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Cohen-Macaulay of Ideal $I_2(G)$

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Abstract: In this paper, we study the Cohen-Macaulay of ideal $I_2(G)$, where $I_2(G) = \langle xyz \mid x-y-z \text{ is } 2-\text{path in } G\rangle$. Also, we determined the 2-projective dimension R-module, $R/I_2(G)$ denoted by $pd_2(G)$ of some graphs.

Key Words: Cohen-Macaulay, projective dimension, ideal, path.

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§1. Introduction

A simple graph is a pair $G = (V, E)$, where $V = V(G)$ and $E = E(G)$ are the sets of vertices and edges of G, respectively. A walk is an alternating sequence of vertices and connecting edges. A path is a walk that does not include any vertex twice, except that its first vertex might be the same as its last. A path with length n denotes by P_n. In a graph G, the distance between two distinct vertices x and y, denoted by $d(x,y)$, is the length of the shortest path connecting x and y, if such a path exists: otherwise, we set $d(x,y) = \infty$. The diameter of a graph G is $diam(G) = \sup\{d(x,y) : x \text{ and } y \text{ are distinct vertices of } G\}$. Also, a cycle is a path that begins and ends on the same vertex. A cycle with length n denotes by C_n. A graph G is said to be connected if there exists a path between any two distinct vertices, and it is complete if it is connected with diameter one. We use K_n to denote the complete graph with n vertices. For a positive integer r, a complete r-partite graph is one in which each vertex is joined to every vertex that is not in the some subset. The complete bipartite graph with part sizes m and n is denoted by $K_{m,n}$. The graph $K_{1,n-1}$ is called a star graph in which the vertex with degree $n-1$ is called the center of the graph. For any graph G, we denote $N[x] = \{y \in V(G) : (x,y) \ is\ an\ edge\ of\ G\}$. Recall that the projective dimension of an R-module M, denoted by $pd(M)$, is the length of the minimal free resolution of M, that is, $pd(M) = max\{I|\ \beta_{i,j}(M) \neq 0\ for\ some\ j\}$. There is a strong connection between the topology of the simplicial complex and the structure of the free resolution of $K[\Delta]$. Let $\beta_{i,j}(\Delta)$ denotes the N-graded Betti numbers of the Stanley-Reisner ring $K[\Delta]$.

To any finite simple graph G with the vertex set $V(G) = \{x_1, \cdots, x_n\}$ and the edge set $E(G)$, one can attach an ideal in the Polynomial rings $R = K[x_1, \cdots, x_n]$ over the field K, where ideal $l_2(G)$ is called the edge ideal of G such that $l_2(G) =< xyz|\ x-y-z$ is $2-$

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path in $G >$. Also the edge ring of G, denoted by $K(G)$ is defined to be the quotient ring $K(G) = R/I_2(G)$. Edge ideals and edge rings were first introduced by Villarreal [9] and then they have been studied by many authors in order to examine their algebraic properties according to the combinatorial data of graphs. In this paper, we denote S_n for a star graph with $n+1$ vertices. Let $R = K[x_1, \cdots, x_n]$ be a polynomial ring over a field K with the grading induced by $deg(x_i) = d_i$, where d_i is a positive integer. If $M = \bigoplus_{i=0}^{\infty} M_i$ is a finitely generated N-graded module over R, its Hilbert function and Hilbert series are defined by $H(M,i) = l(M_i)$ and $F(M,t) = \sum_{i=0}^{\infty} H(M,i)t^i$ respectively, where $l(M_i)$ denotes the length of M_i as a K-module, in our case $l(M_i) = dim_K(M_i)$.

§2. Cohen-Macaulay of Ideal $I_2(G)$ and $pd_2(G)$ of Some Graph G

Definition 2.1 *Let G be a graph with vertex set V. Then a subset $A \subseteq V$ is a 2-vertex cover for G if for every path xyz of G we have $\{x,y,z\} \cap A \neq \varnothing$. A 2-minimal vertex cover of G is a subset A of V such that A is a 2-vertex cover, and no proper subset of A is a vertex cover for G. The smallest cardinality of a 2-vertex cover of G is called the 2-vertex covering number of G and is denoted by $a_{02}(G)$.*

Example 2.2 Let G be a graph shown in the figure. Then the set $\{x_2, x_4, x_7\}$ is a 2-minimal vertex cover of G and $a_{02}(G) = 3$.

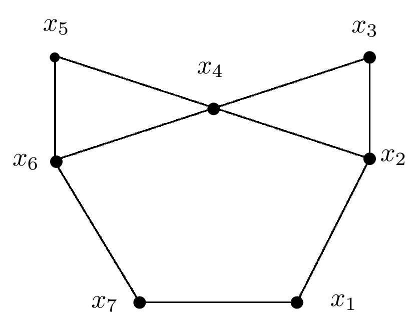

Figure 1

Definition 2.3 *Let G be a graph with vertex set V. A subset $\mathcal{A} \subseteq V$ is a k-independent if for even x of $\mathcal{A}$ we have $deg_{G[S]} \leq k-1$. The maximum possible cardinality of an k-independent set of G, denoted $\beta_{0k}(G)$, is called the k-independence number of G. It is easy see that*

$$\alpha_{02}(G) + \beta_{02}(G) = |V(G)|.$$

Definition 2.4 *Let G be a graph without isolated vertices, Let $\mathcal{S} = K[x_1, \cdots, x_n]$ the polynomial ring on the vertices of G over some fixed field K. The 2-pathes ideal $I_2(G)$ associated to the graph G is the ideal of $\mathcal{S}$ generated by the set of square-free monomials $x_i x_j x_r$ such that $\nu_i \nu_j \nu_r$*

is the path of G, *that is* $I_2(G) =< \{x_i x_j x_r| \ \{\nu_i \nu_j \nu_r\} \in P_2(G)\} >$.

Proposition 2.5 *Let* $\mathcal{S} = K[x_1, \cdots, x_n]$ *be a polynomial ring over a field* K *and* G *a graph with vertices* $\nu_1, \cdots, \nu_n$. *If* P *is an ideal of* R *generated by* $\mathcal{A} = \{x_{i1}, \cdots, x_{ik}\}$ *then* P *is a minimal prime of* $I_2(G)$ *if and only if* $\mathcal{A}$ *is a 2-minimal vertex cover of* G.

Proof It is easy see that $I_2(G) \subseteq P$ if and only if $\mathcal{A}$ is a 2-vertex cover of G. Now, let $\mathcal{A}$ is a 2-minimal vertex cover of G. By Proposition 5.1.3 [9] any minimal prime ideal of $I_2(G)$ is a face ideal thus P is a minimal prime of $I_2(G)$. The converse is clear. □

Corollary 2.6 *If* G *is a graph and* $I_2(G)$ *its 2-path ideal, then*

$$ht\,(I_2(G)) = \alpha_{02}(G).$$

Proof It follows from Proposition 5 and the definition of $\alpha_{02}(G)$. □

Definition 2.7 *A graph* G *is 2-unmixed if all of its 2-minimal vertex covers have the same cardinality.*

Definition 2.8 *A graph* G *with vertex set* $V(G) = \{\nu_1, \nu_2, \cdots, \nu_n\}$ *is 2-cohen-Macaullay over field* K *if the quotient ring* $K[x_1, \cdots, x_n]/I_2(G)$ *is cohen-Macaulay.*

Proposition 2.9 *If* G *is a 2-cohen-Macaulay graph, then* G *is 2-unmixed.*

Proof By Corollary 1.3.6 [9], $I_2(G) = \bigcap_{P \in \min(I_2(G))} P$. Since $R/I_2(G)$ is cohen-Macaullay, all minimal prime ideals of $I_2(G)$ have the same height. Then, by Proposition 5, all 2-minimal vertex cover of G have the same cardinality, as desired. □

Proposition 2.10 *If* G *is a graph and* $G_1, \cdots, G_s$ *are its connected components, then* G *is 2-cohen- Macaulay if and only if for all* i, G_i *is cohen-Macaulay.*

Proof Let $R = K[V(G)]$ and $R_i = L[V(G_i)]$ for all i. Since

$$R/I_2(G) \cong R_1/I_2(G_1) \otimes_K \cdots \otimes_K R_s/I_2(G_s).$$

Hence the results follow from Corollary 2.2.22 [9]. □

Definition 2.11 *For any graph* G *one associates the complementary simplicial complex* $\triangle_2(G)$, *which is defined as*

$$\triangle_2(G) := \{\mathcal{A} \subset V|\ \mathcal{A} \ is\ 2-independent\ set\ in\ G\}.$$

This means that the facets of $\Delta_2(G)$ are precisely the maximal 2-independent sets in G, that is the complements in V of the minimal 2-vertex covers. Thus $\Delta_2(G)$ precisely the Stanley-Reisner complex of $I_2(G)$.

It is easy see that $\omega(\Delta_2(G)) = \{\{x,y,z\} \mid xyz \in P_3(G)\}$. Therefore $I_2(G) = I_{\Delta_2(G)}$, and so G is $2-C-M$ graph if and only if the simplicial complex $\Delta_2(G)$ is cohen-Macaulay.

Now, we can show the following propositiori.

Proposition 2.12 *The following statements hold:*

(a) *For any $n \geq 1$ the complete graph K_n is cohen-Macaulay;*
(b) *The complete bipartite graph $K_{m,n}$ is cohen-Macaulay if and only if $m+n \leq 4$.*

Proof (a) Since $\Delta_2(K_n) =< \{x,y\} \mid x,y \in V(K_n) >$, thus $\Delta_2(K_n)$ is connected l-dimensional simplicial complex, then by Corohary 5.3.7 [9], $\Delta_2(K_n)$ is cohen-Macaula so K_n is cohen-Macaulay.

(b) If $m+n \leq 4$, then $K_{m,n} \cong P_2, P_3, C_4$. It is easy to see that $\Delta_2(K_{m,n})$ is c. So $K_{m,n}$ is cohen-Macaulay.

Conversely, let $K_{m,n}$ is cohen-Macaulay and $m+n \geq 5$. Take $V_1 = \{x_1, \cdots, x_n\}$ and $V_2 = \{y_1, \cdots, y_m\}$ are the partite sets of $K_{m,n}$. One has

$$\Delta_2(K_{m,n}) =< \{x_1, \cdots, x_n\}, \{y_1, \cdots, y_m\}, \{x_i, y_j\} \mid 1 \leq i \leq n, 1 \leq j \leq m >$$

. Since $m+n \geq 5$, $\Delta_2(K_{m,n})$ is not pure simplicial complex. Then, by 5.3.12 [9] $\Delta_2(K_{m,n})$ is not cohen-Macaulay, a contradiction, as desired. □

Now, we present a result about the Hilbert series of $K[\Delta_2(K_n)]$ and $K[\Delta_2(K_{m,n})]$.

Proposition 2.13 *If $\Delta_2(K_n)$ and $\Delta_2(K_{m,n})$ are the complementary simplicial complexes K_n and $K_{m,n}$ respectively, then*

(a) $F(K[\Delta_2(K_n)], z) = 1 + nz/(1-z) + n(n-1)/2(1-z)^2$;
(b) $F(K[\Delta_2(K_{n,m})], z) = 1/(1-z)^n + 1/(1-z)^m + m.nz^2/(1-z)^2 - 1$.

Proof (a) Since $\Delta_2(K_n) =< \{x,y\} \mid x,y \in V(K_n) >$ hence dime $\Delta_2(K_n) = 1$ and $f_{-1}(K_n) = 1$, $f_0,(K_n) = n$ and $f_1(K_n) = \binom{n}{2} = n(n-1)/2$. By Corollary 5.4.5 [9]. We have

$$F(K[\Delta_2(K_n)], z) = 1 + nz/1 - z + n(n-1)/2.z^2/2(1-z)^2.$$

(b) Let $\{x_1, \cdots, x_n\}$ and $\{y_1, \cdots, y_m\}$ are the parties sets of $K_{m,n}$. Since

$$\Delta_2(K_{m,n}) =< \{x_1, \cdots, x_n\}, \{y_1, \cdots, y_m\}, \{x_i, y_j\} \mid 1 \leq i \leq n, 1 \leq j \leq m >$$

Then it is easy see that $f_1(\Delta_2(K_{m,n})) = f_1(\Delta(K_{m,n})) + mn$ and $f_i(\Delta_2(K_{m,n})) = f_i(\Delta(K_{m,n}))$ for all $i \neq 1$. ln the other hand, by 6.6.6[9], $F(K[\Delta_2(K_n)], z) = 1/(1-z)^n - 1.$, Thus

$$F(K[\Delta_2(K_n)], z) = 1/(1-z)^n + 1/(1-z)^m + m.nz^2/(1-z)^2 - 1.$$

This completes the proof. □

Corollary 2.14 $F(K[\Delta_2(S_n)], z) = 1/(1-z)^n + nz^2/(1-z)^2 + z/(1-z)$.

Proof It follows from Proposition 2.13 with assume $m = 1$. □

In this section we mainly present basic properties of 2-shellable graphs.

Lemma 2.15 *Let G be a graph and x be a vertex of degree 1 in G and let $y \in N(x)$ and $G' = G - (\{y\} \cup N(y))$. Then $\Delta_2(G') = lK_{\Delta_2(G)}(\{x,y\})$. Moreover F is a facet of $\Delta_2(G')$ if and only if $F \cup \{x,y\}$ is a facet of $\Delta_2(G)$.*

Proof (a) Let $F \in lK_{\Delta_2(G)}(\{x,y\})$. Then $F \in \Delta_2(G)$, $x,y \notin F$ and $F \cup \{x,y\} \in \Delta_2(G)$. This implies that $(F \cup \{x,y\}) \cap N[y] = \varnothing$ and $F \subseteq (V - \{x,y\}) \cup N[y] = (V - y) \cup N[y] = V(G')$. Thus F is 2-independent in G', it follows that $F \in \Delta_2(G')$. Conversely let $F \in \Delta_2(G')$, then F is 2-independent in G' and $F \cap (x \cup [y]) = \varnothing$. Therefore $F \cup \{x,y\}$ is 2-independent in G and so $F \cup \{x,y\} \in \Delta_2(G)$, $F \cup \{x,y\} = \varnothing$. Thus $F \in lK_{\Delta_2(G)}(\{x,y\})$. Finaly from part one follows that F is a facet of $\Delta_2(G')$ if and only if $F \cup \{x,y\}$ is a facet of $\Delta_2(G)$. □

Definition 2.16 *Fix a field K and set $R = K[x_1, \cdots, x_n]$. If G is a graph with vertex set $V(G) = \{x_1, x_2, \cdots, x_n\}$, we define the projective dimension of G to be the 2-projective dimension R- module $R/I_2(G)$, and we will write $pd_2(G) = pd\,(R/l_2(G))$.*

Proposition 2.17 *If G is a graph and $\{x,y\}$ is a edge of G, then*

$$\begin{aligned} P_2(G) \quad \leq \quad & \max\{P_2\,(G - (N[x] \cup N[y])) + deg(x) + deg(y) \\ & - |N[x] \cap N[y]|, P_2(G - x) + 1, P_2(G - y) + 1\}. \end{aligned}$$

Proof Let $N[x] = \{x_1, \cdots, x_\xi\}$ and $N[y] = \{y_1, \cdots, y_r\}$. It is easy to see that

$$I_2(G) : xy = (I_2(G) - (N[x] \cup N[y]), x_1, \cdots, x_\xi, y_1, \cdots, y_r).$$

Now, let

$$R' = K\left[V\left(G - \left(N[x] \bigcup N[y]\right)\right)\right].$$

Then

$$depth(R/I_2(G) : xy) = depth(R'/I_2(G - (N[x] \cup N[y]).$$

And so by Auslander-Buchsbaum formula, we have

$$\begin{aligned} pd_2(R/I_2(x) : xy) &= pd_2(G - (N[x] \cup N[y]) + deg(x) + deg(y) - |N[x] \cap N[y]|, \\ pd_2(R/I_2(x), x) &= pd_2(G - x) + 1, \\ pd_2(R/I_2(x), y) &= pd_2(G - y) + 1. \end{aligned}$$

On the other hand by Proposition 2.10, together with the exact sequence

$$0 \longrightarrow R/I_2(G) : xy \longrightarrow R/I_2(G) \longrightarrow R/I_2(G)xy \longrightarrow 0,$$

it follows that

$$\begin{aligned} P_2(G) \quad \leq \quad & \max\{P_2(G-(N[x]\cup N[y]))+deg(x)+deg(y) \\ & -|N[x]\cap N[y]|, P_2(G-x)+1, P_2(G-y)+1\}. \end{aligned}$$ □

Proposition 2.18 *Let G be a graph and $I_2(G)$ is path ideal of G. Then*

$$Bight\,(I_2(G)) \leq pd_2(G).$$

Proof Let P be a minimal vertex cover with maximal cardinality. Then by Proposition 2.5, P is an associated prime of $R/I_2(G)$, so

$$pd_2(G) = pd\,(R/I_2(G)) \geq pd_{R_p}\,(R_p/I_2(G)R_p) = dimR_p = htP.$$ □

Proposition 2.19 *Let K_n denote the complete graph on n vertices and let $K_{m,n}$ denote the complete bipartite graph on $m+n$ vertices.*

(*a*) $pd_2(K_n) = n-2$;

(*b*) $pd_2(K_{m,n}) = m+n-2$.

Proof (*a*) The proof is by induction on n. If $n = 2$ or 3, then the result easy follows. Let $n \geq 4$ and suppose that for every complete graphs K_n of other less than n the result is true. Since $Bight\,(I_2\,(K_n)) = n-2$ then by Proposition $pd_2(K_n) \geq n-2$. On the other hand by the inductive hypothesis, we have $pd_2(K_{n-1}) = n-3$. So by Proposition 2.17,

$$pd_2(K_n) \leq \max\{n-2, n-2\}.$$

(*b*) Again we use by induction on $m+n$. If $m+n = 2$ or 3, then it is easy to see that $pd_2(K_{m,n}) = 0$ or 1. Let $m+n \geq 4$ and suppose that for every complete bipartite graph $K_{m,n}$ of order less than $m+n$ the result is true. Since $Bight\,(I_2\,(K_{m,n})) = m+n-2$ then $pd_2(K_{m,n}) \geq m+n-2$. Also, by the inductive hypothesis we have $pd_2(K_{m-1,n}) = m+n-3$ and $pd_2(K_{m,n-1}) = m+n-3$. So by Proposition2.17,

$$pd_2(K_{m,n}) \leq \max\{m+n-2, pd_2(K_{m-1,n})+1, pd_2(K_{m,n-1})+1 = m+n-2\}.$$

This completes the proof. □

Corollary 2.20 *Let S_n denote the star graph on $n+1$ vertices and $S_{m,n}$ denote the double star, then $pd_2(S_{m,n}) = m+n$.*

Proof It follows from Proposition 2.19 with assume $m = 1$ and it is easy to see that $BightI_2\,(S_{m,n}) = m+n$, and so by Proposition 2.17, it follows that

$$pd_2\,(S_{m,n}) = m+n.$$ □

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