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Cordiality in the Context of Duplication in Web and Armed Helm

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Abstract: Let $G = (V(G), E(G))$ be a graph and let $f : V(G) \to \{0, 1\}$ be a mapping from the set of vertices to {0,1} and for each edge $uv \in E$ assign the label $|f(u) - f(v)|$. If the number of vertices labeled with 0 and the number of vertices labeled with 1 differ by at most 1 and the number of edges labled with 0 and the number of edges labeled with 1 differ by at most 1, then f is called a cordial labeling. We discuss cordial labeling of graphs obtained from duplication of certain graph elements in web and armed helm.

Key Words: Graph labeling, cordial labeling, cordial graph, Smarandachely cordial labeling, Smarandachely cordial graph.

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§1. Introduction

We begin with simple, finite, undirected graph $G = (V(G), E(G))$ where $V(G)$ and $E(G)$ denotes the vertex set and the edge set respectively. For all other terminology we follow West [1]. We will give the brief summary of definitions which are useful for the present work.

Definition 1.1 *The graph labeling is an assignment of numbers to the vertices or edges or both subject to certain condition(s).*

A detailed survey of various graph labeling is explained in Gallian [3].

Definition 1.2 *For a graph $G = (V(G), E(G))$, a mapping $f : V(G) \to \{0, 1\}$ is called a binary vertex labeling of G and $f(v)$ is called the label of the vertex v of G under f. For an edge $e = uv$, the induced edge labeling $f^* : E(G) \to \{0, 1\}$ defined as $f^*(uv) = |f(u) - f(v)|$.*

Let $v_f(0), v_f(1)$ be the number of vertices of G having labels 0 and 1 respectively under f and let $e_f(0), e_f(1)$ be the number of edges having labels 0 and 1 respectively under f^*.

Definition 1.3 *Duplication of a vertex v of a graph G produces a new graph G' by adding a new vertex v' such that $N(v') = N(v)$. In other words a vertex v' is said to be duplication of v if all the vertices which are adjacent to v in G are also adjacent to v' in G'.*

Definition 1.4 *Duplication of an edge $e = uv$ of a graph G produces a new graph G' by adding*

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an edge $e' = u'v'$ *such that* $N(u') = N(u) \cup \{v'\} - \{v\}$ *and* $N(v') = N(v) \cup \{u'\} - \{u\}$.

Definition 1.5 *The wheel* W_n*, is join of the graphs* C_n *and* K_1*. i.e* $W_n = C_n + K_1$*. Here vertices corresponding to* C_n *are called rim vertices and* C_n *is called rim of* W_n *while, the vertex corresponding to* K_1 *is called the apex vertex, edges joining the apex vertex and a rim vertex is called spoke.*

Definition 1.6([3]) *The helm* H_n*, is the graph obtained from the wheel* W_n *by adding a pendant edge at each rim vertex.*

Definition 1.7([3]) *The web* Wb_n*, is the graph obtained by joining the pendent points of a helm to form a cycle and then adding a single pendent edge to each vertex of this outer cycle, here vertices corresponding to this outer cycle are called outer rim vertices and vertices corresponding to wheel except the apex vertex are called inner rim vertices.*

We define one new graph family as follows:

Definition 1.8 *An armed helm is a graph in which path* P_2 *is attached at each vertex of wheel* W_n *by an edge. It is denoted by* AH_n *where* n *is the number of vertices in cycle* C_n.

Definition 1.9 *A binary vertex labeling* f *of a graph* G *is called a cordial labeling if* $|v_f(1) - v_f(0)| \leq 1$ *and* $|e_f(1) - e_f(0)| \leq 1$*, and a binary vertex labeling* f *of a graph* G *is called a Smarandachely cordial labeling if* $|v_f(1) - v_f(0)| \geq 1$ *or* $|e_f(1) - e_f(0)| \geq 1$.

A graph G *is said to be cordial if it admits cordial labeling, and Smarandachely cordial if it admits Smarandachely cordial labeling.*

The concept of cordial labeling was introduced by Cahit [2] in which he proved that the wheel W_n is cordial if and only if $n \not\equiv 3(mod 4)$. Vaidya and Dani [4] proved that the graphs obtained by duplication of an arbitrary edge of a cycle and a wheel admit a cordial labeling. Prajapati and Gajjar [5] proved that complement of wheel graph and complement of cycle graph are cordial if $n \not\equiv 4(mod 8)$ or $n \not\equiv 7(mod 8)$. Prajapati and Gajjar [6] proved that cordial labeling in the context of duplication of cycle graph and path graph.

§2. Main Results

Theorem 2.1 *The graph obtained by duplicating all the vertices of the web* Wb_n *is cordial.*

Proof Let $V(Wb_n) = \{t\} \bigcup \{u_i, v_i, w_i, /1 \leq i \leq n\}$ and $E(Wb_n) = \{tu_i, u_iv_i, v_iw_i/1 \leq i \leq n\} \bigcup \{u_nu_1, v_nv_1\} \bigcup \{u_iu_{i+1}, v_iv_{i+1}/1 \leq i \leq n-1\}$. Let G be the graph obtained by duplicating all the vertices in Wb_n. Let $t', u'_1, u'_2, \cdots, u'_n, v'_1, v'_2, \cdots, v'_n, w'_1, w'_2, \cdots, w'_n$ be the new vertices of G by duplicating $t, u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n, w_1, w_2, \cdots, w_n$ respectively. Then $V(G) = \{t, t'\} \cup \{u_i, v_i, w_i, u'_i, v'_i, w'_i/1 \leq i \leq n\}$ and $E(G) = \{tu_i, u_iv_i, v_iw_i, w_iv'_i, u_iv'_i, u'_iv_i, tu'_i, v_iw'_i, t'u_i$ $/1 \leq i \leq n\} \bigcup \{u_nu_1, v_nv_1, v'_nv_1, v_nv'_1, u'_nu_1, u_nu'_1\} \bigcup \{u_iu_{i+1}, v_iv_{i+1}, v'_iv_{i+1}, v_iv'_{i+1}, u'_iu_{i+1}, u_iu'_{i+1}$ $/1 \leq i \leq n-1\}$. Therefore $|V(G)| = 6n + 2$ and $|E(G)| = 15$. Using parity of n, we have the following cases:

Case 1. n is even.

Define a vertex labeling $f : V(G) \to \{0,1\}$ as follows:

$$f(x) = \begin{cases} 1 & \text{if } x = t'; \\ 1 & \text{if } x = w'_i, i \in \{2,4,\ldots,n-2,n\}; \\ 1 & \text{if } x = w_i,\ i \in \{1,3,\ldots,n-3,n-1\}; \\ 1 & \text{if } x \in \{u_i, v'_i\},\ i \in \{1,2,\ldots,n-1,n\}; \\ 0 & \text{if } x = t; \\ 0 & \text{if } x \in \{v_i, u'_i\},\ i \in \{1,2,\ldots,n-1,n\}; \\ 0 & \text{if } x = w_i,\ i \in \{2,4,\ldots,n-2,n\}; \\ 0 & \text{if } x = w'_i,\ i \in \{1,3,\ldots,n-3,n-1\}. \end{cases}$$

Thus $v_f(1) = 3n+1$ and $v_f(0) = 3n+1$. The induced edge labeling $f^* : E(G) \to \{0,1\}$ is $f^*(uv) = |f(u) - f(v)|$, for every edge $e = uv \in E$. Therefore

$$f^*(e) = \begin{cases} 1 & \text{if } \ e \in \{tu_i, u_iv_i\},\ i \in \{1,2,\ldots,n-1,n\}; \\ 0 & \text{if } \ e \in \{u_iu_{i+1}, v_iv_{i+1}\},\ i \in \{1,2,\ldots,n-2,n-1\}; \\ 1 & \text{if } \ e \in \{u_iu'_{i+1}, u'_iu_{i+1}, v_iv'_{i+1}, v'_iv_{i+1}\},\ i \in \{1,2,\ldots,n-2,n-1\}; \\ 0 & \text{if } \ e \in \{t'u_i, tu'_i, u'_iv_i, u_iv'_i\},\ i \in \{1,2,\ldots,n-1,n\}; \\ 0 & \text{if } \ e \in \{v_iw'_i, w_iv'_i\},\ i \in \{1,3,\ldots,n-3,n-1\}; \\ 1 & \text{if } \ e \in \{v_iw'_i, w_iv'_i\},\ i \in \{2,4,\ldots,n-2,n\}; \\ 1 & \text{if } \ e = v_iw_i,\ i \in \{1,3,\ldots,n-3,n-1\}; \\ 0 & \text{if } \ e = v_iw_i,\ i \in \{2,4,\ldots,n-2,n\}; \\ 0 & \text{if } \ e \in \{u_nu_1, v_nv_1\}; \\ 1 & \text{if } \ e \in \{u'_nu_1, u_nu'_1, v'_nv_1, v_nv'_1\}. \end{cases}$$

Thus $e_f(1) = \dfrac{15n}{2}$ and $e_f(0) = \dfrac{15n}{2}$.

Case 2 n is odd.

Define a vertex labeling $f : V(G) \to \{0,1\}$ as follows:

$$f(x) = \begin{cases} 1 & \text{if } x = t'; \\ 1 & \text{if } x = w'_i, i \in \{2,4,\ldots,n-3,n-1\}; \\ 1 & \text{if } x = w_i, i \in \{1,3,\ldots,n-2,n\}; \\ 1 & \text{if } x \in \{u_i, v'_i\}, i \in \{1,2,\ldots,n-1,n\}; \\ 0 & \text{if } x = t; \\ 0 & \text{if } x \in \{v_i, u'_i\},\ i \in \{1,2,\ldots,n-1,n\}; \\ 0 & \text{if } x = w_i,\ i \in \{2,4,\ldots,n-3,n-1\}; \\ 0 & \text{if } x = w'_i,\ i \in \{1,3,\ldots,n-2,n\}. \end{cases}$$

Thus $v_f(1) = 3n+1$ and $v_f(0) = 3n+1$. The induced edge labeling $f^* : E(G) \to \{0,1\}$ is $f^*(uv) = |f(u) - f(v)|$, for every edge $e = uv \in E$. Therefore

$$f^*(e) = \begin{cases} 1 & \text{if } \ e \in \{tu_i, u_iv_i\}, \ i \in \{1,2,\ldots,n-1,n\}; \\ 0 & \text{if } \ e \in \{u_iu_{i+1}, v_iv_{i+1}\}, \ i \in \{1,2,\ldots,n-2,n-1\}; \\ 1 & \text{if } \ e \in \{u_iu'_{i+1}, u'_iu_{i+1}, v_iv'_{i+1}, v'_iv_{i+1}\}, \ i \in \{1,2,\ldots,n-2,n-1\}; \\ 0 & \text{if } \ e \in \{t'u_i, tu'_i, u'_iv_i, u_iv'_i\}, \ i \in \{1,2,\ldots,n-1,n\}; \\ 0 & \text{if } \ e \in \{v_iw'_i, w_iv'_i\}, \ i \in \{1,3,\ldots,n-2,n\}; \\ 1 & \text{if } \ e \in \{v_iw'_i, w_iv'_i\}, \ i \in \{2,4,\ldots,n-3,n-1\}; \\ 1 & \text{if } \ e = v_iw_i, \ i \in \{1,3,\ldots,n-2,n\}; \\ 0 & \text{if } \ e = v_iw_i, \ i \in \{2,4,\ldots,n-3,n-1\}; \\ 0 & \text{if } \ e \in \{u_nu_1, v_nv_1\}; \\ 1 & \text{if } \ e \in \{u'_nu_1, u_nu'_1, v'_nv_1, v_nv'_1\}. \end{cases}$$

Thus $e_f(1) = \dfrac{15n-1}{2}$ and $e_f(0) = \dfrac{15n+1}{2}$.

From both the cases we can conclude $|v_f(1) - v_f(0)| \leq 1$ and $|e_f(1) - e_f(0)| \leq 1$. So, f admits cordial labeling on G. Hence G is cordial. □

Theorem 2.2 *The graph obtained by duplicating all the pendent vertices of the web Wb_n is cordial.*

Proof Let $V(Wb_n) = \{t\} \bigcup \{u_i, v_i, w_i/1 \leq i \leq n\}$ and $E(Wb_n) = \{tu_i, u_iv_i, v_iw_i/1 \leq i \leq n\} \cup \{u_nu_1, v_nv_1\} \bigcup \{u_iu_{i+1}, v_iv_{i+1}/1 \leq i \leq n-1\}$. Let G be the graph obtained by duplicating all the pendent vertices in Wb_n. Let $w'_1, w'_2, \cdots, w'_n$ be the new vertices of G by duplicating $w_1, w_2, \cdots, w_n$ respectively. Then $V(G) = \{t\} \cup \{u_i, v_i, w_i, w'_i/1 \leq i \leq n\}$ and $E(G) = \{tu_i, u_iv_i, v_iw_i, v_iw'_i/1 \leq i \leq n\} \cup \{u_nu_1, v_nv_1\} \bigcup \{u_iu_{i+1}, v_iv_{i+1}/1 \leq i \leq n-1\}$. Therefore $|V(G)| = 4n+1$ and $|E(G)| = 6n$. Define a vertex labeling $f : V(G) \to \{0,1\}$ as follows:

$$f(x) = \begin{cases} 0 & \text{if } x = t; \\ 1 & \text{if } x \in \{w_i, u_i\}, \ i \in \{1,2,\ldots,n-1,n\}; \\ 0 & \text{if } x \in \{v_i, w'_i\}, \ i \in \{1,2,\ldots,n-1,n\}. \end{cases}$$

Thus $v_f(1) = 2n$ and $v_f(0) = 2n+1$. The induced edge labeling $f^* : E(G) \to \{0,1\}$ is $f^*(uv) = |f(u) - f(v)|$, for every edge $e = uv \in E$. Therefore

$$f^*(e) = \begin{cases} 1 & \text{if } \ e \in \{tu_i, u_iv_i, v_iw_i\}, \ i \in \{1,2,\ldots,n-1,n\}; \\ 0 & \text{if } \ e = v_iw'_i, \ i \in \{1,2,\ldots,n-1,n\}; \\ 0 & \text{if } \ e \in \{u_iu_{i+1}, v_iv_{i+1}\}, \ i \in \{1,2,\ldots,n-2,n-1\}; \\ 0 & \text{if } \ e \in \{u_nu_1, v_nv_1\}. \end{cases}$$

Thus $e_f(1) = 3n$ and $e_f(0) = 3n$. Therefore f satisfies the conditions $|v_f(1) - v_f(0)| \leq 1$ and

$|e_f(1) - e_f(0)| \leq 1$. So, f admits cordial labeling on G. Hence G is cordial. □

Theorem 2.3 *The graph obtained by duplicating the outer rim vertices and the apex of the web Wb_n is cordial.*

Proof Let $V(Wb_n) = \{t\} \bigcup \{u_i, v_i, w_i, /1 \leq i \leq n\}$ and $E(Wb_n) = \{tu_i, u_iv_i, v_iw_i/1 \leq i \leq n\} \bigcup \{u_iu_{i+1}, v_iv_{i+1}/1 \leq i \leq n-1\} \bigcup \{u_nu_1, v_nv_1\}$. Let G be the graph obtained by duplicating the outer rim vertices and the apex in Wb_n. Let $t', v_1', v_2', \cdots, v_n'$ be the new vertices of G by duplicating $t, v_1, v_2, \cdots, v_n$ respectively. Then $V(G) = \{t, t'\} \bigcup \{u_i, v_i, w_i, v_i'/1 \leq i \leq n\}$ and $E(G) = \{tu_i, u_iv_i, v_iw_i, w_iv_i', u_iv_i', t'u_i/1 \leq i \leq n\} \bigcup \{u_iu_{i+1}, v_iv_{i+1}, v_i'v_{i+1}, v_iv_{i+1}'/1 \leq i \leq n-1\} \bigcup \{u_nu_1, v_nv_1, v_n'v_1, v_nv_1'\}$. Therefore $|V(G)| = 4n+2$ and $|E(G)| = 10n$. Define a vertex labeling $f : V(G) \to \{0, 1\}$ as follows:

$$f(x) = \begin{cases} 0 & \text{if } x = t; \\ 1 & \text{if } x \in \{u_i, w_i\},\ i \in \{1, 2, \ldots, n-1, n\}; \\ 0 & \text{if } x \in \{v_i, v_i'\},\ i \in \{1, 2, \ldots, n-1, n\}; \\ 1 & \text{if } x = t'. \end{cases}$$

Thus $v_f(1) = 2n+1$ and $v_f(0) = 2n+1$. The induced edge labeling $f^* : E(G) \to \{0, 1\}$ is $f^*(uv) = |f(u) - f(v)|$, for every edge $e = uv \in E$. Therefore

$$f^*(e) = \begin{cases} 1 & \text{if } \ e \in \{tu_i, u_iv_i, v_iw_i, w_iv_i', u_iv_i'\},\ i \in \{1, 2, \ldots, n-1, n\}; \\ 0 & \text{if } \ e \in \{u_iu_{i+1}, v_iv_{i+1}, v_i'v_{i+1}, v_iv_{i+1}'\},\ i \in \{1, 2, \ldots, n-2, n-1\}; \\ 0 & \text{if } \ e = t'u_i,\ i \in \{1, 2, \ldots, n-1, n\}; \\ 0 & \text{if } \ e \in \{u_nu_1, v_nv_1.v_n'v_1, v_nv_1'\}. \end{cases}$$

Thus $e_f(1) = 5n$ and $e_f(0) = 5n$. Therefore f satisfies the conditions $|v_f(1) - v_f(0)| \leq 1$ and $|e_f(1) - e_f(0)| \leq 1$. So, f admits cordial labeling on G. Hence G is cordial. □

Theorem 2.4 *The graph obtained by duplicating all the vertices except the apex vertex of the web Wb_n is cordial.*

Proof Let $V(Wb_n) = \{t\} \bigcup \{u_i, v_i, w_i/1 \leq i \leq n\}$ and $E(Wb_n) = \{tu_i, u_iv_i, v_iw_i/1 \leq i \leq n\} \bigcup \{u_nu_1, v_nv_1\} \bigcup \{u_iu_{i+1}, v_iv_{i+1}/1 \leq i \leq n-1\}$. Let G be the graph obtained by duplicating all the vertices except the apex vertex in Wb_n. Let $u_1', u_2', \cdots, u_n', v_1', v_2', \cdots, v_n', w_1', w_2', \cdots, w_n'$ be the new vertices of G by duplicating $u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n, w_1, w_2, \cdots, w_n$ respectively. Then $V(G) = \{t\} \bigcup \{u_i, v_i, w_i, u_i', v_i', w_i'/1 \leq i \leq n\}$ and $E(G) = \{tu_i, u_iv_i, v_iw_i, w_iv_i', u_iv_i', u_i'v_i, v_iw_i', tu_i'/1 \leq i \leq n\} \bigcup \{u_nu_1, v_nv_1, v_n'v_1, v_nv_1', u_n'u_1, u_nu_1'\} \bigcup \{u_iu_{i+1}, v_iv_{i+1}, v_i'v_{i+1}, v_iv_{i+1}', u_i'u_{i+1}, u_iu_{i+1}'/1 \leq i \leq n-1\}$. Therefore $|V(G)| = 6n+1$ and $|E(G)| = 14n$. Define a vertex labeling $f : V(G) \to \{0,1\}$ as follows:

$$f(x) = \begin{cases} 0 & \text{if } x = t; \\ 1 & \text{if } x \in \{w_i, u_i, u_i'\},\ i \in \{1,2,\ldots,n-1,n\}; \\ 0 & \text{if } x \in \{v_i, w_i', v_i'\},\ i \in \{1,2,\ldots,n-1,n\}. \end{cases}$$

Thus $v_f(1) = 3n$ and $v_f(0) = 3n+1$. The induced edge labeling $f^* : E(G) \to \{0,1\}$ is $f^*(uv) = |f(u) - f(v)|$, for every edge $e = uv \in E$. Therefore

$$f^*(e) = \begin{cases} 1 & \text{if } \ e \in \{tu_i, u_iv_i, v_iw_i, w_iv_i', u_iv_i', u_i'v_i, tu_i'\},\ i \in \{1,2,\ldots,n-1,n\}; \\ 0 & \text{if } \ e = v_iw_i',\ i \in \{1,2,\ldots,n-1,n\}; \\ 0 & \text{if } \ e \in \{u_iu_{i+1}, v_iv_{i+1}, v_i'v_{i+1}, v_iv_{i+1}', u_i'u_{i+1}, u_iu_{i+1}'\},\ i \in \{1,2,\ldots,n-2,n-1\}; \\ 0 & \text{if } \ e \in \{u_nu_1, v_nv_1.v_n'v_1, v_nv_1', u_n'u_1, u_nu_1'\}. \end{cases}$$

Thus $e_f(1) = 7n$ and $e_f(0) = 7n$. Therefore f satisfies the conditions $|v_f(1) - v_f(0)| \le 1$ and $|e_f(1) - e_f(0)| \le 1$. So, f admits cordial labeling on G. Hence G is cordial. □

Theorem 2.5 *The graph obtained by duplicating all the inner rim vertices and the apex vertex of the web Wb_n is cordial.*

Proof Let $V(Wb_n) = \{t\} \bigcup \{u_i, v_i, w_i, /1 \le i \le n\}$ and $E(Wb_n) = \{tu_i, u_iv_i, v_iw_i/1 \le i \le n\} \cup \{u_nu_1, v_nv_1\} \bigcup \{u_iu_{i+1}, v_iv_{i+1}/1 \le i \le n-1\}$. Let G be the graph obtained by duplicating all the inner rim vertices and the apex vertex in Wb_n. Let $t', u_1', u_2', \cdots, u_n'$ be the new vertices of G by duplicating $t, u_1, u_2, \cdots, u_n$ respectively. Then $V(G) = \{t, t'\} \cup \{u_i, v_i, w_i, u_i'/1 \le i \le n\}$ and $E(G) = \{tu_i, u_iv_i, v_iw_i, u_i'v_i, tu_i', t'u_i/1 \le i \le n\} \cup \{u_iu_{i+1}, v_iv_{i+1}, u_i'u_{i+1}, u_iu_{i+1}'/1 \le i \le n-1\} \cup \{u_nu_1, v_nv_1, u_n'u_1, u_nu_1'\}$. Therefore $|V(G)| = 4n+2$ and $|E(G)| = 10n$. Define a vertex labeling $f : V(G) \to \{0,1\}$ as follows:

$$f(x) = \begin{cases} 0 & \text{if } x = t; \\ 1 & \text{if } x \in \{w_i, u_i\},\ i \in \{1,2,\ldots,n-1,n\}; \\ 0 & \text{if } x \in \{v_i, u_i'\},\ i \in \{1,2,\ldots,n-1,n\}; \\ 1 & \text{if } x = t'. \end{cases}$$

Thus $v_f(1) = 2n+1$ and $v_f(0) = 2n+1$. The induced edge labeling $f^* : E(G) \to \{0,1\}$ is $f^*(uv) = |f(u) - f(v)|$, for every edge $e = uv \in E$. Therefore

$$f^*(e) = \begin{cases} 1 & \text{if } \ e \in \{tu_i, u_iv_i, v_iw_i\},\ i \in \{1,2,\ldots,n-1,n\}; \\ 1 & \text{if } \ e \in \{u_i'u_{i+1}, u_iu_{i+1}'\},\ i \in \{1,2,\ldots,n-2,n-1\}; \\ 0 & \text{if } \ e \in \{u_iu_{i+1}, v_iv_{i+1}\},\ i \in \{1,2,\ldots,n-2,n-1\}; \\ 0 & \text{if } \ e \in \{tu_i', t'u_i, u_i'v_i\},\ i \in \{1,2,\ldots,n-1,n\} \end{cases}$$

and

$$f^*(e) = \begin{cases} 0 & \text{if } \ e \in \{u_n u_1, v_n v_1\}; \\ 1 & \text{if } \ e \in \{u'_n u_1, u_n u'_1\}. \end{cases}$$

Thus $e_f(1) = 5n$ and $e_f(0) = 5n$. Therefore f satisfies the conditions $|v_f(1) - v_f(0)| \leq 1$ and $|e_f(1) - e_f(0)| \leq 1$. So, f admits cordial labeling on G. Hence G is cordial. □

Theorem 2.6 *The graph obtained by duplicating all the edges other than spoke edges of the web Wb_n is cordial.*

Proof Let $V(Wb_n) = \{t\} \bigcup \{u_i, v_i, w_i/1 \leq i \leq n\}$ and $E(Wb_n) = \{j_i = tu_i, l_i = u_i v_i, o_i = v_i w_i/1 \leq i \leq n\} \bigcup \{k_i = u_i u_{i+1}, m_i = v_i v_{i+1}/1 \leq i \leq n-1\} \bigcup \{k_n = u_n u_1, m_n = v_n v_1\}$. Let G be the graph obtained by duplicating all the edges other than spoke edges in Wb_n. For each $i \in 1, 2, \cdots, n$, let $k'_i = a_i b_i, l'_i = c_i d_i, m'_i = e_i f_i$ and $o'_i = g_i h_i$ be the new edges of G by duplicating k_i, l_i, m_i and o_i respectively. Then $V(G) = \{t\} \cup \{u_i, v_i, w_i, a_i, b_i, c_i, d_i, e_i, f_i, g_i, h_i/1 \leq i \leq n\}$ and $E(G) = \{b_i u_{i+2}, v_i f_{i+2}/1 \leq i \leq n-2\} \bigcup \{tu_i, u_i v_i, a_i b_i, e_i f_i, c_i d_i, g_i h_i, tc_i, tb_i, g_i u_i, ta_i, a_i v_i, e_i u_i, d_i w_i, v_i w_i, e_i w_i/1 \leq i \leq n\} \bigcup \{b_i v_{i+1}, u_i a_{i+1}, f_i u_{i+1}, v_i e_{i+1}, u_i u_{i+1}, v_i v_{i+1}, d_i v_{i+1}, v_i d_{i+1}, c_i u_{i+1}, u_i c_{i+1}, g_i v_{i+1}, v_i g_{i+1}, f_i w_{i+1}/1 \leq i \leq n-1\} \bigcup \{u_n u_1, v_n v_1, d_n v_1, v_n d_1, c_n u_1, u_n c_1, g_n v_1, v_n g_1, b_{n-1} u_1, b_n u_2, v_{n-1} f_1, f_n v_2, b_n v_1, u_n a_1, f_n u_1, v_n e_1, f_n w_1\}$. Therefore $|V(G)| = 11n + 1$ and $|E(G)| = 30n$. Using parity of n, we have the following cases:

Case 1 n is even.

Define a vertex labeling $f : V(G) \to \{0, 1\}$ as follows:

$$f(x) = \begin{cases} 0 & \text{if } x = t; \\ 0 & \text{if } x \in \{v_i, a_i, d_i, f_i, g_i\}, \ i \in \{1, 2, \ldots, n-1, n\}; \\ 1 & \text{if } x \in \{u_i, b_i, c_i, e_i, h_i\}, \ i \in \{1, 2, \ldots, n-1, n\}; \\ 1 & \text{if } x = w_i, \ i \in \{1, 3, \ldots, n-3, n-1\}; \\ 0 & \text{if } x = w_i, \ i \in \{2, 4, \ldots, n-2, n\}. \end{cases}$$

Thus $v_f(1) = \dfrac{11n}{2}$ and $v_f(0) = \dfrac{11n}{2} + 1$. The induced edge labeling $f^* : E(G) \to \{0, 1\}$ is $f^*(uv) = |f(u) - f(v)|$, for every edge $e = uv \in E$. Therefore

$$f^*(e) = \begin{cases} 1 & \text{if } \ e \in \{tu_i, u_i v_i, a_i b_i, e_i f_i, c_i d_i, g_i h_i, tc_i, tb_i, g_i u_i\}, \ i \in \{1, 2, \ldots, n-1, n\}; \\ 0 & \text{if } \ e \in \{ta_i, a_i v_i, e_i u_i\}, \ i \in \{1, 2, \ldots, n-1, n\}; \\ 0 & \text{if } \ e \in \{u_i u_{i+1}, v_i v_{i+1}, d_i v_{i+1}, v_i d_{i+1}, c_i u_{i+1}, u_i c_{i+1}, g_i v_{i+1}, v_i g_{i+1}\}, 1 \leq i \leq n-1; \\ 1 & \text{if } \ e \in \{b_i v_{i+1}, u_i a_{i+1}, f_i u_{i+1}, v_i e_{i+1}\}, \ i \in \{1, 2, \ldots, n-2, n-1\}; \\ 0 & \text{if } \ e \in \{b_i u_{i+2}, v_i f_{i+2}\}, \ i \in \{1, 2, \ldots, n-3, n-2\}; \\ 1 & \text{if } \ e \in \{d_i w_i, v_i w_i\}, \ i \in \{1, 3, \ldots, n-3, n-1\} \end{cases}$$

and

$$f^*(e) = \begin{cases} 0 & \text{if } e \in \{d_iw_i, v_iw_i\},\ i \in \{2,4,\ldots,n-2,n\}; \\ 0 & \text{if } e \in \{e_iw_i, f_iw_{i+1}\},\ i \in \{1,3,\ldots,n-3,n-1\}; \\ 1 & \text{if } e = e_iw_i,\ i \in \{2,4,\ldots,n-2,n\}; \\ 1 & \text{if } e = f_iw_{i+1},\ i \in \{2,4,\ldots,n-4,n-2\}; \\ 0 & \text{if } e \in \{u_nu_1, v_nv_1, d_nv_1, v_nd_1, c_nu_1, u_nc_1, g_nv_1, v_ng_1, b_{n-1}u_1, b_nu_2, v_{n-1}f_1, f_nv_2\}; \\ 1 & \text{if } e \in \{b_nv_1, u_na_1, f_nu_1, v_ne_1, f_nw_1\}. \end{cases}$$

Thus $e_f(1) = 15n$ and $e_f(0) = 15n$.

Case 2 n is odd.

Define a vertex labeling $f : V(G) \to \{0,1\}$ as follows:

$$f(x) = \begin{cases} 0 & \text{if } x = t; \\ 0 & \text{if } x \in \{v_i, a_i, d_i, f_i, g_i\},\ i \in \{1,2,\ldots,n-1,n\}; \\ 1 & \text{if } x \in \{u_i, b_i, c_i, e_i, h_i\},\ i \in \{1,2,\ldots,n-1,n\}; \\ 1 & \text{if } x = w_i,\ i \in \{1,3,\ldots,n-2,n\}; \\ 0 & \text{if } x = w_i,\ i \in \{2,4,\ldots,n-3,n-1\}. \end{cases}$$

Thus $v_f(1) = \dfrac{11n+1}{2}$ and $v_f(0) = \dfrac{11n+1}{2}$. The induced edge labeling $f^* : E(G) \to \{0,1\}$ is $f^*(uv) = |f(u) - f(v)|$, for every edge $e = uv \in E$. Therefore

$$f^*(e) = \begin{cases} 1 & \text{if } e \in \{tu_i, u_iv_i, a_ib_i, e_if_i, c_id_i, g_ih_i, tc_i, tb_i, g_iu_i\},\ i \in \{1,2,\ldots,n-1,n\}; \\ 0 & \text{if } e \in \{ta_i, a_iv_i, e_iu_i\},\ i \in \{1,2,\ldots,n-1,n\}; \\ 0 & \text{if } e \in \{u_iu_{i+1}, v_iv_{i+1}, d_iv_{i+1}, v_id_{i+1}, c_iu_{i+1}, u_ic_{i+1}, g_iv_{i+1}, v_ig_{i+1}\}, 1 \le i \le n-1; \\ 1 & \text{if } e \in \{b_iv_{i+1}, u_ia_{i+1}, f_iu_{i+1}, v_ie_{i+1}\},\ i \in \{1,2,\ldots,n-2,n-1\}; \\ 0 & \text{if } e \in \{b_iu_{i+2}, v_if_{i+2}\},\ i \in \{1,2,\ldots,n-3,n-2\}; \\ 1 & \text{if } e \in \{d_iw_i, v_iw_i\},\ i \in \{1,3,\ldots,n-2,n\}; \\ 0 & \text{if } e \in \{d_iw_i, v_iw_i\},\ i \in \{2,4,\ldots,n-3,n-1\}; \\ 0 & \text{if } e = e_iw_i,\ i \in \{1,3,\ldots,n-2,n\}; \\ 1 & \text{if } e \in \{e_iw_i, f_iw_{i+1}\},\ i \in \{2,4,\ldots,n-3,n-1\}; \\ 0 & \text{if } e = f_iw_{i+1},\ i \in \{1,3,\ldots,n-4,n-2\}; \\ 0 & \text{if } e \in \{u_nu_1, v_nv_1, d_nv_1, v_nd_1, c_nu_1, u_nc_1, g_nv_1, v_ng_1, b_{n-1}u_1, b_nu_2, v_{n-1}f_1, f_nv_2, f_nw_1\}; \\ 1 & \text{if } e \in \{b_nv_1, u_na_1, f_nu_1, v_ne_1\}. \end{cases}$$

Thus $e_f(1) = 15n$ and $e_f(0) = 15n$.

Therefore f satisfies the conditions $|v_f(1) - v_f(0)| \le 1$ and $|e_f(1) - e_f(0)| \le 1$. So, f admits cordial labeling on G. Hence G is cordial. $\square$

Theorem 2.7 *The graph obtained by duplicating all the vertices of the armed helm AH_n is cordial.*

Proof Let $V(AH_n) = \{t\} \bigcup \{u_i, v_i, w_i/1 \le i \le n\}$ and $E(AH_n) = \{tu_i, u_iv_i, v_iw_i/1 \le i \le n\} \bigcup \{u_iu_{i+1};\ 1 \le i \le n-1\} \bigcup \{u_nu_1\}$. Let G be the graph obtained by duplicating all the vertices in AH_n. Let $t', u_1', u_2', \cdots, u_n', v_1', v_2', \cdots, v_n', w_1', w_2', \cdots, w_n'$ be the new vertices of G by duplicating $t, u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n, w_1, w_2, \cdots, w_n$ respectively. Then $V(G) = \{t, t'\} \bigcup \{u_i, v_i, w_i, u_i', v_i', w_i'/1 \le i \le n\}$ and $E(G) = \{tu_i, u_iv_i, v_iw_i, w_i'v_i, w_iv_i', u_iv_i', t'u_i, u_i'v_i, tu_i';\ 1 \le i \le n\} \bigcup \{u_iu_{i+1}, u_i'u_{i+1}, u_iu_{i+1}';\ 1 \le i \le n-1\} \bigcup \{u_nu_1, u_n'u_1, u_nu_1'\}$. Therefore $|V(G)| = 6n+2$ and $|E(G)| = 12n$. Define a vertex labeling $f : V(G) \to \{0,1\}$ as follows:

$$f(x) = \begin{cases} 1 & \text{if } x = t; \\ 0 & \text{if } x \in \{u_i, u_i', w_i'\},\ i \in \{1, 2, \ldots, n-1, n\}; \\ 1 & \text{if } x \in \{v_i, w_i, v_i'\},\ i \in \{1, 2, \ldots, n-1, n\}; \\ 0 & \text{if } x = t'. \end{cases}$$

Thus $v_f(1) = 3n+1$ and $v_f(0) = 3n+1$. The induced edge labeling $f^* : E(G) \to \{0,1\}$ is $f^*(uv) = |f(u) - f(v)|$, for every edge $e = uv \in E$. Therefore

$$f^*(e) = \begin{cases} 1 & \text{if } \ e \in \{tu_i, u_iv_i, u_iv_i', v_iw_i', u_i'v_i, tu_i'\},\ i \in \{1, 2, \ldots, n-1, n\}; \\ 0 & \text{if } \ e \in \{v_iw_i, t'u_i, w_iv_i'\},\ i \in \{1, 2, \ldots, n-1, n\}; \\ 0 & \text{if } \ e \in \{u_iu_{i+1}, u_i'u_{i+1}, u_iu_{i+1}'\},\ i \in \{1, 2, \ldots, n-2, n-1\}; \\ 0 & \text{if } \ e \in \{u_nu_1, u_n'u_1, u_nu_1'\}. \end{cases}$$

Thus $e_f(1) = 6n$ and $e_f(0) = 6n$. Therefore, f satisfies the conditions $|v_f(1) - v_f(0)| \le 1$ and $|e_f(1) - e_f(0)| \le 1$. So, f admits cordial labeling on G. Hence G is cordial. □

Theorem 2.8 *The graph obtained by duplicating all the vertices other than the rim vertices of the armed helm AH_n is cordial.*

Proof Let $V(AH_n) = \{t\} \bigcup \{u_i, v_i, w_i/1 \le i \le n\}$ and $E(AH_n) = \{tu_i, u_iv_i, v_iw_i/1 \le i \le n\} \cup \{u_nu_1\} \bigcup \{u_iu_{i+1}/1 \le i \le n-1\}$. Let G be the graph obtained by duplicating all the vertices other than the rim vertices in AH_n. Let $t', v_1', v_2', \cdots, v_n', w_1', w_2', \cdots, w_n'$ be the new vertices of G by duplicating $t, v_1, v_2, \cdots, v_n, w_1, w_2, \cdots, w_n$ respectively. Then $V(G) = \{t, t'\} \bigcup \{u_i, v_i, w_i, v_i', w_i'/1 \le i \le n\}$ and $E(G) = \{tu_i, u_iv_i, v_iw_i, w_i'v_i, w_iv_i', u_iv_i', t'u_i/1 \le i \le n\} \bigcup \{u_iu_{i+1}/1 \le i \le n-1\} \cup \{u_nu_1\}$. Therefore $|V(G)| = 5n+2$ and $|E(G)| = 8n$. Using parity of n, we have the following cases:

Case 1 n is even.

Define a vertex labeling $f : V(G) \to \{0,1\}$ as follows:

$$f(x) = \begin{cases} 0 & \text{if } x = t; \\ 0 & \text{if } x \in \{u_i, w_i'\},\ i \in \{1, 2, \ldots, n-1, n\} \end{cases}$$

and

$$f(x)=\begin{cases}1 & \text{if } x\in\{v_i,w_i\},\ i\in\{1,2,\ldots,n-1,n\};\\ 1 & \text{if } x=t';\\ 1 & \text{if } x=v_i',\ i\in\{1,3,\ldots,n-3,n-1\};\\ 0 & \text{if } x=v_i',\ i\in\{2,4,\ldots,n-2,n\}.\end{cases}$$

Thus $v_f(1)=\dfrac{5n+2}{2}$ and $v_f(0)=\dfrac{5n+2}{2}$. The induced edge labeling $f^*:E(G)\to\{0,1\}$ is $f^*(uv)=|f(u)-f(v)|$, for every edge $e=uv\in E$.
Therefore

$$f^*(e)=\begin{cases}1 & \text{if } e\in\{u_iv_i,v_iw_i',t_i'u_i\},\ i\in\{1,2,\ldots,n-1,n\};\\ 1 & \text{if } e=u_iv_i',\ i\in\{1,3,\ldots,n-3,n-1\};\\ 1 & \text{if } e=w_iv_i',\ i\in\{2,4,\ldots,n-2,n\};\\ 0 & \text{if } e\in\{tu_i,v_iw_i\},\ i\in\{1,2,\ldots,n-1,n\};\\ 0 & \text{if } e=u_iu_{i+1},\ i\in\{1,2,\ldots,n-2,n-1\};\\ 0 & \text{if } e=u_iv_i',\ i\in\{2,4,\ldots,n-2,n\};\\ 0 & \text{if } e=w_iv_i',\ i\in\{1,3,\ldots,n-3,n-1\};\\ 0 & \text{if } e\in\{u_nu_1\}.\end{cases}$$

Thus $e_f(1)=4n$ and $e_f(0)=4n$.

Case 2 n is odd.

Define a vertex labeling $f:V(G)\to\{0,1\}$ as follows:

$$f(x)=\begin{cases}1 & \text{if } x=t';\\ 1 & \text{if } x\in\{v_i,w_i\},\ i\in\{1,2,\ldots,n-1,n\};\\ 1 & \text{if } x=v_i',\ i\in\{1,3,\ldots,n-2,n\};\\ 0 & \text{if } x=t;\\ 0 & \text{if } x\in\{u_i,w_i'\},\ i\in\{1,2,\ldots,n-1,n\};\\ 0 & \text{if } x=v_i',\ i\in\{2,4,\ldots,n-3,n-1\}.\end{cases}$$

Thus $v_f(1)=\dfrac{5n+3}{2}$ and $v_f(0)=\dfrac{5n+1}{2}$. The induced edge labeling $f^*:E(G)\to\{0,1\}$ is $f^*(uv)=|f(u)-f(v)|$, for every edge $e=uv\in E$. Therefore

$$f^*(e)=\begin{cases}1 & \text{if } e\in\{u_iv_i,v_iw_i',t_i'u_i\},\ i\in\{1,2,\ldots,n-1,n\};\\ 1 & \text{if } e=u_iv_i',\ i\in\{1,3,\ldots,n-2,n\};\\ 1 & \text{if } e=w_iv_i',\ i\in\{2,4,\ldots,n-3,n-1\};\\ 0 & \text{if } e\in\{tu_i,v_iw_i\},\ i\in\{1,2,\ldots,n-1,n\};\\ 0 & \text{if } e=u_iu_{i+1},\ i\in\{1,2,\ldots,n-2,n-1\}\end{cases}$$

and

$$f^*(e) = \begin{cases} 0 & \text{if } \ e = u_iv_i', \ i \in \{2, 4, \ldots, n-3, n-1\}; \\ 0 & \text{if } \ e = w_iv_i', \ i \in \{1, 3, \ldots, n-2, n\}; \\ 0 & \text{if } \ e \in \{u_nu_1\}. \end{cases}$$

Thus $e_f(1) = 4n$ and $e_f(0) = 4n$.

From both the cases we can conclude $|v_f(1) - v_f(0)| \leq 1$ and $|e_f(1) - e_f(0)| \leq 1$. So, f admits cordial labeling on G. Hence G is cordial. □

Theorem 2.9 *The graph obtained by duplicating all the rim vertices of the armed helm AH_n is cordial.*

Proof Let $V(AH_n) = \{t\} \bigcup \{u_i, v_i, w_i/1 \leq i \leq n\}$ and $E(AH_n) = \{tu_i, u_iv_i, v_iw_i//1 \leq i \leq n\} \bigcup \{u_nu_1\} \bigcup \{u_iu_{i+1}/1 \leq i \leq n-1\}$. Let G be the graph obtained by duplicating all the rim vertices in AH_n. Let $u_1', u_2', \cdots, u_n'$ be the new vertices of G by duplicating $u_1, u_2, \cdots, u_n$ respectively. Then $V(G) = \{t\} \cup \{u_i, v_i, w_i, u_i'/1 \leq i \leq n\}$ and $E(G) = \{tu_i, u_iv_i, v_iw_i, u_i'v_i, tu_i'/1 \leq i \leq n\} \cup \{u_iu_{i+1}, u_i'u_{i+1}, u_iu_{i+1}'/1 \leq i \leq n-1\} \cup \{u_nu_1, u_n'u_1, u_nu_1'\}$. Therefore $|V(G)| = 4n + 1$ and $|E(G)| = 8n$. Define a vertex labeling $f : V(G) \rightarrow \{0, 1\}$ as follows:

$$f(x) = \begin{cases} 1 & \text{if } x = t; \\ 0 & \text{if } x \in \{u_i, u_i'\}, \ i \in \{1, 2, \ldots, n-1, n\}; \\ 1 & \text{if } x \in \{v_i, w_i\}, \ i \in \{1, 2, \ldots, n-1, n\}. \end{cases}$$

Thus $v_f(1) = 2n + 1$ and $v_f(0) = 2n$. The induced edge labeling $f^* : E(G) \rightarrow \{0, 1\}$ is $f^*(uv) = |f(u) - f(v)|$, for every edge $e = uv \in E$. Therefore

$$f^*(e) = \begin{cases} 1 & \text{if } \ e \in \{tu_i, u_iv_i, tu_i', u_i'v_i\}, \ i \in \{1, 2, \ldots, n-1, n\}; \\ 0 & \text{if } \ e = v_iw_i, \ i \in \{1, 2, \ldots, n-1, n\}; \\ 0 & \text{if } \ e \in \{u_iu_{i+1}, u_iu_{i+1}', u_i'u_{i+1}\}, \ i \in \{1, 2, \ldots, n-2, n-1\}; \\ 0 & \text{if } \ e \in \{u_nu_1, u_n'u_1, u_nu_1'\}. \end{cases}$$

Thus $e_f(1) = 4n$ and $e_f(0) = 4n$. Therefore, f satisfies the conditions $|v_f(1) - v_f(0)| \leq 1$ and $|e_f(1) - e_f(0)| \leq 1$. So, f admits cordial labeling on G. Hence G is cordial. □

Theorem 2.10 *The graph obtained by duplicating all the vertices except the apex vertex of the armed helm AH_n is cordial.*

Proof Let $V(AH_n) = \{t\} \bigcup \{u_i, v_i, w_i/1 \leq i \leq n\}$ and $E(AH_n) = \{tu_i, u_iv_i, v_iw_i/1 \leq i \leq n\} \bigcup \{u_nu_1\} \bigcup \{u_iu_{i+1}/1 \leq i \leq n-1\}$. Let G be the graph obtained by duplicating all the vertices except the apex vertex in AH_n. Let $u_1', u_2', \cdots, u_n', v_1', v_2', \cdots, v_n', w_1', w_2', \cdots, w_n'$ be the new vertices of G by duplicating $u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n, w_1, w_2, \cdots, w_n$ respectively. Then $V(G) = \{t\} \bigcup \{u_i, v_i, w_i, u_i', v_i', w_i'/1 \leq i \leq n\}$ and $E(G) = \{tu_i, u_iv_i, v_iw_i, w_i'v_i, w_iv_i', u_iv_i', u_i'v_i, tu_i'/1 \leq i \leq n\} \bigcup \{u_iu_{i+1}, u_i'u_{i+1}, u_iu_{i+1}'/1 \leq i \leq n-1\} \bigcup \{u_nu_1, u_n'u_1, u_nu_1'\}$. Therefore

$|V(G)| = 6n+1$ and $|E(G)| = 11n$. Using parity of n, we have the following cases:

Case 1. n is even.

Define a vertex labeling $f : V(G) \to \{0, 1\}$ as follows:

$$f(x) = \begin{cases} 1 & \text{if } x = t; \\ 1 & \text{if } x \in \{v_i, u'_i\},\ i \in \{1, 2, \dots, n-1, n\}; \\ 1 & \text{if } x = w_i,\ i \in \{2, 4, \dots, n-2, n\}; \\ 1 & \text{if } x = w'_i,\ i \in \{1, 3, \dots, n-3, n-1\}; \\ 0 & \text{if } x \in \{u_i, v'_i\},\ i \in \{1, 2, \dots, n-1, n\}; \\ 0 & \text{if } x = w_i, i \in \{1, 3, \dots, n-3, n-1\}; \\ 0 & \text{if } x = w'_i, i \in \{2, 4, \dots, n-2, n\}. \end{cases}$$

Thus $v_f(1) = 3n+1$ and $v_f(0) = 3n$. The induced edge labeling $f^* : E(G) \to \{0, 1\}$ is $f^*(uv) = |f(u) - f(v)|$, for every edge $e = uv \in E$. Therefore

$$f^*(e) = \begin{cases} 1 & \text{if } \ e \in \{tu_i, u_iv_i\},\ i \in \{1, 2, \dots, n-1, n\}; \\ 1 & \text{if } \ e = v_iw_i,\ i \in \{1, 3, \dots, n-3, n-1\}; \\ 1 & \text{if } \ e \in \{w_iv'_i, v_iw'_i\},\ i \in \{2, 4, \dots, n-2, n\}; \\ 1 & \text{if } \ e \in \{u'_iu_{i+1}, u_iu'_{i+1}\},\ i \in \{1, 2, \dots, n-2, n-1\}; \\ 0 & \text{if } \ e = v_iw_i,\ i \in \{2, 4, \dots, n-2, n\}; \\ 0 & \text{if } \ e = u_iu_{i+1},\ i \in \{1, 2, \dots, n-2, n-1\}; \\ 0 & \text{if } \ e = u_iv'_i,\ i \in \{1, 2, \dots, n-1, n\}; \\ 0 & \text{if } \ e \in \{w_iv'_i, v_iw'_i\},\ i \in \{1, 3, \dots, n-3, n-1\}; \\ 0 & \text{if } \ e \in \{u'_iv_i, tu'_i\},\ i \in \{1, 2, \dots, n-1, n\}; \\ 0 & \text{if } \ e = u_nu_1; \\ 1 & \text{if } \ e \in \{u'_nu_1, u_nu'_1\}. \end{cases}$$

Thus $e_f(1) = \dfrac{11n}{2}$ and $e_f(0) = \dfrac{11n}{2}$.

Case 2. n is odd.

Define a vertex labeling $f : V(G) \to \{0, 1\}$ as follows:

$$f(x) = \begin{cases} 1 & \text{if } x = t; \\ 1 & \text{if } x = w_i,\ i \in \{2, 4, \dots, n-3, n-1\}; \\ 1 & \text{if } x \in \{u'_i, v_i\},\ i \in \{1, 2, \dots, n-1, n\}; \\ 1 & \text{if } x = w'_i,\ i \in \{1, 3, \dots, n-2, n\} \end{cases}$$

and

$$f(x)=\begin{cases}0 & \text{if } x\in\{u_i,v'_i\},\ i\in\{1,2,\dots,n-1,n\};\\ 0 & \text{if } x=w_i,\ i\in\{1,3,\dots,n-2,n\};\\ 0 & \text{if } x=w'_i,\ i\in\{2,4,\dots,n-3,n-1\}.\end{cases}$$

Thus $v_f(1)=3n+1$ and $v_f(0)=3n$. The induced edge labeling $f^*:E(G)\to\{0,1\}$ is $f^*(uv)=|f(u)-f(v)|$, for every edge $e=uv\in E$. Therefore

$$f^*(e)=\begin{cases}1 & \text{if } e\in\{tu_i,u_iv_i\},\ i\in\{1,2,\dots,n-1,n\};\\ 1 & \text{if } e=v_iw_i,\ i\in\{1,3,\dots,n-2,n\};\\ 1 & \text{if } e\in\{w_iv'_i,v_iw'_i\},\ i\in\{2,4,\dots,n-3,n-1\};\\ 1 & \text{if } e\in\{u'_iu_{i+1},u_iu'_{i+1}\},\ i\in\{1,2,\dots,n-2,n-1\};\\ 0 & \text{if } e=v_iw_i,\ i\in\{2,4,\dots,n-3,n-1\};\\ 0 & \text{if } e=u_iu_{i+1},\ i\in\{1,2,\dots,n-2,n-1\};\\ 0 & \text{if } e=u_iv'_i,\ i\in\{1,2,\dots,n-1,n\};\\ 0 & \text{if } e\in\{w_iv'_i,v_iw'_i\},\ i\in\{1,3,\dots,n-2,n\};\\ 0 & \text{if } e\in\{u'_iv_i,tu'_i\},\ i\in\{1,2,\dots,n-1,n\};\\ 0 & \text{if } e=u_nu_1;\\ 1 & \text{if } e\in\{u'_nu_1,u_nu'_1\}.\end{cases}$$

Thus $e_f(1)=\dfrac{11n-1}{2}$ and $e_f(0)=\dfrac{11n+1}{2}$.

From both the cases we can conclude $|v_f(1)-v_f(0)|\le 1$ and $|e_f(1)-e_f(0)|\le 1$. So, f admits cordial labeling on G. Hence G is cordial. □

Theorem 2.11 *The graph obtained by duplicating all the edges other than spoke edges of of the armed helm AH_n is cordial.*

Proof Let $V(AH_n)=\{t\}\bigcup\{u_i,v_i,w_i/1\le i\le n\}$ and $E(AH_n)=\{j_i=tu_i,l_i=u_iv_i,m_i=u_iw_i/1\le i\le n\}\bigcup\{k_i=u_iu_{i+1}/1\le i\le n-1\}\bigcup\{k_n=u_nu_1\}$. Let G be the graph obtained by duplicating all the edges other than spoke edges in AH_n. For each $i\in 1,2,\cdots,n$, let $k'_i=a_ib_i, l'_i=c_id_i$ and $m'_i=e_if_i$ be the new edges of G by duplicating $k_i,l_i$and m_i respectively. Then $V(G)=\{t\}\bigcup\{u_i,v_i,w_i,a_i,b_i,c_i,d_i,e_i,f_i/1\le i\le n\}$ and $E(G)=\{b_iu_{i+2}/1\le i\le n-2\}\bigcup\{tu_i,u_iv_i,v_iw_i,c_id_i,a_ib_i,a_iv_i,ta_i,tb_i,tc_i,e_if_i,e_iu_i,d_iw_i/1\le i\le n\}\bigcup\{u_iu_{i+1},b_iv_{i+1},u_ia_{i+1},c_iu_{i+1},u_ic_{i+1}/1\le i\le n-1\}\bigcup\{u_nu_1,b_nv_1,u_na_1,c_nu_1,u_nc_1,b_{n-1}u_1,b_nu_2\}$. Therefore $|V(G)|=9n+1$ and $|E(G)|=18n$. Using parity of n, we have the following cases:

Case 1. n is even.

Define a vertex labeling $f : V(G) \to \{0,1\}$ as follows:

$$f(x) = \begin{cases} 1 & \text{if } x = t; \\ 0 & \text{if } x \in \{u_i, v_i, c_i, d_i\},\ i \in \{1,2,\ldots,n-1,n\} \end{cases}$$

and

$$f(x) = \begin{cases} 1 & \text{if } x \in \{w_i, a_i, b_i, f_i\},\ i \in \{1,2,\ldots,n-1,n\}; \\ 0 & \text{if } x = e_i,\ i \in \{1,3,\ldots,n-3,n-1\}; \\ 1 & \text{if } x = e_i,\ i \in \{2,4,\ldots,n-2,n\}. \end{cases}$$

Thus $v_f(1) = \dfrac{9n}{2} + 1$ and $v_f(0) = \dfrac{9n}{2}$. The induced edge labeling $f^* : E(G) \to \{0,1\}$ is $f^*(uv) = |f(u) - f(v)|$, for every edge $e = uv \in E$. Therefore

$$f^*(e) = \begin{cases} 1 & \text{if } \ e \in \{tu_i, v_iw_i, a_iv_i, tc_i, d_iw_i\},\ i \in \{1,2,\ldots,n-1,n\}; \\ 0 & \text{if } \ e \in \{u_iv_i, , ta_i, tb_i, a_ib_i, c_id_i\},\ i \in \{1,2,\ldots,n-1,n\}; \\ 1 & \text{if } \ e \in \{b_iv_{i+1}, u_ia_{i+1}\},\ i \in \{1,2,\ldots,n-2,n-1\}; \\ 1 & \text{if } \ e = b_iu_{i+2},\ i \in \{1,2,\ldots,n-3,n-2\}; \\ 0 & \text{if } \ e \in \{u_iu_{i+1}, c_iu_{i+1}, u_ic_{i+1}\},\ i \in \{1,2,\ldots,n-2,n-1\}; \\ 0 & \text{if } \ e = u_ie_i,\ i \in \{1,3,\ldots,n-3,n-1\}; \\ 1 & \text{if } \ e = u_ie_i,\ i \in \{2,4,\ldots,n-2,n\}; \\ 1 & \text{if } \ e = e_if_i,\ i \in \{1,3,\ldots,n-3,n-1\}; \\ 0 & \text{if } \ e = e_if_i,\ i \in \{2,4,\ldots,n-2,n\}; \\ 0 & \text{if } \ e \in \{u_nu_1, c_nu_1, u_nc_1\}; \\ 1 & \text{if } \ e \in \{b_nv_1, u_na_1, b_n - 1u_1, b_nu_2\}. \end{cases}$$

Thus $e_f(1) = 9n$ and $e_f(0) = 9n$.

Case 2. n is odd.

Define a vertex labeling $f : V(G) \to \{0,1\}$ as follows:

$$f(x) = \begin{cases} 1 & \text{if } x = t; \\ 0 & \text{if } x \in \{u_i, v_i, c_i, d_i\},\ i \in \{1,2,\ldots,n-1,n\}; \\ 1 & \text{if } x \in \{w_i, a_i, b_i, f_i\},\ i \in \{1,2,\ldots,n-1,n\}; \\ 0 & \text{if } x = e_i,\ i \in \{1,3,\ldots,n-2,n\}; \\ 1 & \text{if } x = e_i,\ i \in \{2,4,\ldots,n-3,n-1\}. \end{cases}$$

Thus $v_f(1) = \dfrac{9n+1}{2}$ and $v_f(0) = \dfrac{9n+1}{2}$. The induced edge labeling $f^* : E(G) \to \{0,1\}$

is $f^*(uv) = |f(u) - f(v)|$, for every edge $e = uv \in E$. Therefore

$$f^*(e) = \begin{cases} 1 & \text{if } \ e \in \{tu_i, v_iw_i, a_iv_i, tc_i, d_iw_i\}, \ i \in \{1, 2, \ldots, n-1, n\}; \\ 0 & \text{if } \ e \in \{u_iv_i, , ta_i, tb_i, a_ib_i, c_id_i\}, \ i \in \{1, 2, \ldots, n-1, n\}; \\ 1 & \text{if } \ e \in \{b_iv_{i+1}, u_ia_{i+1}\}, \ i \in \{1, 2, \ldots, n-2, n-1\}; \\ 1 & \text{if } \ e = b_iu_{i+2}, \ i \in \{1, 2, \ldots, n-3, n-2\} \end{cases}$$

and

$$f^*(e) = \begin{cases} 0 & \text{if } \ e \in \{u_iu_{i+1}, c_iu_{i+1}, u_ic_{i+1}\}, \ i \in \{1, 2, \ldots, n-2, n-1\}; \\ 0 & \text{if } \ e = u_ie_i, \ i \in \{1, 3, \ldots, n-2, n\}; \\ 1 & \text{if } \ e = u_ie_i, \ i \in \{2, 4, \ldots, n-3, n-1\}; \\ 1 & \text{if } \ e = e_if_i, \ i \in \{1, 3, \ldots, n-2, n\}; \\ 0 & \text{if } \ e = e_if_i, \ i \in \{2, 4, \ldots, n-3, n-1\}; \\ 0 & \text{if } \ e \in \{u_nu_1, c_nu_1, u_nc_1\}; \\ 1 & \text{if } \ e \in \{b_nv_1, u_na_1, b_{n-1}u_1, b_nu_2\}. \end{cases}$$

Thus $e_f(1) = 9n$ and $e_f(0) = 9n$.

From both the cases we can conclude $|v_f(1) - v_f(0)| \leq 1$ and $|e_f(1) - e_f(0)| \leq 1$. So, f admits cordial labeling on G. Hence G is cordial. □

§3. Conclusion

we have derived eleven new results by investigating cordial labeling in the context of duplication in Web and Armed Helm. More exploration is possible for other graph families and in the context of different graph labeling problems.

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