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A Note on Slant and Hemislant Submanifolds of an (ϵ)-Para Sasakian Manifold

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Abstract: In the present note we have derived some basic results pertaining to the geometry of slant and hemislant submanifolds of an (ϵ)-para Sasakian manifold. In particular, we have obtained few results on a totally umbilical slant submanifold of an (ϵ)-para Sasakian manifold. In the next section, we have obtained the integrability conditions of the involved distributions of hemislant submanifold of an (ϵ)-para Sasakian manifold. Finally we have verified the theorems by providing an example of three dimensional hemislant submanifold of (ϵ)-para Sasakian manifold.

Key Words: (ϵ)-Para Sasakian manifold, totally umbilical submanifold, slant submanifold, hemislant submanifold.

AMS(2010): 53B25, 53C15, 53C40, 53C42.

§1. Introduction

Connected almost contact metric manifold was classified by S.Tanno [13], as those automorphism group has maximum dimension. He has given following classifications:

(i) Homogeneous normal contact Riemannian manifolds with constant ϕ holomorphic sectional curvature if the sectional curvature of the plane section containing ξ, say $K(X, \xi) > 0$;

(ii) Global Riemannian product of a line (or a circle) and a Kaehlerian manifold with constant holomorphic sectional curvature, if $K(X, \xi) = 0$;

(iii) A warped product space RX_fC_n, if $K(X, \xi) < 0$.

Manifold of class (i) has Sasakian structure. The manifold of class (ii) are characterized by a tensorial relation admitting a cosymplectic structure. The manifold of class (iii) are characterized by some tensorial equations, attaining a Kenmotsu structure.

An almost paracontact structure (ϕ, ξ, η) satisfying $\phi^2 = I - \eta \otimes \xi$ and $\eta(\xi) = 1$ on a differentiable manifold was introduced by Sato [11] in 1976. After him Takahashi [14] in 1969, gave the notion of almost contact manifold equipped with an associated pseudo-Riemannian metric. Later on, motivated by these circumstances, M.M.Tripathi et.al.([15]) has drawn a relation between a semi-Riemannian metric (not necessarily Lorentzian) and an almost paracontact structure, and he named this indefinite almost paracontact metric structure an (ϵ)-

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almost paracontact structure, where the structure vector field ξ will be spacelike or timelike according as $\epsilon = 1$ or $\epsilon = -1$. Authors have discussed (ϵ)- almost paracontact manifolds and in particular (ϵ)- Sasakian manifolds in([15]).

On the other hand, the study of slant submanifolds in complex spaces was initiated by B.Y.Chen as a natural generalization of both holomorphic and totally real submanifolds in ([4]). After him, A.Lotta in 1996 extended the notion to the setting of almost contact metric manifolds [8]. Further modifications regarding semislant submanifolds were introduced by N.Papaghiuc [10]. These submanifolds are a generalized version of CR-submanifolds. After him, J.L.Cabrerizo et.al. ([2]) extended the study of semislant submanifolds of Kaehler manifold to the setting of Sasakian manifolds. The idea of hemislant submanifold was introduced by Carriazo as a particular class of bislant submanifolds, and he called them antislant submanifolds in [3]. Recently, B.Sahin extended the study of pseudo-slant submanifolds in Kaehler setting for their warped product. Totally umbilical proper slant submanifold of a Kaehler manifold has also been discussed in [12].

This paper contains the analysis about slant and pseudo-slant submanifolds of an (ϵ)-para Sasakian manifold. Section (1) is introductory. Section (2) gives us a view of (ϵ)-para Sasakian manifold. In section (3) we have obtained some results on a totally umbilical proper slant submanifold M of an (ϵ)-para Sasakian manifold. Finally, in section (4) we have derived some conditions for the integrability of the distributions on the hemislant submanifolds of an (ϵ)-para Sasakian manifold.

§2. Preliminaries

Let $\tilde{M}$ be an n-dimensional almost paracontact manifold [11] endowed with an almost paracontact structure (ϕ, ξ, η) consisting of a tensor field ϕ of type $(1, 1)$, a structure vector field ξ and 1-form η satisfying:

$$\phi^2 = I - \eta \otimes \xi, \tag{2.1}$$

$$\eta(\xi) = 1, \tag{2.2}$$

$$\phi(\xi) = 0 \tag{2.3}$$

and

$$\eta \circ \phi = 0 \tag{2.4}$$

for any vector field $X, Y \in \tilde{M}$. A semi-Riemannian metric [9] on a manifold $\tilde{M}$, is a non-degenerate symmetric tensor field g of type $(0, 2)$. If this metric is of index 1 then it is called Lorentzian metric ([1]). Let g be semi-Riemannian metric with index 1 in an n-dimensional almost paracontact manifold $\tilde{M}$ such that,

$$\tilde{g}(\phi X, \phi Y) = \tilde{g}(X, Y) - \epsilon\eta(X)\eta(Y), \tag{2.5}$$

where $\epsilon = +1$ or -1. Then $\tilde{M}$ is called an almost paracontact metric manifold associated with an (ϵ)-almost paracontact metric structure $(\phi, \xi, \eta, g, \epsilon)$([?]). In case, if index $(g) = 1$ then an

(ϵ)-almost paracontact metric manifold is defined as a Lorentzian paracontact manifold and if the metric is positive definite, then an (ϵ)-almost paracontact metric manifold is the usual almost paracontact metric manifold [**?**].

The condition (2.5) is equivalent to

$$g(X, \phi Y) = g(\phi X, Y) \tag{2.6}$$

equipped with

$$g(X, \xi) = \epsilon\eta(X). \tag{2.7}$$

From (2.7), it can be easily observed that

$$g(\xi, \xi) = \epsilon, \tag{2.8}$$

i.e. structure vector field ξ is never lightlike. We define

$$\Phi(X, Y) = g(X, \phi Y) \tag{2.9}$$

and we can obtain

$$\Phi(X, \xi) = 0. \tag{2.10}$$

From (2.9), we can also calculate

$$(\nabla_X \Phi)(Y, Z) = g(\nabla_X \phi)(Y, Z) = (\nabla_X \Phi)(Z, Y). \tag{2.11}$$

An (ϵ)-almost paracontact metric manifold $\tilde{M}$ satisfying

$$2\Phi(X, Y) = (\nabla_X \eta)(Y) + (\nabla_Y \eta)(X) \tag{2.12}$$

$\forall\ X, Y \in TM$, then $\tilde{M}$ is called an (ϵ)-paracontact metric manifold ([15]).

An (ϵ)-almost paracontact metric structure $(\phi, \xi, \eta, g, \epsilon)$ is called an (ϵ)-S-paracontact metric structure if

$$\tilde{\nabla}_X \xi = \epsilon\phi X \tag{2.13}$$

for $\forall\ X \in T\tilde{M}$. A manifold endowed with an (ϵ)-S-paracontact metric structure is called an (ϵ)-S-paracontact metric manifold. Equation (2.13) can be written as

$$\Phi(X, Y) = g(\phi X, Y) = \epsilon g(\tilde{\nabla}_X \xi, Y) = (\nabla_X \eta)(Y) \tag{2.14}$$

for $\forall\ X, Y \in TM$.

An (ϵ)-almost paracontact metric structure is called an (ϵ)-para Sasakian structure if the following relation holds

$$(\tilde{\nabla}_X \phi)(Y) = -g(\phi X, \phi Y)\xi - \epsilon\eta(Y)\phi^2 X, \tag{2.15}$$

where $\tilde{\nabla}$ is the Levi-Civita connection with respect to g on $\tilde{M}$. A manifold equipped with an

(ϵ)-para Sasakian structure is called an (ϵ)-para Sasakian manifold.

From the definition of contact CR-submanifolds of an (ϵ)-paracontact Sasakian manifold we have

Definition 2.1([7]) *An n-dimensional Riemannian submanifold M of an (ϵ)-para Sasakian manifold $\tilde{M}$ is called a contact CR-submanifold if*

(i) ξ is tangent to M;

(ii) there exists on M a differentiable distribution $D : x \longrightarrow D_x \subset T_x(M)$, such that D_x is invariant under ϕ; i.e., $\phi D_x \subset D_x$, for each $x \in M$ and the orthogonal complementary distribution $D^\perp : x \longrightarrow D_x^\perp \subset {T_x}^\perp(M)$ of the distribution D on M is totally real; i.e., $\phi D_x^\perp \subset T_x^\perp(M)$, where $T_x(M)$ and $T_x^\perp(M)$ are the tangent space and the normal space of M at x. D (resp.$D^\perp$) is the horizontal (resp. vertical) distribution. The contact CR-submanifold of an (ϵ)-para Sasakian manifold is called ξ-horizontal (resp. ξ-vertical) if $\xi_x \in D_x$ (resp. $\xi_x \in D_x^\perp$) for each $x \in M$.

Let TM and $T^\perp M$ be the Lie algebras of vector fields tangential to M and normal to M respectively. h and A denote the second fundamental form and the shape operator of the immersion of M into $\tilde{M}$ respectively. If ∇ is the induced connection on M, the Gauss and Weingarten formulae of M into $\tilde{M}$ are characterized by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.16}$$

$$\tilde{\nabla}_X V = -A_V X + \nabla_X^\perp N \tag{2.17}$$

for any X, Y in TM and N in $T^\perp M$. $\nabla^\perp$ is the connection on normal bundle and A_N is the Weingarten endomorphism associated with N by

$$g(A_V X, Y) = g(h(X, Y), V). \tag{2.18}$$

For any $x \in M$ and $X \in T_x M$ we decompose it as

$$\phi X = TX + NX, \tag{2.19}$$

where $TX \in T_x M$ and $NX \in T_x^\perp M$.

Similarly for $V \in T_x^\perp M$ we know

$$\phi V = tV + nV, \tag{2.20}$$

where tV (resp.nV) is vertical (resp. normal) component of ϕV.

Now, for any $X, Y \in TM$, comparing the tangential and normal parts of $(\tilde{\nabla}_X \phi)Y$ by $\mathbf{P}_X Y$ and $\mathbf{Q}_X Y$ respectively. After having some brief calculation, we obtain

$$\mathbf{P}_X Y = (\tilde{\nabla}_X T)Y - A_{NY} X - th(X, Y), \tag{2.21}$$

$$\mathbf{Q}_X Y = (\tilde{\nabla}_X N)Y + h(X, TY) - nh(X, Y) \tag{2.22}$$

for any $X, Y \in TM$.

Again for any $V \in T^{\perp}M$, denoting tangential and normal parts of $(\tilde{\nabla}_X \phi)V$ by $\mathbf{P}_X V$ and $\mathbf{Q}_X V$ respectively, we have

$$\mathbf{P}_X V = (\tilde{\nabla}_X t)V - A_{nV}X + TA_V X, \tag{2.23}$$

$$\mathbf{Q}_X V = (\tilde{\nabla}_X n)V + h(tV, X) + NA_V X, \tag{2.24}$$

where the covariant derivatives of T, N, t and n are given by

$$(\tilde{\nabla}_X T)Y = \nabla_X TY - T\nabla_X Y, \tag{2.25}$$

$$(\tilde{\nabla}_X N)Y = \nabla^{\perp}_X NY - N\nabla_X Y, \tag{2.26}$$

$$(\tilde{\nabla}_X t)V = \nabla_X tV - t\nabla^{\perp}_X V, \tag{2.27}$$

$$(\tilde{\nabla}_X n)V = \nabla^{\perp}_X nV - n\nabla^{\perp}_X V \quad \forall \;\; X, Y \in TM, V \in T^{\perp}M. \tag{2.28}$$

A submanifold M of an almost contact metric manifold $\tilde{M}$ is called totally umbilical if

$$h(X, Y) = g(X, Y)H \tag{2.29}$$

for any $X, Y \in \Gamma(TM)$, where H is the mean curvature. A submanifold M is said to be totally geodesic if $h(X, Y) = 0$ for each $X, Y \in \Gamma(TM)$ and is minimal if $H = 0$ on M.

§3. Slant Submanifolds

The slant submanifold of a para contact Lorentzian manifold were first defined by [5]. Hereafter, for a submanifold M of an almost contact manifold, authors in [6] assumed that the structure vector field ξ is tangential to the submanifold M, whence the tangent bundle TM can be decomposed as

$$(a) TM = D \bigoplus < \xi >,$$

where the orthogonal complementary distribution D of $< \xi >$ is known as the slant distribution on M and $< \xi >$ is the 1-dimensional distribution on M spanned by the structure vector field ξ, and they also assumed that $g(X, X) \geq 0 \;\; \forall X \;\in TM \setminus \xi$. Let M be an immersed submanifold of $\tilde{M}$. For any $x \in M$ and $X \in T_x M$, if the vectors X and ξ are linearly independent, then the angle $\theta(X) \in [0, \pi/2]$ between ϕX and $T_x M$ is well defined, if $\theta(X)$ does not depend on the choice of $x \in M$ and $X \in T_x M$, then M is slant in $\tilde{M}$. The constant angle $\theta(X)$ is then called the slant angle of M in $\tilde{M}$ by [5] and which in short we denote by $Sla(M)$. If μ is ϕ-invariant of the normal bundle $T^{\perp}M$, then

$$(b) T^{\perp}M = FTM \oplus < \mu > .$$

Defining the endomorphism $P : TM \longrightarrow TM$, whose square P^2 will be denoted by Q. Then the tensor fields on M of type $(1, 1)$ determined by these endomorphism will be denoted by the same letters, respectively P and Q.

It is proved the following theorem in [6]:

For a proper slant submanifold M of an (ϵ)-para Sasakian manifold $\tilde{M}$ with slant angle θ, then

$$QX = \lambda(X - \eta(X)\xi). \tag{3.1}$$

From this theorem we can state our next theorem,

Theorem 3.1 *Let M be a submanifold of an (ϵ)-para Sasakian manifold $\tilde{M}$ such that $\xi \in TM$. Then, M is slant iff there exists a constant $\lambda \in [0,1]$ such that*

$$T^2 = \lambda(I - \eta \bigotimes \xi). \tag{3.2}$$

Furthermore, in such case, if θ is the slant angle of M, then $\lambda = cos^2\theta$. Hence for a slant manifold we have

$$g(TX, TX) = cos^2\theta(g(X,Y) - \epsilon\eta(X)\eta(Y)), \tag{3.3}$$

$$g(NX, NY) = sin^2\theta(g(X,Y) - \epsilon\eta(X)\eta(Y)) \tag{3.4}$$

for $\forall$ X, $Y \in TM$.

Proof Follows from [5]. □

Assuming M to be totally umbilical proper slant submanifold of an (ϵ)-para Sasakian manifold, we can obtain the following theorem.

Theorem 3.2 *Let M be a totally umbilical proper slant submanifold of an (ϵ)-paracontact Sasakian manifold $\tilde{M}$, then for any $X \in TM$ following conditions are equivalent:*

(i) $H \in \mu$;
(ii) $g(\nabla_{TX}\xi, X) = \epsilon[||X||^2 - \eta^2(X)]$.

Proof For any $X \in TM$ we know $h(X, TX) = g(X, TX)H = 0$. Then from (2.16) and (2.17) and the structure equation of (ϵ)-para Sasakian manifold for any vector field $X \in TM$, we calculate

$$\begin{aligned} 0 \;&=\; \phi(\nabla_X X + h(X,X)) - \nabla_X TX + A_{NX}X - \nabla^{\perp}_X NX \\ &\quad -g(\phi X, \phi X)\xi - \epsilon\eta(X)\phi^2 X. \end{aligned} \tag{3.5}$$

After using (2.19), and on comparing the tangential component we obtain

$$\begin{aligned} 0 \;&=\; ZT\nabla_X X - \nabla_X TX + th(X,X) + A_{NX}X - g(X,X)\xi \\ &\quad +2\epsilon\eta^2(X)\xi - \epsilon\eta(X)X. \end{aligned} \tag{3.6}$$

As M is totally umbilical submanifold then the term $A_{NX}X$ becomes $Xg(H, NX)$, so using this fact above equation takes the form

$$\begin{aligned} 0 &= T\nabla_X X - \nabla_X TX + th(X,X) + Xg(NX,H) + g(X,X)tH \\ &\quad + g(X,X)\xi - 2\epsilon\eta^2(X)\xi + \epsilon\eta(X)X. \end{aligned} \tag{3.7}$$

If $H \in \mu$ then from (3.6) we get

$$T\nabla_X X - \nabla_X TX = -||X||^2\xi + 2\epsilon\eta(X)[2\eta(X)\xi - X]. \tag{3.8}$$

Taking the inner product in (3.7) by ξ we obtain

$$g(\nabla_X TX, \xi) = \eta^2(X) - \epsilon||X||^2. \tag{3.9}$$

Replacing X by TX, we derive

$$g(\nabla_{TX} T^2X, \xi) = \eta^2(TX) - \epsilon||TX||^2. \tag{3.10}$$

Then from equation (3.3) and (3.4) we calculate

$$cos^2\theta g(X, \nabla_{TX}\xi) - cos^2\theta\eta(X)g(\xi, \nabla_{TX}\xi) = -cos^2\theta[\epsilon||X||^2 - \eta^2(X)]. \tag{3.11}$$

Therefore we can conclude that

$$g(X, \nabla_{TX}\xi) - \eta(X)g(\xi, \nabla_{TX}\xi) = \epsilon||X||^2 - \eta^2(X). \tag{3.12}$$

Now we know that $g(\xi,\xi) = \epsilon$. Taking the covariant derivative of this equation with respect to TX for any $X \in TM$, we obtain

$$g(\nabla_{TX}\xi, \xi) + g(\xi, \nabla_{TX}\xi) = 0, \tag{3.13}$$

which implies $g(\nabla_{TX}\xi, \xi) = 0$. Hence (3.8) becomes

$$g(X, \nabla_{TX}\xi) = \epsilon||X||^2 - \eta^2(X). \tag{3.14}$$

This proves part (ii) of the theorem. If (3.9) holds then equation (3.6) implies $H \in \mu$. This proves theorem (3.2).

Now if $\epsilon||X||^2 - \eta^2(X) = 0$, then from (3.9), we conclude

$$g(X, \nabla_{TX}\xi) = 0. \tag{3.15}$$

Replacing X by TX we have by using (3.3), we get

$$g(TX, \nabla_{T^2X}\xi) = g(\nabla_{cos^2\theta(X-\eta(X)\xi)}\xi, TX) = 0. \tag{3.16}$$

Then the above equation becomes

$$cos^2\theta g(\nabla_X\xi, TX) + cos^2\theta\eta(X)g(\nabla_\xi\xi, TX) = 0. \tag{3.17}$$

From the structure equation (2.4) we have $\nabla_\xi\xi = 0$. Thus we can write

$$cos^2\theta g(\nabla_X\xi, TX) = 0. \tag{3.18}$$

Thus from equation (3.10) we get either M is an anti-invariant submanifold or $\nabla_X\xi = 0$ i.e. ξ is a Killing vector field on M or M is trivial. If ξ is not Killing then we can take at least two linearly independent vectors X and TX to span D_θ i.e. the $dimM \geq 3$.

From above discussion we can conclude the following theorem.

Theorem 3.3 *Let M be a totally umbilical slant submanifold of an ϵ-para Sasakian manifold $\tilde{M}$ such that $\epsilon||X||^2 = \eta^2(X)$ on M then one of the following statements is true:*

(i) $H \in \Gamma(\mu)$;

(ii) M is an anti-invariant submanifold;

(iii) If M is a proper slant submanifold then $dimM \geq 3$;

(iv) M is trivial;

(v) ξ is a Killing vector field on M.

Next we prove

Theorem 3.4 *A totally umbilical proper slant submanifold M of an (ϵ)-para Sasakian manifold $\tilde{M}$ is totally geodesic if $\nabla^\perp_X H \in \Gamma(\mu)$ for any $X \in TM$.*

Proof As $\tilde{M}$ is an (ϵ)-paracontact Sasakian manifold we have

$$(\tilde{\nabla}_X\phi)Y = \tilde{\nabla}_X\phi Y - \phi\tilde{\nabla}_X Y. \tag{3.19}$$

From the fact that $\phi Y = TY + NY$ and $\tilde{M}$ is an (ϵ)-paracontact Sasakian manifold we infer

$$\begin{aligned}\tilde{\nabla}_X TY + \tilde{\nabla}_X NY \quad = \quad & T\nabla_X Y + N\nabla_X Y + \phi h(X,Y) - g(\phi X, \phi Y)\xi \\ & -\epsilon\eta(Y)\phi^2 X.\end{aligned} \tag{3.20}$$

Using (2.25), (2.26) and (2.29) we obtain

$$\begin{aligned}\nabla_X TY + h(X,TY) - A_{NY}X \quad + \quad & \nabla^\perp_X NY = T\nabla_X Y + N\nabla_X Y \\ & + g(X,Y)\phi H - g(\phi X, \phi Y)\xi - \epsilon\eta(Y)\phi^2 X.\end{aligned} \tag{3.21}$$

Taking inner product with ϕH and using the fact that $H \in \Gamma(\mu)$, from (2.5) and (2.29) we get

$$\begin{aligned} g(X,TY)g(H,\phi H)+g(\nabla^{\perp}_{X}NY,\phi H) &= g(X,Y)||H||^2 \\ &\quad -g(\phi X,\phi Y)g(\phi H,\xi)-\epsilon\eta(Y)g(\phi^2X,\phi H). \end{aligned} \tag{3.22}$$

Now we consider

$$\tilde{\nabla}_X\phi H=\phi\tilde{\nabla}_X H+(\tilde{\nabla}_X\phi)H. \tag{3.23}$$

Using the covariant derivative of ∇

$$\begin{aligned} -A_{\phi H}X+\nabla^{\perp}_{X}\phi H &= -TA_HX-NA_HX+t\nabla^{\perp}_{X}H \\ &\quad +n\nabla^{\perp}_{X}H-g(\phi X,\phi H)\xi-\epsilon\eta(H)\phi^2X. \end{aligned} \tag{3.24}$$

Taking inner product with NY, for any $Y\in\Gamma(TM)$ and as the submanifold considered is always tangent to ξ we obtain

$$g(\nabla^{\perp}_{X}\phi H,NY)=-g(NA_HX,NY)+g(n\nabla^{\perp}_{X}H,NY)-\epsilon\eta(H)g(\phi^2X,NY). \tag{3.25}$$

Since $n\nabla^{\perp}_{X}H\in\Gamma(\mu)$, then by (3.5) the above equation takes the form

$$\begin{aligned} g(\nabla^{\perp}_{X}\phi H,NY) &= -sin^2\theta\ [g(A_HX,Y)-\epsilon\eta(A_HX)\eta(Y)] \\ &\quad -\epsilon\eta(H)g(\phi^2X,NY). \end{aligned} \tag{3.26}$$

Using (2.17), (2.18) and (2.29) and having some brief calculations we obtain

$$\begin{aligned} g(\tilde{\nabla}_X\phi H,NY) &= -sin^2\theta[g(X,Y)-\epsilon\eta(X)\eta(Y)]\ ||H||^2 \\ &\quad -\epsilon\eta(H)g(\phi^2X,NY). \end{aligned} \tag{3.27}$$

The above equation can be written as

$$\begin{aligned} g(\tilde{\nabla}_X NY,\phi H) &= sin^2\theta[g(X,Y)-\epsilon\eta(X)\eta(Y)]\ ||H||^2 \\ &\quad +\epsilon\eta(H)g(\phi^2X,NY). \end{aligned} \tag{3.28}$$

Again using the fact that $H\in\Gamma(\mu)$ and by Weingarten formula we have

$$\begin{aligned} g(\nabla^{\perp}_{X}NY,\phi H) &= sin^2\theta[g(X,Y)-\epsilon\eta(X)\eta(Y)]||H||^2 \\ &\quad +\epsilon\eta(H)g(\phi^2X,NY). \end{aligned} \tag{3.29}$$

From (3.14) and (3.21) we get

$$\begin{aligned} sin^2\theta[g(X,Y)-\epsilon\eta(X)\eta(Y)]||H||^2+\epsilon\eta(H)g(\phi^2X,NY) &= g(X,Y)||H||^2 \\ &\quad -\epsilon\eta(Y)g(\phi^2X,\phi H). \end{aligned} \tag{3.30}$$

The equation (3.22) has a solution if $H=0$. Hence M is totally geodesic in $\tilde{M}$. $\square$

§4. Hemislant Submanifolds

A.Carriazo [3] introduced hemi-slant submanifolds as a special case of bislant submanifolds and he called them pseudo-slant submanifolds. This section deals with a special case of hemislant submanifolds which are totally umbilical.

Definition 4.1([5,16]) *A submanifold M of an (ϵ)-para Sasakian manifold $\tilde{M}$ is said to be a hemislant submanifold if there exist two orthogonal complementary distributions D_1 and D_2 satisfying the following properties:*

(*i*) $TM = D_1 \bigoplus D_2 \bigoplus <\xi>$;
(*ii*) D_1 *is a slant distribution with slant angle* $\theta \neq \pi/2$;
(*iii*) D_2 *is totally real i.e.,* $\phi D_2 \subseteq T^{\perp}M$.

A hemislant submanifold is called proper hemislant submanifold if $\theta \neq 0, \frac{\pi}{2}$. Further if μ is ϕ-invariant subspace of the normal bundle $T^{\perp}M$, then for pseudo-slant submanifold, the normal bundle $T^{\perp}M$ can be decomposed as

$$T^{\perp}M = ND_1 \bigoplus ND_2 \bigoplus <\mu> .$$

In this section we will derive some of the integrability conditions of the involved distributions of a hemislant submanifold, which play a crucial role from a geometrical point of view.

Theorem 4.1 *Let M be a hemislant submanifold of an (ϵ)-paracontact Sasakian manifold $\tilde{M}$ then $g([X,Y],\xi) = 0$ for any $X, Y \in D_1 \oplus D_2$.*

Proof We know

$$g(X,\phi Y) = g(Y,\phi X),\ \nabla_X \xi = \epsilon\phi X. \tag{4.1}$$

Taking inner product with Y we obtain

$$g(\nabla_X \xi, Y) = \epsilon g(\phi X, Y). \tag{4.2}$$

We can write

$$g(\nabla_X Y, \xi) = -\epsilon g(\phi X, Y). \tag{4.3}$$

Interchanging X, Y we get

$$g(\nabla_Y X, \xi) = -\epsilon g(\phi Y, X). \tag{4.4}$$

Subtracting equations (4.3) and (4.4) and using (4.1) we have

$$g([X,Y],\xi) = 0. \tag{4.5}$$

This completes the proof. □

From Theorem (4.1) we can deduce the following corollaries.

Corollary 4.1 *The distribution $D_1 \oplus D_2$ on a hemislant submanifold of an (ϵ)-para Sasakian manifold $\tilde{M}$ is integrable.*

Corollary 4.2 *The distribution D_1 and D_2 on a hemislant submanifold of an (ϵ)-para Sasakian manifold $\tilde{M}$ is integrable.*

Proposition 4.1 *Let M be a hemislant submanifold of an (ϵ)-para Sasakian manifold $\tilde{M}$, then for any Z, $W \in D_2$, the anti-invariant distribution $D_2 \oplus < \xi >$ is integrable iff*

$$A_{\phi Z}W - A_{\phi W}Z + \nabla^{\perp}_{Z}\phi W - \nabla^{\perp}_{W}\phi Z - \epsilon\eta(W)Z + \epsilon\eta(Z)W = 0.$$

Proof For any $Z, W \in D_2 \bigoplus \xi$ we know

$$\tilde{\nabla}_Z \phi W = (\tilde{\nabla}_Z \phi W) + \phi\tilde{\nabla}_Z W = (\tilde{\nabla}_Z \phi W) + \phi\nabla_Z W + \phi h(Z, W). \tag{4.6}$$

Using (2.16) and (2.17) we have

$$-A_{\phi W}Z + \nabla^{\perp}_{Z}\phi W = (\tilde{\nabla}_Z \phi W) + \phi\tilde{\nabla}_Z W = (\tilde{\nabla}_Z \phi W) + \phi\nabla_Z W + \phi h(Z, W). \tag{4.6}$$

Interchanging Z and W, we obtain

$$-A_{\phi Z}W + \nabla^{\perp}_{W}\phi Z = (\tilde{\nabla}_W \phi Z) + \phi\tilde{\nabla}_W Z = (\tilde{\nabla}_W \phi Z) + \phi\nabla_W Z + \phi h(W, Z). \tag{4.8}$$

Then from (4.7) and (4.8) we calculate

$$A_{\phi Z}W - A_{\phi W}Z + \nabla^{\perp}_{Z}\phi W - \nabla^{\perp}_{W}\phi Z = (\tilde{\nabla}_Z \phi W) - (\tilde{\nabla}_W \phi Z) + \phi[Z, W]. \tag{4.9}$$

From (2.15) we obtain

$$A_{\phi Z}W - A_{\phi W}Z + \nabla^{\perp}_{Z}\phi W - \nabla^{\perp}_{W}\phi Z = \phi[Z, W] - \epsilon\eta(W)\phi^2 Z + \epsilon\eta(Z)\phi^2 W. \tag{4.10}$$

Taking inner product with ϕX, for any $X \in D_1$ we obtain

$$\begin{aligned} & g(A_{\phi Z}W - A_{\phi W}Z + \nabla^{\perp}_{Z}\phi W - \nabla^{\perp}_{W}\phi Z, \phi X) \\ & = g(\phi[Z, W], \phi X) - \epsilon\eta(W)g(\phi^2 Z, \phi X) + \epsilon\eta(Z)g(\phi^2 W, \phi X). \end{aligned} \tag{4.11}$$

Thus from (2.5) the above equation takes the form

$$\begin{aligned} g(\phi[Z, W], \phi X) \quad = \quad & g(A_{\phi Z}W - A_{\phi W}Z + \nabla^{\perp}_{Z}\phi W - \nabla^{\perp}_{W}\phi Z - \epsilon\eta(W)Z \\ & +\epsilon\eta(Z)W, \phi X). \end{aligned} \tag{4.12}$$

The distribution $D_2 \bigoplus < \xi >$ is integrable iff the right hand side of the above equation is zero.

Proposition 4.2 *Let M be a hemislant submanifold of an (ϵ)-para Sasakian manifold $\tilde{M}$, then the anti-invariant distribution $D_1 \bigoplus < \xi >$ is integrable iff*

$$h(Y,TX) + \nabla^{\perp}_{Y} NX - h(X,TY) - \nabla^{\perp}_{X} NY \in \mu$$

for all $X, Y \in D_1 \bigoplus < \xi >$.

Proof For any $X, Y \in D_1 \bigoplus \xi$, we have

$$\begin{aligned} \phi[X,Y] &= \phi[\tilde{\nabla}_Y X - \tilde{\nabla}_X Y] = \tilde{\nabla}_Y TX + \tilde{\nabla}_Y NX - \tilde{\nabla}_X TY \\ &\quad -\tilde{\nabla}_X NY - \epsilon\eta(Y)\phi^2 X + \epsilon\eta(X)\phi^2 Y. \end{aligned} \tag{4.13}$$

$$\begin{aligned} \phi[X,Y] &= \nabla_Y TX + h(Y,TX) - A_{NX} Y + \nabla^{\perp}_{Y} NX - \nabla_X TY - h(X,TY) \\ &\quad +A_{NY} X - \nabla^{\perp}_{X} NY - \epsilon\eta(Y)\phi^2 X + \epsilon\eta(X)\phi^2 Y. \end{aligned} \tag{4.14}$$

Taking the product with ϕZ, for any $Z \in D_2$, we obtain on solving

$$\begin{aligned} g(\phi[X,Y], \phi Z) &= g(h(Y,TX), \phi Z) + g(\nabla^{\perp}_{Y} NX, \phi Z) - g(h(X,TY), \phi Z) \\ &\quad -g(\nabla^{\perp}_{X} NY, \phi Z) - \epsilon\eta(Y) g(X - \eta(X)\xi, \phi Z) \\ &\quad +\epsilon\eta(Y) g(Y - \eta(Y)\xi, \phi Z). \end{aligned} \tag{4.15}$$

$$g([X,Y], Z) = g(h(Y,TX) + \nabla^{\perp}_{Y} NX - h(X,TY) - \nabla^{\perp}_{X} NY, \phi Z). \tag{4.16}$$

Thus our assertion follows from equation (4.16). □

Theorem 4.2 *Let M be a hemislant submanifold of an (ϵ)-para Sasakian manifold $\tilde{M}$, then at least one of the following statements is true:*

(i) $dim D_2 = 1$;

(ii) $H \in \mu$;

(iii) *M is proper slant.*

Proof For any $Z, W \in TM$, we have

$$(\tilde{\nabla}_Z \phi W) + (\tilde{\nabla}_W \phi Z) = -2g(\phi Z, \phi W)\xi - \epsilon\eta(Z)\phi^2 W + \epsilon\eta(W)\phi^2 Z. \tag{4.17}$$

If we assume the vector fields $Z, W \in D_2$, then the above equation reduces to

$$(\tilde{\nabla}_Z \phi) W + (\tilde{\nabla}_W \phi) Z + 2g(\phi Z, \phi W)\xi = 0. \tag{4.18}$$

In particular if we take the above equation for one vector $Z \in D_2$, i.e

$$(\tilde{\nabla}_Z \phi) Z + g(\phi Z, \phi Z)\xi = 0. \tag{4.19}$$

Again using (2.6) we have

$$(\tilde{\nabla}_Z\phi)Z + ||Z||^2\xi = 0. \tag{4.20}$$

Therefore the tangential and normal components of the above equation are $\mathbf{P}_Z Z = ||Z||^2\xi$ and $\mathbf{Q}_Z Z = 0$ respectively. From (2.21) and tangential component of (4.20) we get

$$(\tilde{\nabla}_Z T)Z = -T\nabla_Z Z = A_{NZ}Z + th(Z,Z) - ||Z||^2\xi. \tag{4.21}$$

Taking the product with $W \in D_2$, we get from (2.18)

$$g(T\nabla_Z Z, W) = g(h(Z,W), NZ) + g(th(Z,Z), W). \tag{4.22}$$

Using the fact that M is totally umbilical submanifold and for any $W \in D_2$, then the above equation takes the form

$$g(Z,W)g(H,NZ) + ||Z||^2 g(tH,W) = -g(T\nabla_Z Z, W) = 0. \tag{4.23}$$

Thus the equation (4.10) has a solution if either dim$D_2 = 1$ or $H \in \mu$ or $D_2 = 0$, i.e. M is proper slant. □

From the above conclusions we can obtain the following theorem

Theorem 4.3 *Let M be a totally umbilical hemislant submanifold of an (ϵ)-para Sasakian manifold $\tilde{M}$. Then at least one of the following statements is true:*

(i) *M is an anti-invariant submanifold;*
(ii) $g(\nabla_{TX}\xi, X) = \epsilon[||X||^2 - \eta^2(X)]$;
(iii) *M is totally geodesic submanifold;*
(iv) *dim D_2=1;*
(v) *M is a proper slant submanifold.*

Proof If $H \neq 0$ then from equation (3.19), we can conclude that the slant distribution D_1 =0 i.e. M is anti-invariant submanifold which is case (i). If $D_1 \neq 0$ and $H \in \mu$, then from theorem (3.2) we get (ii) for any $X \in TM$. Again if $H \in \mu$ then by theorem (3.4), M is totally geodesic. Lastly if $H \notin \mu$, then the equation (4.23) has a solution if either dim$D_2 = 1$ or $D_2 = 0$. Hence the theorem follows. □

Next we have the following theorem.

Theorem 4.4 *Let M be a submanifold of an almost contact metric manifold $\tilde{M}$, such that $\xi \in TM$. Then M is a pseudo-slant submanifold iff there exists a constant $\lambda \in (0,1]$ such that*

(i) D=$\{X \in TM | T^2X = -\lambda X\}$ is a distribution on M;
(ii) *For case $X \in TM$, orthogonal to D, $TX = 0$.*

Furthermore in this case $\lambda = cos^2\theta$, where θ denotes the slant angle of D.

Proof: Follows from [11]. □

Again we prove

Theorem 4.5 *Let M be a hemi-slant submanifold of an (ϵ)-para Sasakian manifold $\tilde{M}$. Then $\nabla Q = 0$ iff M is an anti-invariant submanifold.*

Proof Considering the distribution $D_2 \bigoplus < \xi >$, from (4.4) we can write

$$T^2 X = \lambda(X - \eta(X)\xi). \tag{4.24}$$

Denoting the slant angle of M by θ. Then, replacing X by $\nabla_X Y$, we obtain from (4.24)

$$Q\nabla_X Y = cos^2\theta[\nabla_X Y - \eta(\nabla_X Y)\xi]. \tag{4.25}$$

for any $X, Y \in D_2 \bigoplus < \xi >$. After taking the covariant derivative of equation (4.24) we have

$$\nabla_X QY = cos^2\theta[\nabla_X Y - \eta(\nabla_X Y)\xi) - g(Y, \nabla_X \xi)\xi - \eta(Y)\nabla_X \xi]. \tag{4.26}$$

Adding equations (4.25) and (4.26) we obtain

$$\begin{aligned}(\tilde{\nabla}_X Q)Y &= cos^2\theta[\nabla_X Y - \eta(\nabla_X Y)\xi) + g(Y, \epsilon TX)\xi + \eta(Y)\epsilon TX] \\ &\quad -cos^2\theta\nabla_X Y + cos^2\theta\eta(\nabla_X Y)\xi \qquad (4.27)\end{aligned}$$

for any $X, Y \in D_2 \bigoplus < \xi >$.

Here we observe that $g(Y, TX)\xi + TX\eta(Y) \neq 0$. Therefore $(\tilde{\nabla}_X Q) = 0$ iff $\theta = \frac{\pi}{2}$ holds in $D_2 \bigoplus < \xi >$. Again D_1 is anti-invariant by definition. Thus, the theorem follows. □

§5. An Example

Let us give an example of a three dimensional submanifold of (ϵ)-paracontact Sasakian manifold which is pseudo slant so as to verify the above results. Let $\mathbf{R}^3$ be a 3-dimensional Euclidean space with a rectangular coordinates (x, y, z), we put

$$\eta = dy \qquad \xi = \frac{\partial}{\partial y}. \tag{5.1}$$

We define the $(1,1)$ tensor ϕ as:

$$\phi(\frac{\partial}{\partial x}) = \frac{\partial}{\partial z} \quad \phi(\frac{\partial}{\partial z}) = \frac{\partial}{\partial x} \quad \phi(\frac{\partial}{\partial y}) = 0 \tag{5.2}$$

and we define the Riemannian metric g as

$$g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence we can easily see that (ϕ, ξ, η, g) is an (ϵ)-paracontact Sasakian manifold on $\mathbf{R}^3$.

The vector fields $e_1 = \frac{\partial}{\partial x}$, $e_2 = \frac{\partial}{\partial y}$, $e_3 = \frac{\partial}{\partial z}$, forms a frame of TM. We have

$$\phi e_1 = e_3, \quad \phi e_2 = 0, \quad \phi e_3 = e_1.$$

Let $D_1 =< e_2 >$, $D_2 =< e_1 >$ and $\xi =< e_3 >$. We know

$$\cos\angle(\phi X, Y) = \frac{g(\phi X, Y)}{|\phi X||Y|}.$$

Suppose $X \in D_1$ and $Y \in TM$. Then we can write $X = Ke_2$ where K is a scalar and $Y = re_1 + se_2 + te_3$ where r, s, t are scalars. Notice that

$$g(\phi X, Y) = g(\phi e_2, re_1 + se_2 + te_3) = rg(0, e_1) + sg(0, e_2) + tg(0, e_3) = 0.$$

Hence $cos\angle(\phi X, Y) = 0$ implies $\theta = \frac{\pi}{2}$ Hence the distribution D_1 is anti-invariant.

Again let us assume $U \in D_1$ and $V \in TM$. Then we can write $U = ae_1$, where a is a scalar and $V = ke_1 + le_2 + me_3$ where k, l, m are scalars. Using the formula above we get that

$$g(\phi U, V) = g(\phi(ae_1), ke_1 + le_2 + me_3) = am.$$

Hence $cos\angle(\phi U, V)$=constant. So we have obtained that the distribution D_2 is slant.

In this case, the distribution D_1 is anti-invariant while D_2 is slant. Hence the submanifold under consideration is hemislant.

References

[1] J.K.Beem and P.E..Ehrlich, *Global Lorentzian Geometry*, Marcel Dekker, New York, 1981.

[2] J.L.Cabrerizo, A.Carriazo, L.M.Fernandez, M.Fernandez, Slant submanifolds in Sasakian manifolds, *Glasgow. Math. J.* 42(2000).

[3] A.Carriazo, *New Developments in Slant Submanifolds Theory*, Narosa Publishing House, New Delhi, India, 2002.

[4] B.Y.Chen, Slant immersions, *Bulletin of the Australian Mathematical Society*, Vol.41, no.1, pp. 135-147, 1990.

[5] M.A.Khan, Khushwant singh and V.A.Khan, Slant submanifold of LP-contact manifolds, *Differential Geometry-Dynamical System, Balkan Society of Geometers*, Vol.12, pp. 102-108, 2010.

[6] M.A.Khan, K.Singh and V.A.Khan, Slant Submanifolds of LP-contact manifolds, *Differential Geometry Dynamical Systems*, Vol.12, pp. 102-108, 2010.

[7] M. Kobayashi, CR-submanifolds of a Sasakian manifold, *Tensor N.S.* Vol. 35, 297-307, 1981.

[8] A.Lotta, Slant Submanifolds in contact geometry, *Bull.Math.Soc. Roumania*, 39, 183-198, 1996.

[9] B.O'Neill, *Semi-Riemannian Geometry with Application to Relativity*, Academic Press, 1983.

[10] N.Papaghiuc, Semi-slant submanifolds of Kaehlerian manifold, *An.Stiint.Univ.Al.I.CuzaIasi*, Ser.Noua, mat. 40(1), 55-61, 1994.

[11] I.Sato, On a structure similar to the almost contact structure, *Tensor N.S.* 30(3), 219-224, 1976.

[12] B.Sahin, Every totally umbilical proper slant submanifold of a Kaehler manifold is totally geodesic, *Results in Mathematics*, Vol.54, No.1-2, pp. 167-172, 2009.

[13] S.Tanno, The automorphism groups of almost contact Riemannian manifolds, *Tohoku Math. J.*, 21, 21-38, 1969.

[14] T.Takahashi, Sasakian manifold with Pseudo-Riemannian metric, *Tohoku Math.J.*, 21(2), 644-653, 1969.

[15] M.M.Tripathi, E.Kilic, S.Y.Perktas and S.Keles, Indefinite almost paracontact metric manidolds, *arXiv: 0908.2729 vi[math.DG]*, 2009.

[16] Siraj Uddin, Meraj Ali Khan, Khuswant Singh, *Totally Umbilicxal Proper Slant and Hemislant Submanifolds of an LP-Cosymplectic Manifold*, Hindawi Publishing Corporation, Mathematical Problems in Engineering, Volume 2011, Article ID 516238, 9 pages, 2011.