International J.Math. Combin. Vol.4(2017), 121-137

Clique-to-Clique Monophonic Distance in Graphs

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Abstract: In this paper we introduce the clique-to-clique $C - C'$ monophonic path, the clique-to-clique monophonic distance $d_m(C, C')$, the clique-to-clique $C - C'$ monophonic, the clique-to-clique monophonic eccentricity $e_{m_3}(C)$, the clique-to-clique monophonic radius R_{m_3}, and the clique-to-clique monophonic diameter D_{m_3} of a connected graph G, where C and C' are any two cliques in G. These parameters are determined for some standard graphs. It is shown that $R_{m_3} \leq D_{m_3}$ for every connected graph G and that every two positive integers a and b with $2 \leq a \leq b$ are realizable as the clique-to-clique monophonic radius and the clique-to-clique monophonic diameter, respectively, of some connected graph. Further it is shown that for any three positive integers a, b, c with $3 \leq a \leq b \leq c$ are realizable as the clique-to-clique radius, the clique-to-clique monophonic radius, and the clique-to-clique detour radius, respectively, of some connected graph and also it is shown that for any three positive integers a, b, c with $4 \leq a \leq b \leq c$ are realizable as the clique-to-clique diameter, the clique-to-clique monophonic diameter, and the clique-to-clique detour diameter, respectively, of some connected graph. The clique-to-clique monophonic center $C_{m_3}(G)$ and the clique-to-clique monophonic periphery $P_{m_3}(G)$ are introduced. It is shown that the clique-to-clique monophonic center a connected graph does not lie in a single block of G.

Key Words: Clique-to-clique distance, clique-to-clique detour distance, clique-to-clique monophonic distance.

AMS(2010): 05C12.

§1. Introduction

By a graph $G = (V, E)$ we mean a finite undirected connected simple graph. For basic graph theoretic terminologies, we refer to Chartrand and Zhang [2]. If $X \subseteq V$, then $\langle X \rangle$ is the subgraph induced by X. A clique C of a graph G is a maximal complete subgraph and we denote it by its vertices. A $u - v$ path P beginning with u and ending with v in G is a sequence of distinct vertices such that consecutive vertices in the sequence are adjacent in G. A chord of a path $u_1, u_2, ..., u_n$ in G is an edge $u_i u_j$ with $j \geq i + 2$. For a graph G, the length of a path is the number of edges on the path. In 1964, Hakimi [3] considered the facility location problems as vertex-to-vertex distance in graphs. For any two vertices u and v in a connected graph G, the

[1]Received August 17, 2016, Accepted November 26, 2017.

distance $d(u, v)$ is the length of a shortest $u-v$ path in G. For a vertex v in G, the eccentricity of v is the distance between v and a vertex farthest from v in G. The minimum eccentricity among the vertices of G is its radius and the maximum eccentricity is its diameter, denoted by $rad(G)$ and $diam(G)$ respectively. A vertex v in G is a central vertex if $e(v) = rad(G)$ and the subgraph induced by the central vertices of G is the center $Cen(G)$ of G. A vertex v in G is a peripheral vertex if $e(v) = diam(G)$ and the subgraph induced by the peripheral vertices of G is the periphery $Per(G)$ of G. If every vertex of a graph is central vertex then G is called self-centered.

In 2005, Chartrand et. al. [1] introduced and studied the concepts of detour distance in graphs. For any two vertices u and v in a connected graph G, the detour distance $D(u, v)$ is the length of a longest $u - v$ path in G. For a vertex v in G, the detour eccentricity of v is the detour distance between v and a vertex farthest from v in G. The minimum detour eccentricity among the vertices of G is its detour radius and the maximum detour eccentricity is its detour diameter, denoted by $rad_D(G)$ and $diam_D(G)$ respectively. Detour center, detour self-centered and detour periphery of a graph are defined similarly to the center, self-centered and periphery of a graph respectively.

In 2011, Santhakumaran and Titus [7] introduced and studied the concepts of monophonic distance in graphs. For any two vertices u and v in G, a $u-v$ path P is a $u-v$ monophonic path if P contains no chords. The monophonic distance $d_m(u, v)$ from u to v is defined as the length of a longest $u - v$ monophonic path in G. For a vertex v in G, the monophonic eccentricity of v is the monophonic distance between v and a vertex farthest from v in G. The minimum monophonic eccentricity among the vertices of G is its monophonic radius and the maximum monophonic eccentricity is its monophonic diameter, denoted by $rad_m(G)$ and $diam_m(G)$ respectively. Monophonic center, monophonic self-centered and monophonic periphery of a graph are defined similar to the center and periphery respectively of a graph.

In 2002, Santhakumaran and Arumugam [6] introduced the facility locational problem as clique-to-clique distance $d(C, C')$ in graphs as follows. Let ζ be the set of all cliques in a connected graph G the clique-to-clique distance is defined by $d(C, C') = \min\{d(u, v) : u \in C, v \in C'\}$. For our convenience a $C - C'$ path of length $d(C, C')$ is called a clique-to-clique $C - C'$ geodesic or simply $C - C'$ geodesic. The clique-to-clique eccentricity $e_3(C)$ of a clique C in G is the maximum clique-to-clique distance from C to a clique $C' \in \zeta$ in G. The minimum clique-to-clique eccentricity among the cliques of G is its clique-to-clique radius and the maximum clique-to-clique eccentricity is its clique-to-clique diameter, denoted by r_3 and d_3 respectively. A clique C in G is called a clique-to-clique central clique if $e_3(C) = r_3$ and the subgraph induced by the clique-to-clique central cliques of G are clique-to-clique center of G. A clique C in G is called a clique-to-clique peripheral clique if $e_3(C) = d_3$ and the subgraph induced by the clique-to-clique peripheral cliques of G are clique-to-clique periphery of G. If every clique of G is clique-to-clique central clique then G is called clique-to-clique self-centered.

In 2015, Keerthi Asir and Athisayanathan [4] introduced and studied the concepts of clique-to-clique detour distance $D(C, C')$ in graphs as follows. Let ζ be the set of all cliques in a connected graph G and $C, C' \in \zeta$ in G. A clique-to-clique $C - C'$ path P is a $u - v$ path, where $u \in C$ and $v \in C'$, in which P contains no vertices of C and C' other than u and v and the clique-to-clique detour distance, $D(C, C')$ is the length of a longest $C - C'$ path in G. A $C - C'$ path of length $D(C, C')$ is called a $C - C'$ detour. The clique-to-clique detour eccentricity of a clique C in G is the maximum clique-to-clique detour distance from C to a clique $C' \in \zeta$ in G. The minimum clique-to-clique detour eccentricity among the cliques of G is its clique-to-clique detour radius and the maximum clique-to-clique detour eccentricity is its clique-to-clique detour diameter, denoted by R_3 and D_3 respectively. The clique-to-clique detour center $C_{D3}(G)$, the clique-to-clique detour self-centered, the clique-to-clique detour periphery $P_{D3}(G)$ are defined similar to the clique-to-clique center. the clique-to-clique self-centered and the clique-to-clique periphery of a graph respectively.

These motivated us to introduce the concepts of clique-to-clique monophonic distance in graphs and investigate certain results related to clique-to-clique monophonic distance and other distances in graphs. These ideas have intresting applications in channel assignment problem in radio technologies and capture different aspects of certain molecular problems in theoretical chemistry. Also there are useful applications of these concepts to security based communication network design. In a social network a clique represents a group of individuals having a common interest. Thus the clique-to-clique monophonic centrality have intresting application in social networks. Throughout this paper, G denotes a connected graph with at least two vertices.

§2. Clique-to-Clique Monophonic Distance

Definition 2.1 *Let ζ be the set of all cliques in a connected graph G and $C, C' \in \zeta$. A clique-to-clique $C - C'$ path P is said to be a clique-to-clique $C - C'$ monophonic path if P contains no chords in G. The clique-to-clique monophonic distance $d_m(C, C')$ is the length of a longest $C - C'$ monophonic path in G. A $C - C'$ monophonic path of length $d_m(C, C')$ is called a clique-to-clique $C - C'$ monophonic or simply $C - C'$ monophonic.*

Example 2.2 Consider the graph G given in Fig 2.1. For the cliques $C = \{u, w\}$ and $C' = \{v, z\}$ in G, the $C - C'$ paths are $P_1 : u, v$, $P_2 : w, x, z$ and $P_3 : w, x, y, z$. Now P_1 and P_2 are $C - C'$ monophonic paths, while P_3 is not so. Also the clique-to-clique distance $d(C, C') = 1$, the clique-to-clique monophonic distance $d_m(C, C') = 2$, and the clique-to-clique detour distance $D(C, C') = 3$. Thus the clique-to-clique monophonic distance is different from both the clique-to-clique distance and the clique-to-clique detour distance. Now it is clear that P_1 is a $C - C'$ geodesic, P_2 is a $C - C'$ monophonic, and P_3 is a $C - C'$ detour.

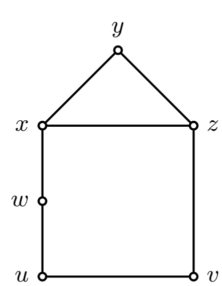

Fig.2.1

Keerthi Asir and Athisayanathan [4] showed that for any two cliques C and C' in a non-trivial connected graph G of order n, $0 \leq d(C,C') \leq D(C,C') \leq n-2$. Now we have the following theorem.

Theorem 2.3 *For any two cliques C and C' in a non-trivial connected graph G of order n, $0 \leq d(C,C') \leq d_m(C,C') \leq D(C,C') \leq n-2$.*

Proof By definition $d(C,C') \leq d_m(C,C')$. If P is a unique $C-C'$ path in G, then $d(C,C') = d_m(C,C') = D(C,C')$. Suppose that G contains more than one $C-C'$ path. Let Q be a longest $C-C'$ path in G.

Case 1. If Q does not contain a chord, then $d_m(C,C') = D(C,C')$.

Case 2. If Q contains a chord, then $d_m(C,C') < D(C,C')$. □

Remark 2.4 The bounds in Theorem 2.3 are sharp. If $G = K_2$, then $0 = d(C,C) = d_m(C,C) = D(C,C) = n-2$. Also if G is a tree, then $d(C,C') = d_m(C,C') = D(C,C')$ for every cliques C and C' in G and the graph G given in Fig. 2.1, $0 < d(C,C') < d_m(C,C') < D(C,C') < n-2$.

Theorem 2.5 *Let C and C' be any two adjacent cliques ($C \neq C'$) in a connected graph G. Then $d_m(C,C') = n-2$ if and only if G is a cycle $C_n (n > 3)$.*

Proof Assume that G is cycle $C_n : u_1, u_2, \cdots, u_{n-1}, u_n, u_1 (n \geq 4)$. Since any edge in G is a clique, without loss of generality we assume that $C = \{u_1, u_2\}$, $C' = \{u_n, u_1\}$ be any two adjacent cliques. Then there exists two distinct $C-C'$ paths, say P_1 and P_2 such that $P_1 : u_1$ is a trivial $C-C'$ path of length 0 and $P_2 : u_2, u_3, \cdots, u_{n-1}, u_n$ is $C-C'$ monophonic path of length $n-2$. It is clear that $d_m(C,C') = n-2$. Conversely assume that for any two distinct adjacent cliques C and C' in a connected graph G, $d_m(C,C') = n-2$. We prove that G is a cycle. Suppose that G is not a cycle. Then G must be either a tree or a cyclic graph.

Case 1. If G is a tree, then $C-C'$ path is trivial. So that $d_m(C,C') = 0 < n-2$, which is a contradiction.

Case 2. If G is a cyclic graph, then G must contain a cycle $C_d : x_1, x_2, \cdots, x_d, x_1$ of length $d < n$. If $C = \{x_1, x_2\}$ and $C' = \{x_n, x_1\}$ then $d_m(C,C') < n-2$, which is a contradiction. □

Since the length of a clique-to-clique monophonic path between any two cliques in $K_{n,m}$ is 2, we have the following theorem.

Theorem 2.6 *Let $K_{n,m} (n \leq m)$ be a complete bipartite graph with the partition V_1, V_2 of $V(K_{n,m})$ such that $|V_1| = n$ and $|V_2| = m$. Let C and C' be any two cliques in $K_{n,m}$, then $d_m(C,C') = 2$.*

Since every tree has unique clique-to-clique monophonic path, we have the following theorem.

Theorem 2.7 *If G is a tree, then $d(C,C') = d_m(C,C') = D(C,C')$ for every cliques C and C' in G.*

The converse of the Theorem 2.7 is not true. For the graph G obtained from a complete bipartite graph $K_{2,n}(n \geq 2)$ by joining the vertices of degree n by an edge. In such a graph every clique C is isomorphic to K_3 and so for any two cliques C and C', $d(C, C') = d_m(C, C') = D(C, C') = 0$, but G is not tree.

§3. Clique-to-Clique Monophonic Center

Definition 3.1 *Let G be a connected graph and let ζ be the set of all cliques in G. The clique-to-clique monophonic eccentricity $e_{m_3}(C)$ of a clique C in G is defined by $e_{m_3}(C) = \max\{d_m(C, C') : C' \in \zeta\}$. A clique C' for which $e_{m_3}(C) = d_m(C, C')$ is called a clique-to-clique monophonic eccentric clique of C. The clique-to-clique monophonic radius of G is defined as, $R_{m_3} = rad_{m_3}(G) = \min\{e_{m_3}(C) : C \in \zeta\}$ and the clique-to-clique monophonic diameter of G is defined as, $D_{m_3} = diam_{m_3}(G) = \max\{e_{m_3}(C) : C \in \zeta\}$. A clique C in G is called a clique-to-clique monophonic central clique if $e_{m_3}(C) = R_{m_3}$ and the clique-to-clique monophonic center of G is defined as, $C_{m_3}(G) = Cen_{m_3}(G) = \langle\{C \in \zeta : e_{m_3}(C) = R_{m_3}\}\rangle$. A clique C in G is called a clique-to-clique monophonic peripheral clique if $e_{m_3}(C) = D_{m_3}$ and the clique-to-clique monophonic periphery of G is defined as, $P_{m_3}(G) = Per_{m_3}(G) = \langle\{C \in \zeta : e_{m_3}(C) = D_{m_3}\}\rangle$. If every clique of G is a clique-to-clique monophonic central clique, then G is called a clique-to-clique monophonic self centered graph.*

Example 3.2 For the graph G given in Fig.3.1, the set of all cliques are given by, $\zeta = \{C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_8, C_9\}$ where $C_1 = \{v_1, v_2, v_3\}$, $C_2 = \{v_3, v_4\}$, $C_3 = \{v_4, v_5, v_6\}$, $C_4 = \{v_6, v_7\}$, $C_5 = \{v_7, v_8\}$, $C_6 = \{v_8, v_{10}\}$, $C_7 = \{v_9, v_{10}\}$, $C_8 = \{v_4, v_9\}$, $C_9 = \{v_{10}, v_{11}, v_{12}, v_{13}, v_{14}\}$.

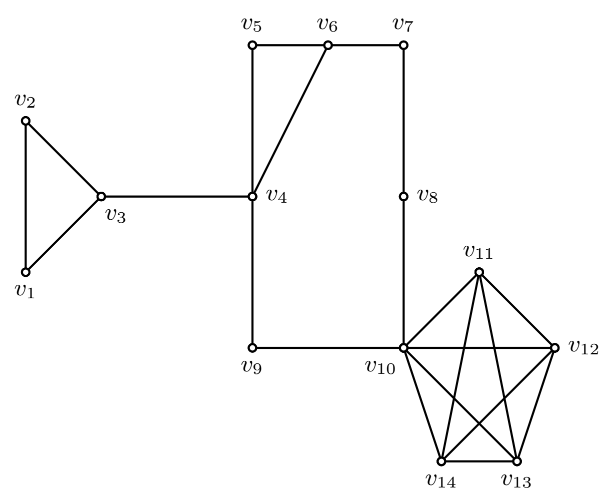

Fig.3.1

The clique-to-clique eccentricity $e_3(C)$, the clique-to-clique detour eccentricity $e_{D3}(C)$, the clique-to-clique monophonic eccentricity $e_{m_3}(C)$ of all the cliques of G are given in Table 1.

Cliques C	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
$e_3(C)$	3	2	2	2	3	3	2	2	3
$e_{m_3}(C)$	5	4	4	5	4	4	5	4	5
$e_{D3}(C)$	6	5	4	5	5	5	6	5	6

Table 1

The clique-to-clique monophonic eccentric clique of all the cliques of G are given in Table 2.

Cliques C	Clique-to-Clique Monophonic Eccentric Cliques
C_1	C_4, C_5, C_6, C_7, C_9
C_7	C_1, C_2, C_3, C_8
C_9	C_1, C_2, C_3, C_8

Table 2

The clique-to-clique radius $r_3 = 2$, the clique-to-clique diameter $d_3 = 3$, the clique-to-clique detour radius $R_3 = 4$, the clique-to-clique detour diameter $D_3 = 6$, the clique-to-clique monophonic radius $R_{m_3} = 4$ and the clique-to-clique monophonic diameter $D_{m_3} = 5$. Also it is clear that the clique-to-clique center $C_3(G) = \langle\{C_2, C_3, C_4, C_7, C_8\}\rangle$, the clique-to-clique periphery $P_3(G) = \langle\{C_1, C_5, C_6, C_9\}\rangle$, the clique-to-clique detour center $C_{D3}(G) = \langle\{C_3\}\rangle$, the clique-to-clique detour periphery $P_{D3}(G) = \langle\{C_1, C_7, C_9\}\rangle$, the clique-to-clique monophonic center $C_{m_3}(G) = \langle\{C_2, C_3, C_5, C_6, C_8\}\rangle$, the clique-to-clique monophonic periphery $P_{m_3}(G) = \langle\{C_1, C_4, C_7, C_9\}\rangle$.

The clique-to-clique monophonic radius R_{m_3} and the clique-to-clique monophonic diameter D_{m_3} of some standard graphs are given in Table 3.

Graph G	K_n	$P_n(n \geq 3)$	$C_n(n \geq 4)$	$W_n(n \geq 5)$	$K_{n,m}(m \geq n)$
R_{m_3}	0	$\lfloor \frac{n-3}{2} \rfloor$	$n-2$	$n-3$	2
D_{m_3}	0	$n-3$	$n-2$	$n-3$	2

Table 3

Remark 3.3 The complete graph K_n, the cycle C_n, the wheel W_n and the complete bipartite graph $K_{n,m}$ are the clique-to-clique monophonic self centered graphs.

Remark 3.4 In a connected graph G, $C_3(G)$, $C_{D3}(G)$, $C_{m_3}(G)$ and $P_3(G)$, $P_{D3}(G)$, $P_{m_3}(G)$ need not be same. For the graph G given in Fig 3.1, it is shown that $C_3(G)$, $C_{D3}(G)$, $C_{m_3}(G)$ and $P_3(G)$, $P_{D3}(G)$, $P_{m_3}(G)$ are distinct.

Theorem 3.5 *Let G be a connected graph of order n. Then*

(i) $0 \leq e_3(C) \leq e_{m_3}(C) \leq e_{D3}(C) \leq n-2$ *for every clique C in G;*

(ii) $0 \leq r_3 \leq R_{m_3} \leq R_3 \leq n-2$*;*

(iii) $0 \leq d_3 \leq D_{m_3} \leq D_3 \leq n-2$.

Proof This follows from Theorem 2.3. □

Remark 3.6 The bounds in Theorem 3.5(*i*) are sharp. If $G = K_2$, then $0 = e_3(C) = e_{m_3}(C) = e_{D3}(C) = n-2$. Also if G is a tree, then $e_3(C) = e_{m_3}(C) = e_{D3}(C)$ for every clique C in G and the graph G given in Fig. 2.1, $e_3(C) < e_{m_3}(C) < e_{D3}(C)$, where $C = \{u, w\}$.

In [1, 2] it is shown that in a connected graph G, the radius and diameter are related by $rad(G) \leq diam(G) \leq 2rad(G)$, the detour radius and detour diameter are related by $rad_D(G) \leq diam_D(G) \leq 2rad_D(G)$, and Santhakumaran et. al. [7]showed that the monophonic radius and monophonic diameter are related by $rad_m(G) \leq diam_m(G)$. Also Santhakumaran et. al. [6] showed that the clique-to-clique radius and clique-to-clique diameter are related by $r_3 \leq d_3 \leq 2r_3+1$ and Keerthi Asir et. al. [4] showed that the upper inequality does not hold for the clique-to-clique detour distance. The following example shows that the similar inequality does not hold for the clique-to-clique monophonic distance.

Remark 3.7 For the graph G of order $n \geq 7$ obtained by identifying the central vertex of the wheel $W_{n-1} = K_1 + C_{n-2}$ and an end vertex of the path P_2. It is easy to verify that $D_{m_3} > 2R_{m_3}$ and $D_{m_3} > 2R_{m_3} + 1$.

Ostrand [5] showed that every two positive integers a and b with $a \leq b \leq 2a$ are realizable as the radius and diameter respectively of some connected graph, Chartrand et. al. [1] showed that every two positive integers a and b with $a \leq b \leq 2a$ are realizable as the detour radius and detour diameter respectively of some connected graph, and Santhakumaran et. al. [7] showed that every two positive integers a and b with $a \leq b$ are realizable as the monophonic radius and monophonic diameter respectively of some connected graph. Also Santhakumaran et. al. [6] showed that every two positive integers a and b with $a \leq b \leq 2a+1$ are realizable as the clique-to-clique radius and clique-to-clique diameter respectively of some connected graph. Keerthi Asir et. al. [4] showed that every two positive integers a and b with $2 \leq a \leq b$ are realizable as the clique-to-clique detour radius and clique-to-clique detour diameter respectively of some connected graph. Now we have a realization theorem for the clique-to-clique monophonic radius and the clique-to-clique monophonic diameter for some connected graph.

Theorem 3.8 *For each pair a, b of positive integers with $2 \leq a \leq b$, there exists a connected graph G with $R_{m_3} = a$ and $D_{m_3} = b$.*

Proof Our proof is divided into cases following.

Case 1. $a = b$.

Let $G = C_{a+2} : u_1, u_2, \cdots, u_{a+2}, u_1$ be a cycle of order $a+2$. Then $e_{m_3}(u_i u_{i+1}) = a$ for $1 \leq i \leq a+2$. It is easy to verify that every clique S in G with $e_{m_3}(S) = a$. Thus $R_{m_3} = a$

and $D_{m_3} = b$ as $a = b$.

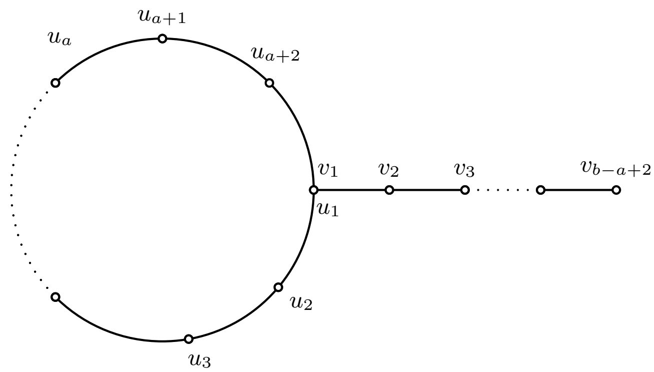

Fig. 3.2

Case 2. $2 \leq a < b \leq 2a$.

Let $C_{a+2} : u_1, u_2, \cdots, u_{a+2}, u_1$ be a cycle of order $a+2$ and $P_{b-a+2} : v_1, v_2, \cdots, v_{b-a+2}$ be a path of order $b-a+2$. We construct the graph G of order $b+3$ by identifying the vertex u_1 of C_{a+2} and v_1 of P_{b-a+2} as shown in Fig. 3.2. It is easy to verify that

$$e_{m_3}(u_i u_{i+1}) = \begin{cases} b-i+2, & \text{if } \ 2 \leq i \leq \left\lceil \frac{a+2}{2} \right\rceil \\ b-a+i-1, & \text{if } \ \left\lceil \frac{a+2}{2} \right\rceil < i \leq a+1, \end{cases}$$

and $e_{m_3}(v_i v_{i+1}) = a+i-1$ if $2 \leq i \leq b-a+1$, $e_{m_3}(u_2u_3) = e_{m_3}(u_{a+1}u_{a+2}) = e_{m_3}(u_{b-a}u_{b-a+1}) = b$, $e_{m_3}(u_1u_2) = e_{m_3}(u_1u_{a+2}) = e_{m_3}(v_1v_2) = a$. It is easy to verify that there is no clique S in G with $e_{m_3}(S) < a$ and there is no clique S' in G with $e_{m_3}(S') > b$. Thus $R_{m_3} = a$ and $D_{m_3} = b$ as $a < b$.

Case 3. $a < b > 2a$.

Let G be a graph of order $b + 2a + 4$ obtained by identifying the central vertex of the wheel $W_{b+3} = K_1 + C_{b+2}$ and an end vertex of the path P_{2a+2}, where $K_1 : v_1$, $C_{b+1} : u_1, u_2, \cdots, u_{b+2}, u_1$ and $P_{2a+2} : v_1, v_2, \cdots, v_{2a+2}$. The resulting graph G is shown in Fig.3.3.

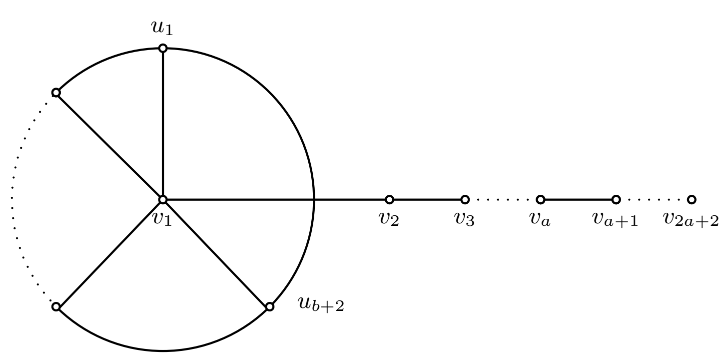

Fig. 3.3

It is easy to verify that $e_{m_3}(v_1u_iu_{i+1}) = b$ if $1 \leq i \leq b+2$ and

$$e_{m_3}(v_iv_{i+1}) = \begin{cases} 2a-i, & \text{if } 1 \leq i \leq a, \\ i-1, & \text{if } a < i < 2a+2. \end{cases}$$

It is also easy to verify that there is no clique S in G with $e_{m_3}(S) < a$ and there is no clique S' in G with $e_{m_3}(S') > b$. Thus $R_{m_3} = a$ and $D_{m_3} = b$ as $b > 2a$. □

Santhakumaran et. al. [7] showed that every three positive integers a, b and c with $3 \leq a \leq b \leq c$ are realizable as the radius, monophonic radius and detour radius respectively of some connected graph. Now we have a realization theorem for the clique-to-clique radius, clique-to-clique monophonic radius and clique-to-clique detour radius respectively of some connected graph.

Theorem 3.9 *For any three positive integers a, b, c with $3 \leq a \leq b \leq c$, there exists a connected graph G such that $r_3 = a$, $R_{m_3} = b$, $R_3 = c$.*

Proof The proof is divided into cases following.

Case 1. $a = b = c$.

Let $P_1 : u_1, u_2, \cdots, u_{a+2}$ and $P_2 : v_1, v_2, \ldots, v_{a+2}$ be two paths of order $a+2$. We construct the graph G of order $2a+4$ by joining u_1 in P_1 and v_1 in P_2 by an edge. It is easy to verify that $e_3(u_1v_1) = e_{m_3}(u_1v_1) = e_{D3}(u_1v_1) = a$, $e_3(u_iu_{i+1}) = e_{m_3}(u_iu_{i+1}) = e_{D3}(u_iu_{i+1}) = a+i$ if $1 \leq i \leq a+1$.

It is also easy to verify that there is no clique S in G with $e_3(S) < a, e_{m_3}(S) < b$ and $e_{D3}(S) < c$. Thus $r_3 = a$, $R_{m_3} = b$ and $R_3 = c$ as $a = b = c$.

Case 2. $3 \leq a \leq b < c$.

Let $P_1 : u_1, u_2, \cdots, u_{a+2}$ and $P_2 : v_1, v_2, \cdots, v_{a+2}$ be two paths of order $a+2$. Let $Q_1 : w_1, w_2, \ldots, w_{b-a+3}$ and $Q_2 : z_1, z_2, \cdots, z_{b-a+3}$ be two paths of order $b-a+3$. Let $K_1 : x_1, x_2, \cdots, x_{c-b+1}$ and $K_2 : y_1, y_2, \cdots, y_{c-b+1}$ be two complete graphs of order $c-b+1$. We construct the graph G of order $2c+4$ as follows: (i) identify the vertices u_1 in P_1 with w_1 in Q_1 and also identify the vertices v_1 in P_2 with z_1 in Q_2; (ii) identify the vertices u_3 in P_1 with w_{b-a+3} in Q_1 and also identify the vertices z_{b-a+3} in Q_2 with v_3 in P_2; (iii) identify the vertices u_{a+1} in P_1 with x_1 in K_1 and also identify the vertices x_{c-b+1} in K_1 with u_a in P_1; (iv) identify the vertices v_{a+1} in P_2 with y_1 in K_2 and also identify the vertices y_{c-b+1} in K_2 with v_a in P_2; (v) join each vertex $w_i (2 \leq i \leq b-a+2)$ in Q_1 with u_2 in P_1 and join each vertex $z_i (2 \leq i \leq b-a+2)$ in Q_2 with v_2 in P_2 (vi) join u_1 in P_1 with v_1 in P_2. The resulting graph G is shown in Fig.3.4.

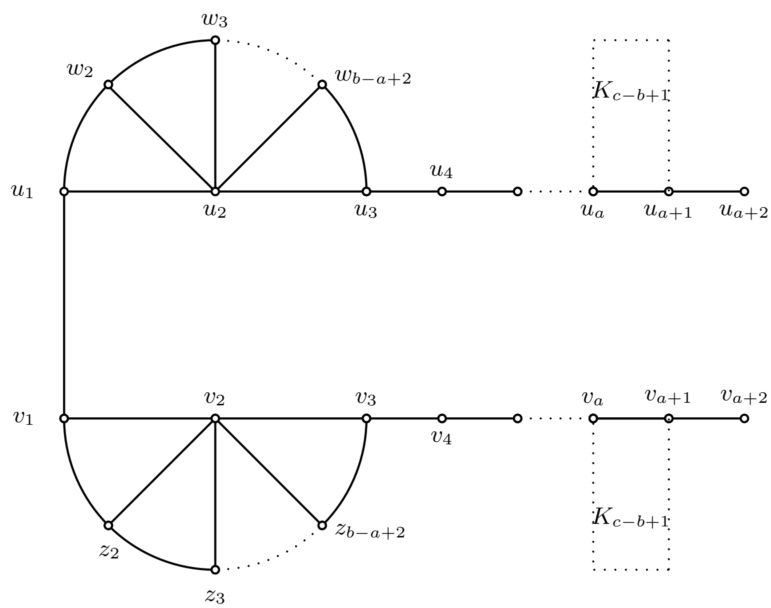

Fig.3.4

It is easy to verify that $e_3(u_1v_1) = a$,

$$e_3(u_2w_iw_{i+1}) = \begin{cases} a+1, & \text{if } \ i = 1, \\ a+2, & \text{if } \ 2 \le i \le b-a+2, \end{cases}$$

$$e_3(v_2z_iz_{i+1}) = \begin{cases} a+1, & \text{if } \ i = 1, \\ a+2, & \text{if } \ 2 \le i \le b-a+2, \end{cases}$$

$$e_3(u_iu_{i+1}) = \begin{cases} a+i, & \text{if } \ 3 \le i < a, \\ 2a+1, & \text{if } \ i = a+1, \end{cases}$$

$$e_3(v_iv_{i+1}) = \begin{cases} a+i, & \text{if } \ 3 \le i < a, \\ 2a+1, & \text{if } \ i = a+1, \end{cases}$$

$$e_3(K_1) = 2a, \quad e_3(K_2) = 2a, \quad e_{m_3}(u_1v_1) = b,$$

and $e_{m_3}(u_2w_iw_{i+1}) = b+i$, if $1 \le i \le b-a+2$, $e_{m_3}(v_2z_iz_{i+1}) = b+i$, if $1 \le i \le b-a+2$,

$$e_{m_3}(u_iu_{i+1}) = \begin{cases} 2b-a+i, & \text{if } \ 3 \le i < a, \\ 2b+1, & \text{if } \ i = a+1, \end{cases}$$

$$e_{m_3}(v_iv_{i+1}) = \begin{cases} 2b-a+i, & \text{if } \ 3 \le i < a, \\ 2b+1, & \text{if } \ i = a+1, \end{cases}$$

$$e_{m_3}(K_1) = 2b, \quad e_{m_3}(K_2) = 2b, \quad e_{D3}(u_1v_1) = c,$$

$e_{D3}(u_2w_iw_{i+1}) = c+i$, if $1 \le i \le b-a+2$, $e_{D3}(v_2z_iz_{i+1}) = c+i$, if $1 \le i \le b-a+2$,

$$e_{D3}(u_iu_{i+1}) = \begin{cases} c+b-a+1+i, & \text{if } 3 \le i < a, \\ c+b+2, & \text{if } i = a+1, \end{cases}$$

$$e_{D3}(v_iv_{i+1}) = \begin{cases} c+b-a+1+i, & \text{if } 3 \le i < a, \\ c+b+2, & \text{if } i = a+1, \end{cases}$$

$$e_{D3}(K_1) = c+b+1, \quad e_{D3}(K_2) = c+b+1.$$

It is also easy to verify that there is no clique S in G with $e_3(S) < a$, $e_{m_3}(S) < b$ and $e_{D3}(S) < c$. Thus $r_3 = a$, $R_{m_3} = b$ and $R_3 = c$ as $a \le b < c$.

Case 3. $3 \le a < b = c$.

Let $P_1 : u_1, u_2, \cdots, u_a, u_{a+2}$ and $P_2 : v_1, v_2, \cdots, v_a, v_{a+2}$ be two paths of order $a+2$. Let $Q_1 : w_1, w_2, \cdots, w_{b-a+3}$ and $Q_2 : z_1, z_2, \cdots, z_{b-a+3}$ be two paths of order $b-a+3$. Let $E_i : x_i(3 \le i \le b-a+2)$ and $F_i : y_i(3 \le i \le b-a+2)$ be $2(b-a)$ copies of K_1. We construct the graph G of order $4b-2a+6$ as follows: (i) identify the vertices u_1 in P_1 with w_1 in Q_1 and also identify the vertices v_1 in P_2 with z_1 in Q_2; (ii) identify the vertices u_3 in P_1 with w_{b-a+3} in Q_1 and also identify the vertices z_{b-a+3} in Q_2 with v_3 in P_2 (iii) join each vertex $x_i(3 \le i \le b-a+2)$ with $w_i(3 \le i \le b-a+2)$ and u_1 and also join each vertex $y_i(3 \le i \le b-a+2)$ with $z_i(3 \le i \le b-a+2)$ and v_1 (iv) join u_1 in P_1 with v_1 in P_2. The resulting graph G is shown in Fig. 3.5.

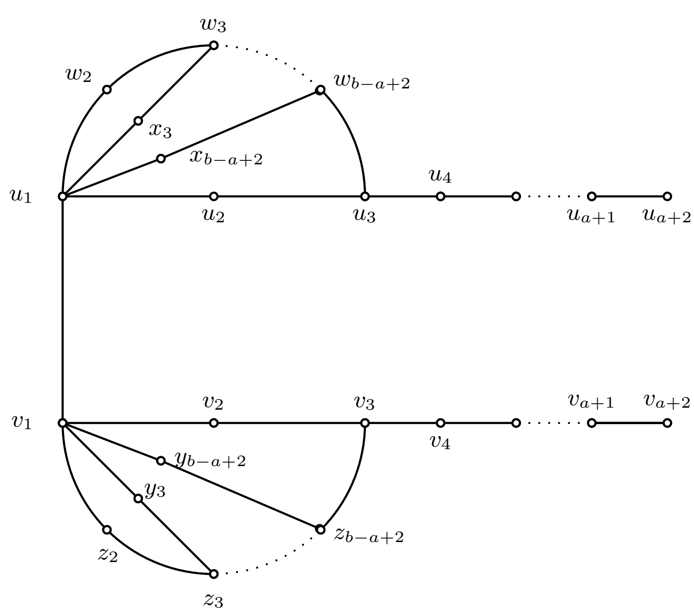

Fig.3.5

It is easy to verify that $e_3(u_1v_1) = a$,

$$e_3(w_iw_{i+1}) = \begin{cases} a+1, & \text{if } \ i = 1, \\ a+2, & \text{if } \ i = 2, \\ a+3, & \text{if } \ 3 \le i \le b-a+2, \end{cases}$$

and $e_3(u_iu_{i+1}) = a+i$, if $1 \le i \le a+1$, $e_3(u_1x_i) = a+1$, if $3 \le i \le b-a+2$, $e_3(w_ix_i) = a+2$, if $3 \le i \le b-a+2$, $e_{m_3}(u_1v_1) = b$,

$$e_{m_3}(w_iw_{i+1}) = \begin{cases} b+1, & \text{if } \ i = 1 \\ 2b-a+5-i, & \text{if } \ 2 \le i \le \left\lfloor \frac{b-a+5}{2} \right\rfloor \\ b+i, & \text{if } \ \left\lfloor \frac{b-a+5}{2} \right\rfloor < i \le b-a+2 \end{cases}$$

$$e_{m_3}(u_iu_{i+1}) = \begin{cases} b+1, & \text{if } \ i = 1, \\ 2b-a+3, & \text{if } \ i = 2, \\ 2b-a+i, & \text{if } \ 3 \le i \le a+1, \end{cases}$$

and $e_{m_3}(u_1x_i) = b+1$, if $3 \le i \le b-a+2$,

$$e_{m_3}(w_ix_i) = \begin{cases} 2b-a+6-i, & \text{if } \ 3 \le i \le \left\lfloor \frac{b-a+5}{2} \right\rfloor, \\ b+i, & \text{if } \ \left\lfloor \frac{b-a+5}{2} \right\rfloor < i \le b-a-2, \end{cases}$$

and $e_{D3}(u_1v_1) = c$,

$$e_{D3}(w_iw_{i+1}) = \begin{cases} c+1, & \text{if } \ i = 1, \\ 2c-a+5-i, & \text{if } \ 2 \le i \le \left\lfloor \frac{b-a+5}{2} \right\rfloor, \\ c+i, & \text{if } \ \left\lfloor \frac{b-a+5}{2} \right\rfloor < i \le b-a+2, \end{cases}$$

$$e_{D3}(u_iu_{i+1}) = \begin{cases} c+1, & \text{if } \ i = 1, \\ 2c-a+3, & \text{if } \ i = 2, \\ 2c-a+i, & \text{if } \ 3 \le i \le a+1, \end{cases}$$

and $e_{D3}(u_1x_i) = c+1$, if $3 \le i \le b-a+2$,

$$e_{D3}(w_ix_i) = \begin{cases} 2c-a+6-i, & \text{if } \ 3 \le i \le \left\lfloor \frac{b-a+5}{2} \right\rfloor, \\ c+i, & \text{if } \ \left\lfloor \frac{b-a+5}{2} \right\rfloor < i \le b-a+2. \end{cases}$$

It is easy to verify that there is no clique S in G with $e_3(S) < a$, $e_{m_3}(S) < b$ and $e_{D3}(S) < c$. Thus $r_3 = a$, $R_{m_3} = b$ and $R_3 = c$ as $a < b = c$. □

Santhakumaran et. al. [7] showed that every three positive integers a, b and c with $5 \le a \le b \le c$ are realizable as the diameter, monophonic diameter and detour diameter respectively of some connected graph. Now we have a realization theorem for the clique-to-clique diameter, clique-to-clique monophonic diameter and clique-to-clique detour diameter respectively of some connected graph.

Theorem 3.10 *For any three positive integers a, b, c with $4 \leq a \leq b \leq c$, there exists a connected graph G such that $d_3 = a$, $D_{m_3} = b$ and $D_3 = c$.*

Proof The proof is divided into cases following.

Case 1. $a = b = c$.

Let $G = P_{a+3} : u_1, u_2, \cdots, u_{a+3}$ be a path. Then

$$e_3(u_iu_{i+1}) = e_{m_3}(u_iu_{i+1}) = e_{D3}(u_iu_{i+1}) = \begin{cases} a - i + 1, & \text{if } 1 \leq i \leq \left\lfloor \frac{a+1}{2} \right\rfloor, \\ i - 2, & \text{if } \left\lfloor \frac{a+1}{2} \right\rfloor < i \leq a + 2. \end{cases}$$

It is easy to verify that there is no clique S in G with $e_3(S) > a$, $e_{m_3}(S) > b$ and $e_{D3}(S) > c$. Thus $d_3 = a$, $D_{m_3} = b$ and $D_3 = c$ as $a = b = c$.

Case 2. $4 \leq a \leq b < c$.

Let $P_1 : u_1, u_2, \cdots, u_{a+2}$ be a path of order $a+2$. Let $P_2 : w_1, w_2, \cdots, w_{b-a+3}$ be a path of order $b - a + 3$. Let $P_3 : x_1, x_2$ be a path of order 2. Let $K_1 : y_1, y_2, \cdots, y_{c-b+1}$ be a complete graph of order $c - b + 1$. We construct the graph G of order $c + 3$ as follows: (i) identify the vertices u_1 in P_1, w_1 in P_2 with x_1 in P_3 and identify the vertices u_3 in P_1 with w_{b-a+3} in P_2; (ii) identify the vertices u_{a+1} in P_1 with y_1 in K_1 and identify the vertices u_a in P_1 with y_{c-b+1} in K_1; (iii) join each vertex $w_i (2 \leq i \leq b - a + 2)$ in P_2 with u_2 in P_1. The resulting graph G is shown in Fig.3.6. It is easy to verify

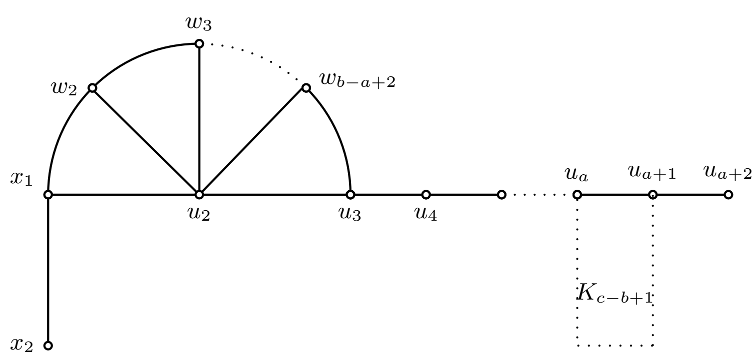

Fig.3.6

that $e_3(x_1x_2) = a$, $e_3(K_1) = a - 1$,

$$e_3(u_iu_{i+1}) = \begin{cases} a - i, & \text{if } 3 \leq i \leq \left\lfloor \frac{a}{2} \right\rfloor, \\ i - 1, & \text{if } \left\lfloor \frac{a}{2} \right\rfloor < i \leq a, \end{cases}$$

$$e_3(u_2w_iw_{i+1}) = \begin{cases} a-1, & \text{if } 1 \le i \le b-a+1, \\ a-2, & \text{if } i = b-a+2, \end{cases}$$

and $e_{m_3}(x_1x_2) = b$, $e_{m_3}(K_1) = b-1$,

$$e_{m_3}(u_iu_{i+1}) = \begin{cases} b-a+i-1, & \text{if } 3 \le i \le a \text{ for } b-a+i \ge a-i, \\ a-i, & \text{if } 3 \le i \le a \text{ for } b-a+i \le a-i, \end{cases}$$

$$e_{m_3}(u_2w_iw_{i+1}) = \begin{cases} b-i, & \text{if } 1 \le i \le \lfloor \frac{b}{2} \rfloor \text{ for } \lfloor \frac{b}{2} \rfloor < b-a+3, \\ i-1, & \text{if } \lfloor \frac{b}{2} \rfloor < i \le b-a+3 \text{ for } \lfloor \frac{b}{2} \rfloor \le b-a+3, \\ b-i, & \text{if } 1 \le i \le b-a+3 \text{ for } \lfloor \frac{b}{2} \rfloor \ge b-a+3, \end{cases}$$

and $e_{D3}(x_1x_2) = c$, $e_{D3}(K_1) = b$,

$$e_{D3}(u_iu_{i+1}) = \begin{cases} b-a+i, & \text{if } 3 \le i \le a \text{ for } b-a+i \ge c-b+a-i-1, \\ c-b+a-i-1, & \text{if } 3 \le i \le a \text{ for } b-a+i \le c-b+a-i-1, \end{cases}$$

$$e_{D3}(u_2w_iw_{i+1}) = \begin{cases} c-i-1, & \text{if } 1 \le i \le \lfloor \frac{b}{2} \rfloor \text{ for } \lfloor \frac{b}{2} \rfloor < c-b+1, \\ i-1, & \text{if } \lfloor \frac{b}{2} \rfloor < i \le b-a+3 \text{ for } \lfloor \frac{b}{2} \rfloor \le c-b+1, \\ c-i-1, & \text{if } 1 \le i \le b-a+3 \text{ for } \lfloor \frac{b}{2} \rfloor \ge c-b+1 \end{cases}$$

It is easy to verify that there is no clique S in G with $e_3(S) > a$, $e_{m_3}(S) > b$ and $e_{D3}(S) > c$. Thus $d_3 = a$, $D_{m_3} = b$ and $D_3 = c$ as $a \le b < c$.

Case 3. $4 \le a < b = c$.

Let $P_1 : u_1, u_2, \cdots, u_{a+2}$ be a path of order $a+2$. Let $P_2 : w_1, w_2, \ldots, w_{b-a+3}$ be a path of order $b-a+3$. Let $P_3 : x_1, x_2$ be a path of order 2. Let $E_i : x_i (3 \le i \le b-a+2)$ be a $b-a$ copies of K_1. We construct the graph G of order $2b-a+4$ as follows: (i) identify the vertices u_1 in P_1,w_1 in P_2 with x_1 in P_3 and also identify the vertices u_3 in P_1 with w_{b-a+3} in P_2; (ii) join each vertex $x_i (3 \le i \le b-a+2)$ with u_3 in P_1 and w_i in P_2. The resulting graph G is shown in Fig.3.7.

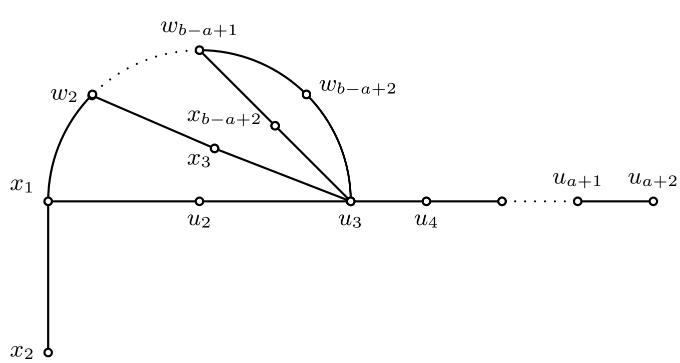

Fig.3.7

It is easy to verify that $e_3(x_1x_2) = a$,

$$e_3(u_iu_{i+1}) = \begin{cases} a-i, & \text{if } 1 \le i \le \left\lfloor \frac{a}{2} \right\rfloor, \\ i-1, & \text{if } \left\lfloor \frac{a}{2} \right\rfloor < i \le a, \end{cases}$$

$e_3(u_3x_i) = a-2, \quad \text{if } \; 3 \le i \le b-a+2,$

$$e_3(w_iw_{i+1}) = \begin{cases} a, & \text{if } 1 \le i \le b-a, \\ a-1, & \text{if } i = b-a+1, \\ a-2, & \text{if } i = b-a+2, \end{cases}$$

and $e_3(x_iw_{i-1}) = a-1, \quad \text{if } \; 3 \le i \le b-a+2$, $e_{m_3}(x_1x_2) = b$,

$$e_{m_3}(u_3x_i) = \begin{cases} b-a+2, & \text{if } 3 \le i \le b-a+2 \text{ for } b-a+3 \ge a-2, \\ a-2, & \text{if } 3 \le i \le b-a+2 \text{ for } b-a+3 \le a-2, \end{cases}$$

$$e_{m_3}(u_iu_{i+1}) = \begin{cases} b, & \text{if } i = 1, \\ b-a+2, & \text{if } i = 2 \text{ for } b-a+3 \ge a-2 \;\;, \\ a-2, & \text{if } i = 2 \text{ for } b-a+3 \le a-2 \;\;, \\ b-a+i-1, & \text{if } 3 \le i \le a \text{ for } b-a+3 \ge a-2 \;\;, \\ a-i, & \text{if } 3 \le i \le a \text{ for } b-a+3 \le a-2 \;\;, \end{cases}$$

$$e_{m_3}(w_iw_{i+1}) = \begin{cases} b-i, & \text{if } 1 \le i \le \left\lfloor \frac{b-a+1}{2} \right\rfloor, \\ a+i-1, & \text{if } \left\lfloor \frac{b-a+1}{2} \right\rfloor < i \le b-a+1, \\ i-1, & \text{if } i = b-a+2 \text{ for } i \ge a-2, \\ a-2, & \text{if } i = b-a+2 \text{ for } i \le a-2, \end{cases}$$

$$e_{m_3}(x_iw_{i-1}) = \begin{cases} e_{m_3}(w_{i-2}w_{i-1}) & \text{if } 3 \le i \le \left\lfloor \frac{b-a+3}{2} \right\rfloor, \\ e_{m_3}(w_{i-1}w_i) & \text{if } \left\lfloor \frac{b-a+3}{2} \right\rfloor < i \le b-a+2, \end{cases}$$

$e_{D3}(x_1x_2) = c$,

$$e_{D3}(u_iu_{i+1}) = \begin{cases} c, & \text{if } i = 1, \\ c-a+2, & \text{if } i = 2 \text{ for } b-a+3 \ge a-2 \;\;, \\ a-2, & \text{if } i = 2 \text{ for } b-a+3 \le a-2 \;\;, \\ c-a+i-1, & \text{if } 3 \le i \le a \text{ for } b-a+3 \ge a-2 \;\;, \\ a-i, & \text{if } 3 \le i \le a \text{ for } b-a+3 \le a-2 \;\;, \end{cases}$$

$$e_{D3}(w_iw_{i+1}) = \begin{cases} c-i, & \text{if } 1 \le i \le \left\lfloor \frac{b-a+1}{2} \right\rfloor, \\ a+i-1, & \text{if } \left\lfloor \frac{b-a+1}{2} \right\rfloor < i \le b-a+1, \\ i-1, & \text{if } i = b-a+2 \text{ for } i \ge a-2, \\ a-2, & \text{if } i = b-a+2 \text{ for } i \le a-2 \end{cases}$$

$$e_{D3}(x_iw_{i-1}) = \begin{cases} e_{D3}(w_{i-2}w_{i-1}) & \text{if } 3 \le i \le \left\lfloor \frac{b-a+3}{2} \right\rfloor, \\ e_{D3}(w_{i-1}w_i) & \text{if } \left\lfloor \frac{b-a+3}{2} \right\rfloor < i \le b-a+2, \end{cases}$$

$$e_{D3}(u_3x_i) = \begin{cases} c-a+2, & \text{if } 3 \le i \le b-a+2 \text{ for } b-a+3 \ge a-2, \\ a-2, & \text{if } 3 \le i \le b-a+2 \text{ for } b-a+3 \le a-2. \end{cases}$$

It is easy to verify that there is no clique S in G with $e_3(S) > a$, $e_{m_3}(S) > b$ and $e_{D3}(S) > c$. Thus $d_3 = a$, $D_{m_3} = b$ and $D_3 = c$ as $a < b = c$. □

In [2], it is shown that the center of every connected graph G lies in a single block of G, Chartrand et. al. [1] showed that the detour center of every connected graph G lies in a single block of G, and Santhakumaran et. al. [7] showed that the monophonic center of every connected graph G lies in a single block of G. But Keerthi Asir et. al. [4] showed that the clique-to-clique detour center of every connected graph G does not lie in a single block of G. However the similar result is not true for the clique-to-clique monophonic center of a graph.

Remark 3.11 The clique-to-clique monophonic center of every connected graph G does not lie in a single block of G. For the Path P_{2n+1}, the clique-to-clique monophonic center is always P_3, which does not lie in a single block.

We leave the following open problems.

Problem 3.12 *Does there exist a connected graph G such that $e_3(C) \neq e_{m_3}(C) \neq e_{D3}(C)$ for every clique C in G?*

Problem 3.13 *Is every graph a clique-to-clique monophonic center of some connected graph?*

Problem 3.14 *Characterize clique-to-clique monophonic self-centered graphs.*

Acknowledgement

We are grateful to the referee whose valuable suggestions resulted in producing an improved paper.

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