vec{d} = \overbrace{(d, d, ..., d)}^{N}$. Therefore, the above summation is zero if $z \neq \vec{d} - g(a)$ and the above summation is d^N if

$z = \vec{d} - g(a)$. Thus we have

$$\begin{aligned}|\psi_3\rangle &= \sum_{z\in K}\sum_{x\in K}\frac{(\omega)^{x\cdot z+g(a)\cdot x}|z\rangle}{d^N}|\phi\rangle \\ &= \sum_{z\in K}\frac{d^N\delta_{z,\vec{d}-g(a)}|z\rangle}{d^N}|\phi\rangle \\ &= |\vec{d} - (g(a_1), g(a_2), g(a_3), \ldots, g(a_N))\rangle|\phi\rangle \qquad (53)\end{aligned}$$

from which

$$|\vec{d} - (g(a_1), g(a_2), g(a_3), \ldots, g(a_N))\rangle \qquad (54)$$

can be obtained. That is to say, if we measure the first N d-dimensional states of the state $|\psi_3\rangle$, that is, $|\vec{d} - (g(a_1), g(a_2), g(a_3), \ldots, g(a_N))\rangle$, then we can retrieve the following values (a natural number string)

$$g(a_1), g(a_2), g(a_3), \ldots, g(a_N) \qquad (55)$$

using a single query.

IV. ALGORITHM FOR DETERMINING AN INTEGER STRING

Let us suppose that the following sequence of real values is given

$$a_1, a_2, a_3, \ldots, a_N. \qquad (56)$$

Let us now introduce a function

$$g : \mathbf{R} \to \mathbf{Z}. \qquad (57)$$

Our goal is of determining the following values (an integer string)

$$g(a_1), g(a_2), g(a_3), \ldots, g(a_N). \qquad (58)$$

Recall that in the best classical case, we need N queries, that is, N separate evaluations of the function (57). In our quantum algorithm, we shall require a single query.

We introduce a positive integer d. Throughout the discussion, we consider the problem in the modulo d. Assume the following

$$-(d-1) \le \overbrace{g(a_1), g(a_2), g(a_3), \ldots, g(a_N)}^{N} \le d-1, \qquad (59)$$

where $g(a_j) \in \{-(d-1), ..., -1, 0, 1, ..., d-1\}$, and we define

$$g(a) = (g(a_1), g(a_2), g(a_3), \ldots, g(a_N)), \qquad (60)$$

where each entry of $g(a)$ is an integer. Here $g(a) \in \{-(d-1), ..., -1, 0, 1, ..., d-1\}^N$. We define $f(x)$ as follows:

$$\begin{aligned}f(x) &= g(a)\cdot x \bmod d \\ &= g(a_1)x_1 + g(a_2)x_2 + \cdots + g(a_N)x_N \bmod d, \qquad (61)\end{aligned}$$

where $x = (x_1, ..., x_N) \in \{-(d-1), ..., -1, 0, 1, ..., d-1\}^N$. Let us follow the quantum states through the algorithm.

The input state is

$$|\psi_0\rangle = |0\rangle^{\otimes N}|d-1\rangle, \qquad (62)$$

where $|0\rangle^{\otimes N}$ means $|\overbrace{0, 0, ..., 0}^{N}\rangle$. Here $|0\rangle$ is a $(2d-1)$-dimensional state and $|d-1\rangle$ is a d-dimensional state. We discuss the general transform of $|0\rangle$

$$|0\rangle \to \sum_{y=-(d-1)}^{d-1}\frac{\omega^{y\cdot 0}|y\rangle}{\sqrt{2d-1}} = \sum_{y=-(d-1)}^{d-1}\frac{|y\rangle}{\sqrt{2d-1}}, \qquad (63)$$

where $\omega^0 = 1$.

Subsequently let us define the d-dimensional state $|\phi\rangle$ as follows:

$$|\phi\rangle = \frac{1}{\sqrt{d}}(\omega^d|0\rangle + \omega^{d-1}|1\rangle + \cdots + \omega|d-1\rangle), \qquad (64)$$

where $\omega = e^{2\pi i/d}$. In the following, we discuss the Fourier transform of $|d-1\rangle$

$$\begin{aligned}|d-1\rangle &\to \sum_{y=0}^{d-1}\frac{\omega^{y\cdot(d-1)}|y\rangle}{\sqrt{d}} = \sum_{y=0}^{d-1}\frac{\omega^{yd-y}|y\rangle}{\sqrt{d}} \\ &= \sum_{y=0}^{d-1}\frac{\omega^{d-y}|y\rangle}{\sqrt{d}} = |\phi\rangle, \qquad (65)\end{aligned}$$

where $\omega^{yd} = \omega^d = 1$.

The general transform of $|x_1...x_N\rangle$ is as follows:

$$\begin{aligned}&|x_1...x_N\rangle \\ &\to \sum_{z_1=-(d-1)}^{d-1}\cdots\sum_{z_N=-(d-1)}^{d-1}\frac{\omega^{z_1x_1}|z_1\rangle}{\sqrt{2d-1}}\cdots\frac{\omega^{z_Nx_N}|z_N\rangle}{\sqrt{2d-1}} \\ &= \sum_{z\in K}\frac{\omega^{z\cdot x}|z\rangle}{\sqrt{(2d-1)^N}}, \qquad (66)\end{aligned}$$

where $K = \{-(d-1), ..., -1, 0, 1, ..., d-1\}^N$ and z is $(z_1, z_2, ..., z_N)$. Hence, for completeness, $\sum_{z\in K}$ is a shorthand to the compound sum

$$\sum_{z_1\in\{-(d-1),...,-1,0,1,...,d-1\}}\cdots\sum_{z_N\in\{-(d-1),...,-1,0,1,...,d-1\}}. \qquad (67)$$

After the componentwise general transforms of the first N $(2d-1)$-dimensional states and after the Fourier transform of the d-dimensional state $|d-1\rangle$ in (62)

$$\overbrace{G|0\rangle \times G|0\rangle \times ... \times G|0\rangle}^{N} \times F|d-1\rangle, \qquad (68)$$

we have

$$|\psi_1\rangle = \sum_{x\in K}\frac{|x\rangle}{\sqrt{(2d-1)^N}}|\phi\rangle. \qquad (69)$$

Here, the notation $G|0\rangle$ means the general transform of $|0\rangle$ and the notation $F|d-1\rangle$ means the Fourier transform of $|d-1\rangle$.

We introduce $SUM_{f(x)}$ gate

$$|x\rangle|j\rangle \rightarrow |x\rangle|(f(x)+j) \bmod d\rangle, \tag{70}$$

where

$$f(x) = g(a)\cdot x \bmod d. \tag{71}$$

We have

$$SUM_{f(x)}|x\rangle|\phi\rangle = \omega^{f(x)}|x\rangle|\phi\rangle. \tag{72}$$

The rationale behind of the above relation (72) is equivalent to it of the relation (43). We have $|\psi_2\rangle$, by operating $SUM_{f(x)}$ to $|\psi_1\rangle$,

$$SUM_{f(x)}|\psi_1\rangle = |\psi_2\rangle = \sum_{x\in K}\frac{\omega^{f(x)}|x\rangle}{\sqrt{(2d-1)^N}}|\phi\rangle. \tag{73}$$

After the general transform of $|x\rangle$, using the previous equations (66) and (73), we can now evaluate $|\psi_3\rangle$ as follows:

$$\begin{aligned}|\psi_3\rangle &= \sum_{z\in K}\sum_{x\in K}\frac{(\omega)^{x\cdot z+f(x)}|z\rangle}{(2d-1)^N}|\phi\rangle \\ &= \sum_{z\in K}\sum_{x\in K}\frac{(\omega)^{x\cdot z+g(a)\cdot x}|z\rangle}{(2d-1)^N}|\phi\rangle.\end{aligned} \tag{74}$$

Notice

$$\begin{aligned}\sum_{x\in K}(\omega)^{x\cdot(z+g(a))} &= (2d-1)^N\delta_{z+g(a),0} \\ &= (2d-1)^N\delta_{z,-g(a)}.\end{aligned} \tag{75}$$

Therefore, the above summation is zero if $z \neq -g(a)$ and the above summation is $(2d-1)^N$ if $z = -g(a)$. Thus we have

$$\begin{aligned}|\psi_3\rangle &= \sum_{z\in K}\sum_{x\in K}\frac{(\omega)^{x\cdot z+g(a)\cdot x}|z\rangle}{(2d-1)^N}|\phi\rangle \\ &= \sum_{z\in K}\frac{(2d-1)^N\delta_{z,-g(a)}|z\rangle}{(2d-1)^N}|\phi\rangle \\ &= |-(g(a_1),g(a_2),g(a_3),\ldots,g(a_N))\rangle|\phi\rangle\end{aligned} \tag{76}$$

from which

$$|-(g(a_1),g(a_2),g(a_3),\ldots,g(a_N))\rangle \tag{77}$$

can be obtained. That is to say, if we measure the first N $(2d-1)$-dimensional states of the state $|\psi_3\rangle$, that is, $|-(g(a_1),g(a_2),g(a_3),\ldots,g(a_N))\rangle$, then we can retrieve the following values (an integer string)

$$g(a_1),g(a_2),g(a_3),\ldots,g(a_N) \tag{78}$$

using a single query.

V. CONCLUSIONS

In conclusion, first, we have discussed the Bernstein-Vazirani algorithm for determining a bit string. Next, we have discussed the generalized Bernstein-Vazirani algorithm for determining a natural number string. Finally, we have discussed the generalized Bernstein-Vazirani algorithm for determining an integer string. All of the generalized algorithms presented here have had the following structure. Given the set of real values $\{a_1,a_2,a_3,\ldots,a_N\}$ and a special function g, we have determined N values of the function $g(a_1),g(a_2),g(a_3),\ldots,g(a_N)$ simultaneously. The speed of determining the strings has been shown to outperform the best classical case by a factor of N in every case.

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