Spin radiation from a rotating dipole

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When the canonical spin tensor is used, a spin is detected along the axis of rotation of a rotating electric dipole. Early this spin radiation was obtained by Feynman in the frame of quantum mechanics. The magnitude of the spin flux is half the flux of the angular momentum that is emitted by a rotating dipole, according to modern electrodynamics, and this angular momentum flux is recognized here as an orbital angular momentum. Thus, the total angular momentum flux exceeds 1.5 times the value now recognized. It is shown that the torque experienced by a rotating dipole from the field is equal in magnitude to this total angular momentum flux.

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1. Introduction. Radiation of energy and angular momentum by a rotating dipole, according to classical electrodynamics

As is known, a rotating electric dipole or two dipole oscillators perpendicular to each other,

$$p^x = p\exp(-i\omega t), \quad p^y = ip\exp(-i\omega t)\,, \tag{1.1}$$

radiate electromagnetic waves. The power and the angular distribution of this power (Fig. 1) are, respectively, [1 § 67, Problem 1; 2]

$$P = \omega^4 p^2 / 6\pi\varepsilon_0 c^3, \quad dP/d\Omega = \omega^4 p^2 (\cos^2\theta + 1)/32\pi^2\varepsilon_0 c^3 \tag{1.2}$$

where $d\Omega = \sin\theta d\theta d\varphi$ (We use the system of units where $\mathrm{div}\mathbf{E} = \rho/\varepsilon_0$). The polarization of the radiation is circular along the axis of rotation and is linear in the plane of rotation (Fig. 3).

The radiation contains angular momentum L_z, which is the moment of linear momentum. This angular momentum flux, i.e. torque, is [1 § 72, § 75]

$$dL_z/dt = \tau_z = \omega^3 p^2 / 6\pi\varepsilon_0 c^3\,, \tag{1.3}$$

But this flux is located in the neighborhood of the plane of rotation where the polarization is near linear. The angular distribution of the angular momentum flux, according to [3-8], see Fig. 2, is

$$dL_z/dtd\Omega = \omega^3 p^2 \sin^2\theta / 16\pi^2\varepsilon_0 c^3\,. \tag{1.4}$$

As was noted [9], “The angular momentum (1.3) is not contained in the pure wave zone, where the field strenghts are perpendicular to $\mathbf{r}$ and behave like $1/r$. In this zone, indeed, L_z vanishes: L_z is proportional to E^r and $E^r \sim 1/r^2$”. So, we must recognize that this flux is not a radiation; this is an orbital angular momentum flux.

The presence of an angular momentum in the field of a rotating dipole is naturally. This field is a multipole field of order $(l = 1, m = 1)$. And equalities (1.2) and (1.3) are in the agreement with formula [9, 10 (9.144)]

$$dL_z/dt = mP/\omega\,. \tag{1.5}$$

Equation (1.5) is an additional proof that the moment of linear momentum L_z is not a spin. According to (1.5), each photon has an angular momentum $L_z = m\hbar$, not $\hbar$.

2. Spin radiation by a rotating dipole in the frame of the electrodynamics

At the same time, the modern electrodynamics does not notice an angular momentum flux in the direction of the axis of rotation, where the radiation is intense and the polarization is circular, although it was suggested as early as 1899 by Sadowsky [11] and as 1909 by Poynting [12] that circularly

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polarized light carries angular momentum volume *density,* and the angular momentum density is proportional to the energy volume density.

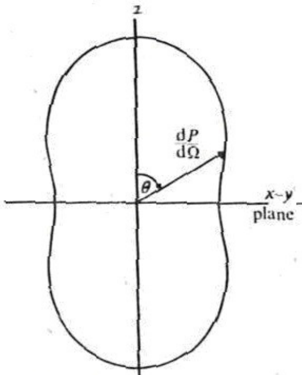

Fig. 1.
Angular distribution of the energy flux.
$dP/d\Omega \propto (\cos^2\theta+1)$

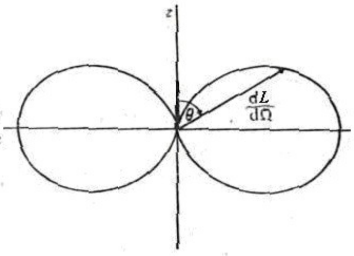

Fig. 2.
Angular distribution of z-component of the moment of momentum flux
$dL_z/dtd\Omega \propto \sin^2\theta$

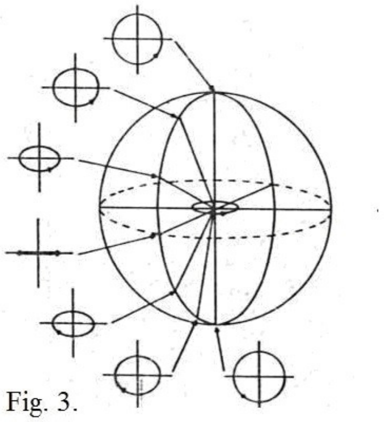

Fig. 3.
Polarization of the electric field seen by looking from different directions at a circular oscillator

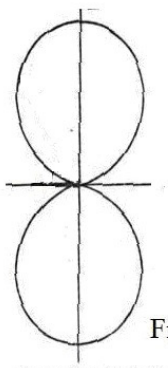

Fig. 4.
Angular distribution of z-component of the spin flux
$dS_Z/dtd\Omega \propto \cos^2\theta$

J.H. Poynting: If we put E for the energy in unit volume and G for the torque per unit area, we have $G = E\lambda/2\pi$ [12, p. 565].

This sentence points that any absorption of a circularly polarized light results in a mechanical torque volume density acting on the absorber (see also [13]).

The classical experiments [14 – 17] confirm that the angular momentum density is proportional to energy density. In these experiments, the angular momentum of the light was transferred to a half-wave plate, which rotated. So, work was performed *in any point of the plate*. This (positive or negative) amount of work reappeared as an alteration in the frequency of the light, which resulted in moving fringes *in any point* of the interference pattern in a suitable interference experiment.

Now, according to the Lagrange formalism, this angular momentum is recognised as *spin* and is described by the canonical spin tensor [18-20]

$$\Upsilon^{\lambda\mu\nu} = -2A^{[\lambda}F^{\mu]\nu}, \quad \Upsilon^{xyt} = -2A^{[x}F^{y]t} = \varepsilon_0\mathbf{E}\times\mathbf{A}, \tag{2.1}$$

where A^λ is the magnetic vector potential and $F^{\mu\nu}$ is the electromagnetic field tensor. The expression $\varepsilon_0\mathbf{E}\times\mathbf{A}$ is also presented in [21,10].

The sense of the spin tensor $\Upsilon^{\lambda\mu\nu}$ is given by the equalities:

$$d^3S^{ij} = \Upsilon^{ijk}da_k dt, \quad d^3S^{ij} = \Upsilon^{ijt}dV; \tag{2.2}$$

where d^3S^{ij} is the spin passing through a surface element da_k, or the spin which is contained in a volume element dV. This spin tensor was successfully used to describe the spin of plane waves [22-24].

And, since a rotating dipole radiates circularly polarized waves along its axis of rotation, it must radiate spin along this direction. A calculation of this spin radiation is presented here.

The spin volume density $\varepsilon_0\mathbf{E}\times\mathbf{A}$ is integrated over a thin spherical layer (of thickness dr), which surrounds the source of the radiation, and then the integral is divided by dt on the assumption $dr/dt = c$. So, the formula for the spin flux is

$$dS^{xy}/dt = \int \Upsilon^{xyt}r^2 d\Omega dr/dt, \tag{2.3}$$

The expression for *radiated* electric field [25, 2] is used

$$\mathbf{E} = \frac{\omega^2(\mathbf{p}r^2 - (\mathbf{pr})\mathbf{r})}{4\pi\varepsilon_0 c^2 r^3}\exp(ikr - i\omega t) \tag{2.4}$$

$$E_x = \frac{\omega^2 p(r^2 - x^2 - ixy)}{4\pi\varepsilon_0 c^2 r^3}\exp(ikz - i\omega t), \quad E_y = \frac{\omega^2 p(ir^2 - xy - iy^2)}{4\pi\varepsilon_0 c^2 r^3}\exp(ikr - i\omega t) \tag{2.5}$$

$$\mathbf{A} = -\int \mathbf{E}dt = -i\mathbf{E}/\omega. \tag{2.6}$$

Inserting (1.1), (2.1), (2.5), (2.6) into (2.3) yields the time averaged spin flux:

$$dS^{xy}/dt = \Re\int \varepsilon_0 (E_x \overline{A}_y - E_y \overline{A}_x) cr^2 d\Omega/2 = \Re\int i\varepsilon_0 (E_x \overline{E}_y - E_y \overline{E}_x) cr^2 d\Omega / 2\omega. \tag{2.7}$$

Here

$$\begin{aligned}
&(E_x \overline{E}_y - E_y \overline{E}_x) \\
&= \frac{\omega^4 p^2}{16\pi^2 \varepsilon_0^2 c^4 r^6}[(r^2 - x^2 - iyx)(-ir^2 - xy + iy^2) - (ir^2 - xy - iy^2)(r^2 - x^2 + iyx)] \\
&= \frac{-i\omega^4 p^2 z^2}{8\pi^2 \varepsilon_0^2 c^4 r^4} = \frac{-i\omega^4 p^2}{8\pi^2 \varepsilon_0^2 c^4 r^2} \cos^2\theta.
\end{aligned} \tag{2.8}$$

Inserting (2.8) into (2.7) yields

$$dS^{xy}/dt = \int \frac{\omega^3 p^2}{16\pi^2 \varepsilon_0 c^3} \cos^2\theta d\Omega. \tag{2.9}$$

So, the angular distribution of the spin flux (see Fig. 4) is

$$dS_z / dtd\Omega = \omega^3 p^2 \cos^2\theta / 16\pi^2 \varepsilon_0 c^3. \tag{2.10}$$

Integration of equality (2.9) gives the spin flux

$$dS_z / dt = \omega^3 p^2 / 12\pi\varepsilon_0 c^3. \tag{2.11}$$

The results (2.10), (2.11) were presented in the works [6-8].

Thus the total angular momentum flux, orbital + spin, (1.3) + (2.11), is

$$dJ_z/dt = dL_z/dt + dS_z/dt = \omega^3 p^2/6\pi\varepsilon_0 c^3 + \omega^3 p^2/12\pi\varepsilon_0 c^3 = \omega^3 p^2/4\pi\varepsilon_0 c^3. \tag{2.12}$$

Note that for $\theta = 0$, i.e. where there is no orbital angular momentum (1.3), according to (1.2) and (2.10), the photon relation is valid:

$$\text{(energy)} = \omega\text{(spin)}, \quad dPdt = \omega dS_z = \omega^4 p^2 / 16\pi^2 \varepsilon_0 c^3 d\Omega dt. \tag{2.13}$$

3. Spin radiation by a rotating dipole in the frame of the quantum mechanics

It is remarkable that the result (2.10), $dS_z / dtd\Omega \propto \cos^2\theta$, for the angular distribution of z-component of the spin flux was obtained by Feynman [26] beyond the standard electrodynamics. Really, the amplitudes that a RHC photon and a LHC photon are emitted in the direction θ into a certain small solid angle $d\Omega$ are [26, (18.1), (18.2)]

$$a(1+\cos\theta)/2 \quad \text{and} \quad -a(1-\cos\theta)/2. \tag{3.1}$$

So, in the direction θ, the spin flux density is proportional to

$$[a(1+\cos\theta)/2]^2 - [a(1-\cos\theta)/2]^2 = a^2\cos\theta. \tag{3.2}$$

The projection of the spin flux density on z-axis is

$$dS_z / dtd\Omega \propto a^2 \cos^2\theta. \tag{3.3}$$

Note that the Feynman's method gives the power distribution (1.2) as well:

$$dP/d\Omega \propto [a(1+\cos\theta)/2]^2 + [a(1-\cos\theta)/2]^2 = a^2(1+\cos^2\theta)/2. \tag{3.4}$$

4. Reaction to the dipole emitting the angular momentum flux

When emitting the angular momentum flux (2.12), the rotating dipole must experience the torque of the opposite direction. Here is the calculation of this torque, which is experienced by the dipoles (1.1).

$$p^x = p\exp(-i\omega t), \quad p^y = ip\exp(-i\omega t),$$

For this calculation, we use the result obtained by considering the absorption of a circularly polarized wave [13,24,27,28]. The mechanical stresses indicated in [13] arise from the action of the volume torque density $\tau_\wedge$ on the absorber. If the absorber is an electrically conductive medium, then the torque density is given by a formula similar to the formula for the density of the Lorentz force $f_\wedge = \mathbf{j}\times\mathbf{B}$:

$$\tau_\wedge = \mathbf{j}\times\mathbf{A}, \tag{4.1}$$

where $\mathbf{j}$ and $\mathbf{A}$ are the electric current density and the magnetic vector potential, respectively, and $\wedge$ means "density". This formula is used in this article in the form

$$d\tau = Id\mathbf{l} \times \mathbf{A} \tag{4.2}$$

to calculate the action on the dipoles (1.1). Here $d\tau$ is the torque acting on an element $d\mathbf{l}$ of wire that carries the current I.

The dipoles considered here are "elementary vibrators" in the sense that the current is the same at all points of the dipole, and the charges are only at the ends. The current of the dipoles is obtained by differentiating the relation $p = ql$. From (1.1), it turns out

$$I^x = \partial_t p^x / l = -ip\omega \exp(-i\omega t)/l, \quad I^y = \partial_t p^y / l = p\omega \exp(-i\omega t)/l. \tag{4.3}$$

To calculate the action on x-dipole (it is located in the Fig. 5 horizontally), an element dy of y-dipole is considered. The current I^y of the element dy creates a retarded vector potential dA^y [1 (66.2)] near element dx of x-dipole:

$$dA^y = I^y dy \exp(i\omega r)/4\pi r = p\omega dy \exp[i\omega(r-t)]/4\pi rl. \tag{4.4}$$

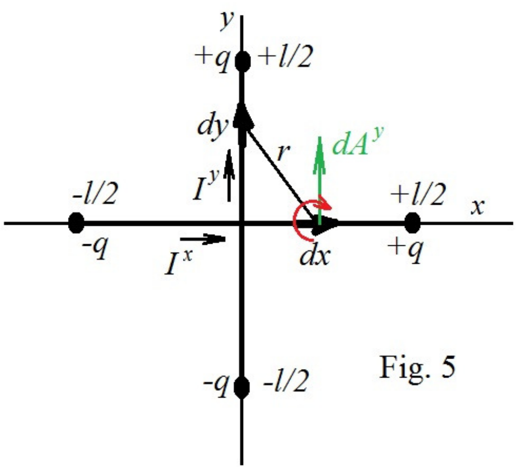

Fig. 5

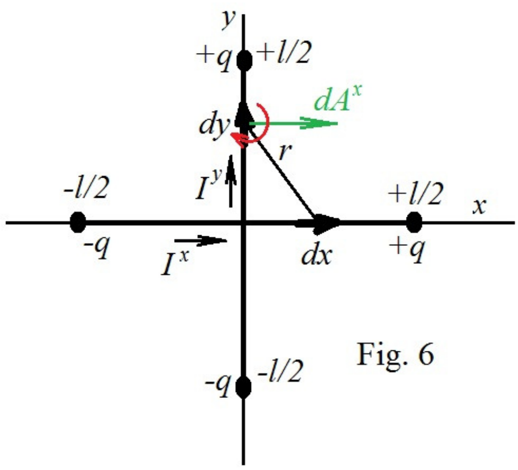

Fig. 6

According to formula (4.2), the torque acting on element dx of x-dipole is equal to

$$d^2\tau^{xy} = \Re\{(I^x)^* dx dA^y\}\frac{1}{2} = \Re\{i\exp(i\omega t)\exp[i\omega(r-t)]\}\frac{p^2\omega^2 dxdy}{8\pi rl^2} = -\sin\omega r \frac{p^2\omega^2 dxdy}{8\pi rl^2}, \tag{4.5}$$

where * means complex conjugation. For a small dipole, we replace $\sin\omega r \to \omega r$, reduce by r (!), integrate over x, y within $-l/2, l/2$, and get

$$\text{for } x\text{-dipole: } \tau^{xy} = -p^2\omega^3/8\pi. \tag{4.6}$$

The same torque is experienced by y-dipole. Really

$$dA^x = I^x dx \exp(i\omega r)/4\pi r = -ip\omega dx \exp[i\omega(r-t)]/4\pi rl. \tag{4.7}$$

$$d^2\tau^{xy} = -\Re\{(I^y)^* dy dA^x\}\frac{1}{2} = \Re\{\exp(i\omega t) i \exp[i\omega(r-t)]\}\frac{p^2\omega^2 dxdy}{8\pi rl^2} = -\sin\omega r \frac{p^2\omega^2 dxdy}{8\pi rl^2}, \tag{4.8}$$

$$\text{for } y\text{-dipole: } \tau^{xy} = -p^2\omega^3/8\pi. \tag{4.9}$$

Adding the results (4.6) and (4.9), we obtain

$$\text{for a rotating dipole } \tau^{xy} = -p^2\omega^3/4\pi. \tag{4.10}$$

This coincides in magnitude with the total angular momentum flux, orbital + spin, emitting from the rotating dipole (2.12).

The Coulomb interaction between the charges of dipoles due to the electric field, as well as the action of the Lorentz force on the dipoles due to the magnetic field, can not produce a nonzero result because these fields, unlike that the field $\mathbf{A}$, increase as $1/r^2$ with decreasing the size of the dipoles, and this action would tend to infinity.

5.Conclusion

The successful use of the canonical spin tensor confirms the validity of the criticism of the modern concept of classical spin, which was previously stated [13,24,29]. In particular, a plane wave of circular polarization contains the spin density

I am eternally grateful to Professor Robert Romer for the courageous publication of my question: "Does a plane wave really not carry spin?" (was submitted on 07 October, 1999) [30].

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