AN ELEMENTARY PROOF OF THE ABC CONJECTURE

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Abstract: In this paper, we gives an elementary proof of the *ABC* conjecture.

An Elementary Proof of the *ABC* Conjecture

To the memory of my Father who taught me arithmetic.

1. Introduction and notations

Let a a positive integer, $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as:

$$a = \prod_i a_i^{\alpha_i} = rad(a).\prod_i a_i^{\alpha_i - 1} \tag{1.1}$$

We note:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a.rad(a) \tag{1.2}$$

The ABC conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph OEsterlé of Pierre et Marie Curie University (Paris 6) ([1]). It describes the distribution of the prime factors of two integers with those of its sum. The definition of the ABC conjecture is given above:

Conjecture 1.3. *(**ABC Conjecture**): Let a, b, c positive integers relatively prime with $c = a + b$, then for each $\varepsilon > 0$, there exists $K(\varepsilon)$ such that :*

$$c < K(\varepsilon).rad(abc)^{1+\varepsilon} \tag{1.4}$$

2. Proof of the conjecture (1.3)

Let a, b, c positive integers, relatively prime, with $c = a + b$. We suppose that $b < a$, we can write that a verifies:

$$a = t.b + (1-t).c \quad t \in]0,1[, t = \frac{c-a}{c-b} \tag{2.1}$$

Let:

$$\beta_0 = \frac{1}{1-t} = \frac{c-b}{a-b} > 1 \Longrightarrow c = a\beta_0 - (\beta_0 - 1)b \Longrightarrow c < a\beta_0 + (\beta_0 - 1)b \tag{2.2}$$

As:

$$a\beta_0 + (\beta_0 - 1)b < \beta_0.(\beta_0 - 1).ab \Longrightarrow c < \beta_0.(\beta_0 - 1).ab \tag{2.3}$$

But $a = \mu_a.rad(a)$ and $b = \mu_b.rad(b)$, we obtain then:

$$c < \beta_0.(\beta_0 - 1).\mu_a.\mu_b rad(ab)$$

If $\beta > \beta_0 \Longrightarrow \beta.(\beta - 1) > \beta_0.(\beta_0 - 1) \Longrightarrow c < \beta.(\beta - 1).\mu_a.\mu_b rad(ab)$. Let $\varepsilon > 0$ a real number, we choose $\beta = \beta_0 + \varepsilon$ and taking :

$$\boxed{K(\varepsilon) = \frac{\beta(\beta - 1)\mu_a.\mu_b}{rad(c).rad(abc)^{\varepsilon}} = \frac{(\beta_0 + \varepsilon)(\beta_0 + \varepsilon - 1)\mu_a.\mu_b}{rad(c).rad(abc)^{\varepsilon}}} \tag{2.4}$$

Then, we obtain:

$$\boxed{c < K(\varepsilon).rad(abc)^{1+\varepsilon}} \tag{2.5}$$

Q.E.D

3. Examples

3.1 Example of Eric Reyssat

We give here the example of Eric Reyssat ([1]), it is given by:

$$3^{10} \times 109 + 2 = 23^5 = 6436343 \tag{3.1}$$

$a = 3^{10}.109 \Rightarrow \mu_a = 3^9 = 19683$ and $rad(a) = 3 \times 109$,
$b = 2 \Rightarrow \mu_b = 1$ and $rad(b) = 2$,
$c = 23^5 = 6436343 \Rightarrow rad(c) = 23$. Then $rad(abc) = 2 \times 3 \times 109 \times 23 = 15042$ and $\beta_0 = \dfrac{6436341}{3^{10}.109 - 2} = 1 + 3.10735652674 \times 10^{-7}$. For example, we take $\varepsilon = \beta_0 - 1 = 3.10735652674 \times 10^{-7}$, the expression of $K(\varepsilon)$ becomes:

$$K(\varepsilon) = K(\beta_0 - 1) = \frac{(2\beta_0 - 1)2\varepsilon.\mu_a.\mu_b}{rad(c).rad(abc)^\varepsilon} = \frac{2\varepsilon(1+2\varepsilon).19683}{23} \times 15042 \tag{3.2}$$

Let us verify (2.5):

$$\begin{aligned} c \overset{?}{<} K(\varepsilon).rad(abc)^\varepsilon \Longrightarrow c = 6436343 &\overset{?}{<} \frac{2\varepsilon(1+2\varepsilon).19683}{23} \times 15042 \\ log6436343 &\overset{?}{<} log2 + log(1+2\varepsilon) + log19683 - log23 + log15042 \\ 6.808\,639 &< 8.672\,425 \end{aligned} \tag{3.3}$$

Hence (2.5) is verified.

3.2 Example of A. Nitaj

The example of Nitaj about the ABC conjecture ([2]) is:

$$a = 11^{16}.13^2.79 = 613\,474\,843\,408\,551\,921\,511 \tag{3.4}$$

$$b = 7^2.41^2.311^3 = 2\,477\,678\,547\,239 \tag{3.5}$$

$$c = 2.3^3.5^{23}.953 = 613\,474\,845\,886\,230\,468\,750 \tag{3.6}$$

$$rad(abc) = 2.3.5.7.11.13.41.79.311.953 = 28\,828\,335\,646\,110 \tag{3.7}$$

we let the verification to the reader of the result given by (2.5).

4. Conclusion

We can announce the theorem:

Theorem 1. *(David Masser, Joseph Œsterlé & Abdelmajid Ben Hadj Salem; 2018) Let a, b, c positive integers relatively prime with $c = a + b$, then for each $\varepsilon > 0$, there exists $K(\varepsilon)$ such that :*

$$c < K(\varepsilon).rad(abc)^{1+\varepsilon} \tag{4.1}$$

where $K(\varepsilon)$ is given by (2.5).

References

[1] M. Waldschmidt. 2013. On the abc Conjecture and some of its consequences presented at The 6th World Conference on 21st Century Mathematics, Abdus Salam School of Mathematical Sciences (ASSMS), Lahore (Pakistan), March 6-9, 2013.

[2] T. Gowers. 2008. The Princeton companion to mathematics. Princeton, ISBN 0691118809, 1027 pages.