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A Note on Hyperstructres and Some Applications

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Abstract: In classical group theory, two elements composed yield another element. This theory, definitely, has limitations in its use in the study of atomic reactions and reproduction in organisms where two elements composed can yield more than one. In this paper, we partly give a review of some properties of hyperstructures with some examples in chemical sciences. On the other hand, we also construct some examples of hyperstructures in genotype, extending the works of Davvaz (2007) to blood genotype. This is to motivate new and collaborative researches in the use of hyperstructures in these related fields.

Key Words: Genotype as a hyperstructure, hypergroup, offspring.

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§1. Introduction

The theory of *hyperstructures* began in 1934 by F. Marty. In his presentation at the 8th congress of Scandinavian Mathematicians, he illustrated the definition of hypergroup and some applications, giving some of its uses in the study of groups and some functions. It is a kind of generalization of the concept of abstract group and an extension of well-known group theory and as well leading to new areas of study.

The study of hypergroups now spans to the investigation and studying of subhypergroups, relations defined on hyperstructures, cyclic hypergroups, canonical hypergroups, P-hypergroups, hyperrings, hyperlattices, hyperfields, hypermodules and H_ν-structures but to mention a few.

A very close concept to this is that of HX Group which was developed by Li [11] in 1985. There have been various studies linking HX Groups to hyperstructures. In the late 20th century, the theory experienced more development in the applications to other mathematical theories such as character theory of finite groups, combinatorics and relation theory. Researchers like P. Corsini, B. Davvaz, T. Vougiouklis, V. Leoreanu, but to mention a few, have done very extensive studies in the theory of hyperstructures and their uses.

§2. Definitions and Examples of Hyperstrutures

Definition 2.1 *Let H be a non empty set. The operation $\circ : H \times H \longrightarrow \mathcal{P}^*(H)$ is called a*

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hyperoperation and $(H, \circ)$ is called a hypergroupoid, where $\mathcal{P}^*(H)$ is the collection of all non empty subsets of H. In this case, for $A, B \subseteq H$, $\;A \circ B = \cup\{a \circ b | a \in A, b \in B\}$.

Remark 2.1 A hyperstructure is a set on which a hyperoperation is defined. Some major kinds of hyperstructures are hypergroups, HX groups, H_ν groups, hyperrings and so on.

Definition 2.2 *A hypergroupoid $(H, \circ)$ is called a semihypergroup if*

$$(a \circ b) \circ c = a \circ (b \circ c) \quad \forall a, b, c \in H \quad (Associativity)$$

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Definition 2.3 *A hypergroupoid $(H, \circ)$ is called a quasihypergroup if*

$$a \circ H = H = H \circ a \quad \forall a \in H \quad (Reproduction \;\; Axiom).$$

Definition 2.4 *A hypergroupoid $(H, \circ)$ is called a hypergroup if it is both a semihypergroup and quasihypergroup.*

Example 2.1 (1) For any group G, if the hyperoperation is defined on the cosets, it generally yields a hypergroup.

(2) If we partition $H = \{1, -1, i, -i\}$ by $K^* = \{\{1, -1\}, \{i, -i\}\}$, then $(H/K^*, \circ)$ is a hypergroup.

(3)([8]) Let $(G, +) = (\mathrm{Z}, +)$ be an abelian group with an equivalence relation ρ partitioning G into $\overline{x} = \{x, -x\}$. Then, if $\overline{x} \circ \overline{y} = \{\overline{x + y}, \overline{x - y}\}$ $\forall \overline{x}, \overline{y} \in G/\rho, (G/\rho, \circ)$ is a hypergroup.

Definition 2.5 *A hypergroupoid $(H, \circ)$ is called a H_ν group if it satisfies*

(1) $(a \circ b) \circ c \cap a \circ (b \circ c) \neq \emptyset \quad \forall a, b, c \in H \quad$ *(Weak Associativity);*

(2) $a \circ H = H = H \circ a \quad \forall a \in H \quad$ *(Reproduction Axiom).*

Remark 2.2 An H_ν group may not be a hypergroup. A subset $K \subseteq H$ is called a subhypergroup if $(K, \circ)$ is also a hypergroup. A hypergroup $(H, \circ)$ is said to have an identity e if $\forall a \in H \quad a \in e \circ a \cap a \circ e \neq \emptyset$.

Example 2.2 Davvaz [8] has given an example of a H_ν group as the chemical reaction

$$A_2 + B_2 \;\underset{\longleftrightarrow}{energy}\; 2AB$$

in which A° and B° are the fragments of A_2, B_2, AB and $\mathcal{H} = \{A^\circ, B^\circ, A_2, B_2, AB\}$.

Definition 2.6 *Let G be a group and $\circ : G \times G \longrightarrow \mathcal{P}^*(G)$ a hyperoperation. Let $\mathcal{C} \subseteq \mathcal{P}^*(G)$ and $A, B \in \mathcal{C}$. If $\mathcal{C}$, under the product $A \circ B = \cup\{a \circ b | a \in A, b \in B\}$, is a group, then $(\mathcal{C}, \circ)$ is a HX group on G with unit element $E \subseteq \mathcal{C}$ such that $E \circ A = A = A \circ E \; \forall A \in \mathcal{C}$.*

It is important to study HX group separately because some hypergroups exist but are not

HX groups. An example is $(\{\{0\},(0,+\infty),(-\infty,0)\},+)$; it a hypergroup but not a HX group. Note that if the unit element E of the quotient group of G by E is a normal subgroup of G, then the quotient group is a HX group.

Definition 2.7 *If for the identity element $e \in G$ we have $e \in E$, then $(\mathcal{C},\circ)$ is a regular HX group on G.*

Theorem 2.1([10]) *If $\mathcal{C}$ is a HX group on G, then $\forall A,B \in \mathcal{C}$*

(1) $|A| = |B|$;
(2) $A \cap B \neq \emptyset \Longrightarrow |A \cap B| = |E|$.

Remark 2.3 Corsini [4] has shown that a HX group, also referred to as *Chinese Hyperstructure* is a H_ν Group and that, under some condition, is a hypergroup. But, trivially, a hypergroup is a H_ν Group since only that associativity was relaxed in a hypergroup to obtain a H_ν Group. Besides, Onasanya [12] has shown that no additional condition is needed by a *Chinese Hyperstructure*, that is a HX group, to become a hypergroup.

§3. Applications and Occurrences of Hyperstrutures in Biological and Chemical Sciences

The chain reactions that occur between hydrogen and halogens, say iodine (I), give interesting examples of hyperstructures [8]. This can be seen in Table 1. Many properties of these reactions can be seen from the study of hyperstructures.

Table 1. Reaction of Hydrogen with Iodine

$+$	H^o	I^o	H_2	I_2	HI
H^o	H^o,H_2	H^o,I^o,HI	H^o,H_2	H^o,I^o,HI,I_2	H^o,I^o,H_2,HI
I^o	I^o,H^o,HI	I^o,I_2	I^o,H^o,HI,H_2	I^o,I_2	H^o,I^o,HI,I_2
H_2	H^o,H_2	H^o,I^o,HI,H_2	H^o,H_2	H^o,I^o,HI,H_2,I_2	H^o,I^o,H_2,HI
I_2	I^o,H^o,I_2,HI	I^o,I_2	H^o,I^o,HI,H_2,I_2	I^o,I_2	H^o,I^o,HI,I_2
HI	H^o,I^o,H_2,HI	H^o,I^o,HI,I_2	H^o,I^o,HI,H_2	H^o,I^o,HI,I_2	H^o,I^o,HI,I_2,H_2

Let $G = \{H^o,I^o,H_2,I_2,HI\}$ so that $(G,\circ)$ is such that $\forall A,B \in G$, we have that $A \circ B$ are the possible product(s) representing the reaction between A and B. Then, $(G,\circ)$ is a H_v-group. The subsets $G_1 = \{H^o,H_2\}$ and $G_2 = \{I^o,I_2\}$ are the only H_v-subgroups of $(G,\circ)$ and indeed they are trivial hypergroups.

Davvaz [6] has the following examples: Dismutation is a kind of chemical reaction. Comproportionation is a kind of dismutation in which two different reactants of the same element having different oxidation numbers combine to form a new product with intermediate oxidation number. An example is the reaction

$$Sn + Sn^{4+} \rightarrow 2Sn^{2+}.$$

In this reaction, letting $\mathcal{G} = \{Sn, Sn^{2+}, Sn^{4+}\}$, the following table shows all possible occurrences.

Table 2. Dismutation Reaction of Tin

$\circ$	Sn	Sn^{2+}	Sn^{4+}
Sn	Sn	Sn, Sn^{2+}	Sn^{2+}
Sn^{2+}	Sn, Sn^{2+}	Sn^{2+}	Sn^{2+}, Sn^{4+}
Sn^{4+}	Sn^{2+}	Sn^{2+}, Sn^{4+}	Sn^{4+}

While it is agreeable that $(\mathcal{G}, \circ)$ is weak associative as claimed by [6], we say further that it is a H_ν group. Also, while $(\{Sn, Sn^{2+}\}, \circ)$ is agreed to be a hypergroup, we say that $(\{Sn^{2+}, Sn^{4+}\}, \circ)$ is not just a H_ν semigroup as claimed by [6] but a H_ν group.

Furthermore, Cu(0), Cu(I), Cu(II) and Cu(III) are the four oxidation states of copper. Its different species can react with themselves (without energy) as defined below

(1) $Cu^{3+} + Cu^{+} \mapsto Cu^{2+}$;

(2) $Cu^{3+} + Cu \mapsto Cu^{2+} + Cu^{+}$.

Table 3. Redox (Oxidation-Reduction) reaction of Copper

$\circ$	Cu	Cu^{+}	Cu^{2+}	Cu^{3+}
Cu	Cu	Cu, Cu^{+}	Cu, Cu^{2+}	Cu^{+}, Cu^{2+}
Cu^{+}	Cu, Cu^{+}	Cu^{+}	Cu^{+}, Cu^{2+}	Cu^{2+}
Cu^{2+}	Cu, Cu^{2+}	Cu^{+}, Cu^{2+}	Cu^{2+}	Cu^{2+}, Cu^{3+}
Cu^{3+}	Cu^{+}, Cu^{2+}	Cu^{2+}	Cu^{2+}, Cu^{3+}	Cu^{3+}

Let $G = \{Cu, Cu^{+}, Cu^{2+}, Cu^{3+}\}$. Then $(G, \circ)$ is weak associative and

$$Cu^{+} \circ X = X \circ Cu^{+} \neq X$$

so that $(G, \circ)$ is an H_v semigroup. $\{Cu, Cu^{+}\}$, $\{Cu, Cu^{2+}\}$ $\{Cu^{+}, Cu^{2+}\}$ and $\{Cu^{2+}, Cu^{3+}\}$ are hypergroups with respect to $\circ$. From Table 4 we also have that $(\{Cu, Cu^{+}, Cu^{2+}\}, \circ)$ is a hypergroup.

Table 4. Another Redox reaction of Cu

$\circ$	Cu	Cu^{+}	Cu^{2+}
Cu	Cu	Cu, Cu^{+}	Cu, Cu^{2+}
Cu^{+}	Cu, Cu^{+}	Cu^{+}	Cu^{+}, Cu^{2+}
Cu^{2+}	Cu, Cu^{2+}	Cu^{+}, Cu^{2+}	Cu^{2+}

It should be noted that $\{Cu, Cu^{+}\}$, $\{Cu, Cu^{2+}\}$ and $\{Cu^{+}, Cu^{2+}\}$ are subhypergroups of $(\{Cu, Cu^{+}, Cu^{2+}\}, \circ)$.

§4. Identities of Hyperstructures

Definition 4.1([8]) *The set $I_p = \{e \in H | \exists x \in H$ such that $x \in x \circ e \cup e \circ x\}$ is refered to as partial identities of H.*

Definition 4.2([3]) *An element $e \in H$ is called the right (analogously the left) identity of H if $x \in x \circ e (x \in e \circ x)$ $\forall x \in H$. It is called an identity of H if it is both right and left identity.*

Definition 4.3([3]) *A hypergroup H is semi regular if each $x \in H$ has at least one right and one left identity.*

It can be seen that every right or left identity of H is in I_p.

4.1 Blood Genotype as a Hyperstructure

Let $G = \{AA, AS, SS\}$ and the hyperoperation $\oplus$ denote mating. The blood genotype is a kind of hyperstructure.

Table 5. Genotype Table [12]

$\oplus$	AA	AS	SS
AA	$\{AA\}$	$\{AA, AS\}$	$\{AS\}$
AS	$\{AA, AS\}$	$\{AA, AS, SS\}$	$\{AS, SS\}$
SS	$\{AS\}$	$\{AS, SS\}$	$\{SS\}$

In Table 5, $\{AA\} \oplus G \neq G \neq G \oplus \{AA\}$; the *reproduction axiom* is not satisfied. Also, it is weak associative. It is a H_ν semigroup.

Note that a lot has been discussed on the occurrence of hyperstructure algebra in inheritance [7]. For most of the monohybrid and dihybrid crossing of the pea plant, they are hypergroups in the second generation. Take for instance, *the monohybrid Crossing in the Pea Plant* , the parents has the RR(Round) and rr(Wrinkled) genes. The first generation has Rr(Round). The second generation has RR(Round), Rr(Round) and rr(Wrinkled). Now consider the set $G = \{R, W\}$; R for Round and W for Wrinkled. Crossing this generation under the operation $\oplus$ for mating, [7] already established it is a hypergroup.

In the following section, a little more information about their properties would be given and an extension to cases which are hypergroups in earlier generations are made.

§5. Main Results

5.1 Hyperstructures in Group Theory

The following example is a construction of an HX group which is also a hypergroup and a H_ν Group by Remark 2.1.

Example 5.1 Let us partition $(\mathrm{Z}_{10}, +)$ by $\rho = \{\{0,5\},\{1,6\},\{2,7\},\{3,8\},\{4,9\}\}$. Then we can see that $E = \{0, 5\}$ is a normal subgroup of $(\mathrm{Z}_{10}, +)$ and that $E^2 = E$. $(\mathrm{Z}_{10}/\rho, \circ)$ is also a regular HX group since $0 \in E$.

We give some further clarifications on Table 5, that this is a H_ν cyclic semigroup, with generator $\{AS\}$. It has no H_ν subsemigroups. The set of partial identites I_p of $(G, \oplus)$ is G itself by Definition 4.1, and the identity (which is both right and left identity) of G is $\{AS\}$ by Definition 4.2. Then, $(G, \oplus)$ is also a semi regular hypergroupoid by Defnition 4.3. Note that if the parents' genotype are $\{AA, AS\}$ or $\{AA, SS\}$ or $\{AS, SS\}$, the first generations of each of these are H_ν semigroups. These can be seen in the tables below.

Table 6. Parents with the genotype AA and AS

$\oplus$	AA	AS
AA	$\{AA\}$	$\{AA, AS\}$
AS	$\{AA, AS\}$	$\{AA, AS, SS\}$

The first generation $H_1 = \{AA, AS, SS\}$ is a H_ν semigroup under $\oplus$.

Table 7. Parents with the genotype AA and SS

$\oplus$	AA	SS
AA	$\{AA\}$	$\{AS\}$
SS	$\{AS\}$	$\{SS\}$

The first generation $H_2 = \{AA, AS, SS\}$ is a H_ν semigroup under $\oplus$.

Table 8. Parents with the genotype AA and SS

$\oplus$	AS	SS
AS	$\{AA, AS, SS\}$	$\{AS, SS\}$
SS	$\{AS, SS\}$	$\{SS\}$

The first generation $H_3 = \{AA, AS, SS\}$ is a H_ν semigroup under $\oplus$.

It is established in this work that the case of crossing between organism which have lethal genes (i.e. the genes that cause the death of the carrier at homozygous condition), such as the crossing of mice parents with traits Yellow(Yy) and Grey(yy), is a semi regular hypergroup at all generations, including the parents' generation. However, the parents with traits Yellow(Yy) and Yellow(Yy) have their first generation and the generations of all other offsprings to be semi regular hypergyoups. These are summarized in the tables below.

Table 9. Parents with the genotype Yellow(Yy) and Grey(yy)

$\oplus$	Yy	yy
Yy	$\{Yy, yy\}$	$\{Yy, yy\}$
yy	$\{Yy, yy\}$	$\{yy\}$

They produce the offspring Yy and yy like themselves in the first generation in the ratio 2:3. Let $G = \{Yy, yy\}$, $(G, \oplus)$ is a semi regular hypergroup.

Table 10. Parents with the genotype Yellow(Yy) and Yellow(Yy)

$\oplus$	Yy	Yy
Yy	$\{Yy, yy\}$	$\{Yy, yy\}$
Yy	$\{Yy, yy\}$	$\{yy\}$

They produce the offspring Yy and yy in the first generation in the ratio 2:1 but is not a hypergroupoid for the occurrence of yy. But crossing this first generation produces the result of Table 9, showing that the first generation with $\oplus$ is a hypergroup. This same result is obtained for all other generations in this crossing henceforth.

It is important to note that the monohybrid and dihybrid mating of pea plant considered in [7] are not just hypergroups but semi regular hypergroups. The particular case mentioned above has a right and a left identity $I = \{W\}$.

§6. Conclusions

The following is just to make some conclusions. Far reaching ones can be made from the in-depth studies and applications of the theory of hyperstructures. The algebraic properties of these hyperstructures can be used to gain insight into what happens in the biological situations and chemical reactions which they have modelled. For instance, the *weak associativity*, in case it is a case of H_ν group, of some of the chemical reactions suggests that, given reactants A, B, and C, one must be careful in the order of mixture as you may not always have the same product when $A + B$ is done before adding C as in when $B + C$ is done before adding A. In other words, $A+(B+C)$ does not always equal $(A+B)+C$. Moreover, the *strong associativity*, in the case of hypergroup, indicates that same products are obtained in both orders.

From blood the genotype table of $G = \{AA, AS, SS\}$, reproduction axiom is not satisfied with the element $\{SS\}$, meaning that if marriages are only contracted between any member of the group and someone with $\{SS\}$ genotype, all offsprings shall be carriers of sickle cell in all subsequent generations. Besides, its *weak associativity* property indicates that if there were to be marriages between individuals with genotypes A, B, and C so that those with the genotypes A and B marry and produce offsprings which now marry those with genotype C, then some of the offsprings of this marriage will always have the same genotype as some of the offsprings of those with genotype A marrying the offsprings produced by the marriages of people with the genotypes B and C.

If the operation $\oplus$ denotes cross breeding, it should also be noted that genetic crossing (in terms of genotype or phenotype) is not always, at the parents level, a hyperstructure. This is because in the collection of all traits $\mathcal{P}^*(T)$ of *Parents*, there sometimes will be *trait* A and *trait* B which combine to form a *trait* C but such that $C \notin \mathcal{P}^*(T)$. An example is in the *incomplete dominance* reported when Mendel crossed the four O' clock plant (Mirabilis jalapa) which produced an intermediate flower colour (Pink) from parents having Red and White colours. Not even at any generation will it be a hyperstructure as long as there is incomplete dominance. Hence, the theory of hyperstructures should not be applied in this case.

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