International J.Math. Combin. Vol.4(2017), 68-74

A Class of Lie-admissible Algebras

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Abstract: In this paper, we study nonassociative algebras which satisfy the following identities: $(xy)z = (yx)z, x(yz) = x(zy)$. These algebras are Lie-admissible algebras i.e., they become Lie algebras under the commutator $[f, g] = fg - gf$. We obtain a nonassociative Gröbner-Shirshov basis for the free algebra $LA(X)$ with a generating set X of the above variety. As an application, we get a monomial basis for $LA(X)$. We also give a characterization of the elements of $S(X)$ among the elements of $LA(X)$, where $S(X)$ is the Lie subalgebra, generated by X, of $LA(X)$.

Key Words: Nonassociative algebra, Lie admissible algebra, Gröbner-Shirshov basis.

AMS(2010): 17D25, 13P10, 16S15.

§1. Introduction

In 1948, A. A. Albert introduced a new family of (nonassociative) algebras whose commutator algebras are Lie algebras [1]. These algebras are called Lie-admissible algebras, and they arise naturally in various areas of mathematics and mathematical physics such as differential geometry of affine connections on Lie groups. Examples include associative algebras, pre-Lie algebras and so on.

Let $k\langle X\rangle$ be the free associative algebra generated by X. It is well known that the Lie subalgebra, generated X, of $k\langle X\rangle$ is a free Lie algebra (see for example [6]). Friedrichs [15] has given a characterization of Lie elements among the set of noncommutative polynomials. A proof of characterization theorem was also given by Magnus [18], who refers to other proofs by P. M. Cohn and D. Finkelstein. Later, two short proofs of the characterization theorem were given by R. C. Lyndon [17] and A. I. Shirshov [21], respectively.

Pre-Lie algebras arise in many areas of mathematics and physics. As was pointed out by D. Burde [8], these algebras first appeared in a paper by A. Cayley in 1896 (see [9]). Survey [8] contains detailed discussion of the origin, theory and applications of pre-Lie algebras in geometry and physics together with an extensive bibliography. Free pre-Lie algebras had already

[1]Supported by the NNSF of China (Nos. 11401246; 11426112; 11501237), the NSF of Guangdong Province (Nos. 2014A030310087; 2014A030310119; 2016A030310099), the Outstanding Young Teacher Training Program in Guangdong Universities (No. YQ2015155), the Research Fund for the Doctoral Program of Huizhou University (Nos. C513.0210; C513.0209; 2015JB021) and the Scientific Research Innovation Team Project of Huizhou University (hzuxl201523).

[2]Received May 25, 2017, Accepted November 15, 2017.

been studied as early as 1981 by Agrachev and Gamkrelidze [2]. They gave a construction of monomial bases for free pre-Lie algebras. Segal [20] in 1994 gave an explicit basis (called good words in [20]) for a free pre-Lie algebra and applied it for the PBW-type theorem for the universal pre-Lie enveloping algebra of a Lie algebra. Linear bases of free pre-Lie algebras were also studied in [3, 10, 11, 14, 25]. As a special case of Segal's latter result, the Lie subalgebra, generated by X, of the free pre-Lie algebra with generating set X is also free. Independently, this result was also proved by A. Dzhumadil'daev and C. Löfwall [14]. M. Markl [19] gave a simple characterization of Lie elements in free pre-Lie algebras as elements of the kernel of a map between spaces of trees.

Gröbner bases and Gröbner-Shirshov bases were invented independently by A.I. Shirshov for ideals of free (commutative, anti-commutative) non-associative algebras [22, 24], free Lie algebras [23, 24] and implicitly free associative algebras [23, 24] (see also [4, 5, 12, 13]), by H. Hironaka [16] for ideals of the power series algebras (both formal and convergent), and by B. Buchberger [7] for ideals of the polynomial algebras.

In this paper, we study a class of Lie-admissible algebras. These algebras are nonassociative algebras which satisfy the following identities: $(xy)z = (yx)z, x(yz) = x(zy)$. Let $LA(X)$ be the free algebra with a generating set X of the above variety. We obtain a nonassociative Gröbner-Shirshov basis for the free algebra $LA(X)$. Using the Composition-Diamond lemma of nonassociative algebras, we get a monomial basis for $LA(X)$. Let $S(X)$ be the Lie subalgebra, generated by X, of $LA(X)$. We get a linear basis of $S(X)$. As a corollary, we show that $S(X)$ is not a free Lie algebra when the cardinality of X is greater than 1. We also give a characterization of the elements of $S(X)$ among the elements of $LA(X)$. For the completeness of this paper, we formulate the Composition-Diamond lemma for free nonassociative algebras in Section 2.

§2. Composition-Diamond Lemma for Nonassociative Algebras

Let X be a well ordered set. Each letter $x \in X$ is a nonassociative word of degree 1. Suppose that u and v are nonassociative words of degrees m and n respectively. Then uv is a nonassociative word of degree $m + n$. Denoted by $|uv|$ the degree of uv, by X^* the set of all associative words on X and by X^{**} the set of all nonassociative word on X. If $u = (p(v)q)$, where $p, q \in X^*, u, v \in X^{**}$, then v is called a subword of u. Denote u by $u|_v$, if this is the case.

The set X^{**} can be ordered by the following way: $u > v$ if either

(1) $|u| > |v|$; or

(2) $|u| = |v|$ and $u = u_1u_2$, $v = v_1v_2$, and either

(2a) $u_1 > v_1$; or

(2b) $u_1 = v_1$ and $u_2 > v_2$.

This ordering is called degree lexicographical ordering and used throughout this paper.

Let k be a field and $M(X)$ be the free nonassociative algebra over k, generated by X. Then each nonzero element $f \in M(X)$ can be presented as

$$f = \alpha\overline{f} + \sum_i \alpha_i u_i,$$

where $\overline{f} > u_i, \alpha, \alpha_i \in k, \alpha \neq 0,\ u_i \in X^{**}$. Then $\overline{f}$, α are called the leading term and leading coefficient of f respectively and f is called monic if $\alpha = 1$. Denote by $d(f)$ the degree of f, which is defined by $d(f) = |\bar{f}|$.

Let $S \subset M(X)$ be a set of monic polynomials, $s \in S$ and $u \in X^{**}$. We define S-word $(u)_s$ in a recursive way:

(*i*) $(s)_s = s$ is an S-word of s-length 1;

(*ii*) If $(u)_s$ is an S-word of s-length k and v is a nonassociative word of degree l, then

$$(u)_s v \quad \text{and} \quad v(u)_s$$

are S-words of s-length $k + l$.

Note that for any S-word $(u)_s = (asb)$, where $a, b \in X^*$, we have $\overline{(asb)} = (a(\bar{s})b)$.

Let f, g be monic polynomials in $M(X)$. Suppose that there exist $a, b \in X^*$ such that $\bar{f} = (a(\bar{g})b)$. Then we define the composition of inclusion

$$(f, g)_{\bar{f}} = f - (agb).$$

The composition $(f, g)_{\bar{f}}$ is called trivial modulo $(S, \bar{f})$, if

$$(f, g)_{\bar{f}} = \sum_i \alpha_i (a_i s_i b_i)$$

where each $\alpha_i \in k,\ a_i, b_i \in X^*,\ s_i \in S,\ (a_i s_i b_i)$ an S-word and $(a_i(\bar{s_i})b_i) < \bar{f}$. If this is the case, then we write $(f, g)_{\bar{f}} \equiv 0 \bmod(S, \bar{f})$. In general, for $p, q \in M(X)$ and $w \in X^{**}$, we write

$$p \equiv q \mod(S, w)$$

which means that $p - q = \sum \alpha_i (a_i s_i b_i)$, where each $\alpha_i \in k, a_i, b_i \in X^*,\ s_i \in S,\ (a_i s_i b_i)$ an S-word and $(a_i(\bar{s_i})b_i) < w$.

Definition 2.1([22,24]) *Let $S \subset M(X)$ be a nonempty set of monic polynomials and the ordering $>$ defined as before. Then S is called a Gröbner-Shirshov basis in $M(X)$ if any composition $(f, g)_{\bar{f}}$ with $f, g \in S$ is trivial modulo $(S, \bar{f})$, i.e., $(f, g)_{\bar{f}} \equiv 0$ mod$(S, \bar{f})$.*

Theorem 2.2([22,24]) (Composition-Diamond lemma for nonassociative algebras) *Let $S \subset M(X)$ be a nonempty set of monic polynomials, $Id(S)$ the ideal of $M(X)$ generated by S and the ordering $>$ on X^{**} defined as before. Then the following statements are equivalent:*

(*i*) *S is a Gröbner-Shirshov basis in $M(X)$;*

(*ii*) *$f \in Id(S) \Rightarrow \bar{f} = (a(\bar{s})b)$ for some $s \in S$ and $a, b \in X^*$, where (asb) is an S-word;*

(iii) $Irr(S) = \{u \in X^{**} | u \neq (a(\bar{s})b)\ a, b \in X^*,\ s \in S$ *and* (asb) *is an* S*-word*$\}$ *is a linear basis of the algebra* $M(X|\ S) = M(X)/Id(S)$.

§3. A Nonassociative Gröbner-Shirshov Basis for the Algebra $LA(X)$

Let $\mathcal{LA}$ be the variety of nonassociative algebras which satisfy the following identities: $(xy)z = (yx)z, x(yz) = x(zy)$. Let $LA(X)$ be the free algebra with a generating set X of the variety $\mathcal{LA}$. It's clear that the free algebra $LA(X)$ is isomorphic to $M(X|(uv)w - (vu)w, w(uv) - w(vu), u, v, w \in X^{**})$.

Theorem 3.1 *Let* $S = \{(uv)w - (vu)w, w(uv) - w(vu), u > v, u, v, w \in X^{**}\}$. *Then* S *is a Gröbner-Shirshov basis of the algebra* $M(X|\ (uv)w - (vu)w, w(uv) - w(uv), u, v, w \in X^{**})$.

Proof It is clear that $Id(S)$ is the same as the ideal generated by the set $\{(uv)w - (vu)w, w(uv) - w(vu), u, v, w \in X^{**}\}$ of $M(X)$. Let $f_{123} = (u_1u_2)u_3 - (u_2u_1)u_3, g_{123} = v_1(v_2v_3) - v_1(v_3v_2), u_1 > u_2, v_2 > v_3, u_i, v_i \in X^{**}, 1 \leq i \leq 3$. Clearly, $\overline{f_{123}} = (u_1u_2)u_3$ and $\overline{g_{123}} = v_1(v_2v_3)$. Then all possible compositions in S are the following:

(c_1) $(f_{123}, f_{456})_{(u_1|_{(u_4u_5)u_6}u_2)u_3}$;
(c_2) $(f_{123}, f_{456})_{(u_1u_2|_{(u_4u_5)u_6})u_3}$;
(c_3) $(f_{123}, f_{456})_{(u_1u_2)u_3|_{(u_4u_5)u_6}}$;
(c_4) $(f_{123}, f_{456})_{((u_4u_5)u_6)u_3}, u_1u_2 = (u_4u_5)u_6$;
(c_5) $(f_{123}, f_{456})_{(u_1u_2)u_3}, (u_1u_2)u_3 = (u_4u_5)u_6$;
(c_6) $(f_{123}, g_{123})_{(u_1|_{v_1(v_2v_3)}u_2)u_3}$;
(c_7) $(f_{123}, g_{123})_{(u_1u_2|_{v_1(v_2v_3)})u_3}$;
(c_8) $(f_{123}, g_{123})_{(u_1u_2)u_3|_{v_1(v_2v_3)}}$;
(c_9) $(f_{123}, g_{123})_{(v_1(v_2v_3))u_3}, u_1u_2 = v_1(v_2v_3)$;
(c_{10}) $(f_{123}, g_{123})_{(u_1u_2)(v_2v_3)}, u_1u_2 = v_1, u_3 = v_2v_3$;
(c_{11}) $(g_{123}, f_{123})_{v_1|_{(u_1u_2)u_3}(v_2v_3)}$;
(c_{12}) $(g_{123}, f_{123})_{v_1(v_2|_{(u_1u_2)u_3}v_3)}$;
(c_{13}) $(g_{123}, f_{123})_{v_1(v_2v_3|_{(u_1u_2)u_3})}$;
(c_{14}) $(g_{123}, f_{123})_{v_1((u_1u_2)u_3)}, v_2v_3 = (u_1u_2)u_3$;
(c_{15}) $(g_{123}, g_{456})_{v_1|_{v_4(v_5v_6)}(v_2v_3)}$;
(c_{16}) $(g_{123}, g_{456})_{v_1(v_2|_{v_4(v_5v_6)}v_3)}$;
(c_{17}) $(g_{123}, g_{456})_{v_1(v_2v_3|_{v_4(v_5v_6)})}$;
(c_{18}) $(g_{123}, g_{456})_{v_1(v_4(v_5v_6))}, v_2v_3 = v_4(v_5v_6)$;
(c_{19}) $(g_{123}, g_{456})_{v_1(v_2v_3)}, v_1(v_2v_3) = v_4(v_5v_6)$.

The above compositions in S all are trivial module S. Here, we only prove the following cases: (c_1), (c_4), (c_9), (c_{10}), (c_{14}), (c_{18}). The other cases can be proved similarly.

$$\begin{aligned}(f_{123}, f_{456})_{(u_1|_{(u_4u_5)u_6}u_2)u_3} \equiv& (u_2u_1|_{(u_4u_5)u_6})u_3 - (u_1'|_{(u_5u_4)u_6}u_2)u_3\\ \equiv& (u_2u_1'|_{(u_5u_4)u_6})u_3 - (u_1'|_{(u_5u_4)u_6}u_2)u_3 \equiv 0,\end{aligned}$$

$$\begin{aligned}(f_{123}, f_{456})_{((u_4u_5)u_6)u_3}, u_1u_2 = (u_4u_5)u_6 =&(u_6(u_4u_5))u_3 - ((u_5u_4)u_6)u_3\\ \equiv&(u_6(u_5u_4))u_3 - ((u_5u_4)u_6)u_3 \equiv 0,\end{aligned}$$

$$\begin{aligned}(f_{123}, g_{123})_{(v_1(v_2v_3))u_3}, u_1u_2 = v_1(v_2v_3) =&((v_2v_3)v_1)u_3 - (v_1(v_3v_2))u_3\\ \equiv&((v_3v_2)v_1)u_3 - (v_1(v_3v_2))u_3 \equiv 0,\end{aligned}$$

$$\begin{aligned}(f_{123}, g_{123})_{(u_1u_2)(v_2v_3)}, u_1u_2 = v_1, u_3 = v_2v_3 =&(u_2u_1)(v_2v_3) - (u_1u_2)(v_3v_2)\\ \equiv&(u_2u_1)(v_3v_2) - (u_2u_1)(v_3v_2) = 0,\end{aligned}$$

$$\begin{aligned}(g_{123}, f_{123})_{v_1((u_1u_2)u_3)}, v_2v_3 =&(u_1u_2)u_3 = v_1(u_3(u_1u_2)) - v_1((u_2u_1)u_3)\\ \equiv&v_1(u_3(u_2u_1)) - v_1((u_2u_1)u_3) \equiv 0,\end{aligned}$$

$$\begin{aligned}(g_{123}, g_{456})_{v_1(v_4(v_5v_6))}, v_2v_3 =&(v_4(v_5v_6)) = v_1((v_5v_6)v_4) - v_1(v_4(v_6v_5))\\ \equiv&v_1((v_6v_5)v_4) - v_1(v_4(v_6v_5)) \equiv 0.\end{aligned}$$

Therefore S is a Gröbner-Shirshov basis of the algebra $M(X|(uv)w - (vu)w, w(uv) - w(uv), u, v, w \in X^{**})$. □

Definition 3.2 *Each letter $x_i \in X$ is called a regular word of degree 1. Suppose that $u = vw$ is a nonassociative word of degree $m, m > 1$. Then $u = vw$ is called a regular word of degree m if it satisfies the following conditions:*

($S1$) *both v and w are regular words;*

($S2$) *if $v = v_1v_2$, then $v_1 \le v_2$;*

($S3$) *if $w = w_1w_2$, then $w_1 \le w_2$.*

Lemma 3.3 *Let $N(X)$ be the set of all regular words on X. Then $Irr(S) = N(X)$.*

Proof Suppose that $u \in Irr(S)$. If $|u| = 1$, then $u = x \in N(X)$. If $|u| > 1$ and $u = vw$, then by induction $v, w \in N(X)$. If $v = v_1v_2$, then $v_1 \le v_2$, since $u \in Irr(S)$. If $w = w_1w_2$, then $w_1 \le w_2$, since $u \in Irr(S)$. Therefore $u \in N(X)$.

Suppose that $u \in N(X)$. If $|u| = 1$, then $u = x \in Irr(S)$. If $u = vw$, then v, w are regular and by induction $v, w \in Irr(S)$. If $v = v_1v_2$, then $v_1 \le v_2$, since $u \in N(X)$. If $w = w_1w_2$, then $w_1 \le w_2$, since $u \in N(X)$. Therefore $u \in Irr(S)$. □

From Theorems 2.2, 3.1 and Lemma 3.3, the following result follows.

Theorem 3.4 *The set $N(X)$ of all regular words on X forms a linear basis of the free algebra $LA(X)$.*

§4. A Characterization Theorem

Let X be a well ordered set, $S(X)$ the Lie subalgebra, generated by X, of $LA(X)$ under the commutator $[f,g]=fg-gf$. Let $T=\{[x_i,x_j]|x_i>x_j,x_i,x_j\in X\}$ where $[x_i,x_j]=x_ix_j-x_jx_i$.

Lemma 5.1 *The set $X\bigcup T$ forms a linear basis of the Lie algebra $S(X)$.*

Proof Let $u\in X\bigcup T$. If $u=x_i$, then $\bar{u}=x_i$. If $u=[x_i,x_j],x_i>x_j$, then $u=x_ix_j-x_jx_i$ and thus $\bar{u}=x_ix_j$. Then we may conclude that if $u,v\in X\bigcup T$ and $u\neq v$, then $\bar{u}\neq\bar{v}$. Therefore the elements in $X\bigcup T$ are linear independent. Since $[[f,g],h]=(fg)h-(gf)h-h(fg)+h(gf)=0=-[h,[f,g]]$, then all the Lie words with degree ≥ 3 equal zero. Therefore, the set $X\bigcup T$ forms a linear basis of the Lie algebra $S(X)$. □

Corollary 5.2 *Let $|X|>1$. Then the Lie subalgebra $S(X)$ of $LA(X)$ is not a free Lie algebra.*

Theorem 5.3 *An element $f(x_1,x_2,\cdots,x_s)$ of the algebra $LA(X)$ belongs to $S(X)$ if and only if $d(f)<3$ and the relations $x_ix'_j=x'_jx_i,i,j=1,2,\cdots,n$ imply the equation $f(x_1+x'_1,x_2+x'_2,\cdots,x_s+x'_s)=f(x_1,x_2,\cdots,x_s)+f(x'_1,x'_2,\cdots,x'_s)$.*

Proof Suppose that an element $f(x_1,x_2,\cdots,x_s)$ of the algebra $LA(X)$ belongs to $S(X)$. From Lemma 4.1, it follows that $d(f)<3$ and it suffices to prove that if $u(x_1,x_2,\cdots,x_s)\in X\bigcup T$, then the relations $x_ix'_j=x'_jx_i$ imply the equation $u(x_1+x'_1,x_2+x'_2,\cdots,x_s+x'_s)=u(x_1,x_2,\cdots,x_s)+u(x'_1,x'_2,\cdots,x'_s)$. This holds since $d(f)<3$ and $[x'_i,x_j]=[x_j,x'_i]=0$, $x'_i,x_j,1\leq i,j\leq s$.

Let d_1 be an element of the algebra $LA(X)$ that does not belong to $S(X)$. If $\bar{d_1}=x_ix_j$ where $x_i>x_j$, then let $d_2=d_1-[x_i,x_j]$. Clearly, d_2 is also an element of the algebra $LA(X)$ that does not belong to $S(X)$. Then after a finite number of steps of the above algorithm, we will obtain an element d_t whose leading term is u_t where $u_t=x_px_q,x_p\leq x_q$. It's easy to see that in the expression

$$d_t(x_1+x'_1,x_2+x'_2,\cdots,x_s+x'_s)-d_t(x_1,x_2,\cdots,x_s)-d_t(x'_1,x'_2,\cdots,x'_s)$$

the element x'_qx_p occurs with nonzero coefficient. □

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