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Direct Product of Multigroups and Its Generalization

P.A. Ejegwa

(Department of Mathematics/Statistics/Computer Science, University of Agriculture, P.M.B. 2373, Makurdi, Nigeria)

A. M. Ibrahim

(Department of Mathematics, Ahmadu Bello University, Zaria, Nigeria)

E-mail: ejegwa.augustine@uam.edu.ng, amibrahim@abu.edu.ng

Abstract: This paper proposes the concept of direct product of multigroups and its generalization. Some results are obtained with reference to root sets and cuts of multigroups. We prove that the direct product of multigroups is a multigroup. Finally, we introduce the notion of homomorphism and explore some of its properties in the context of direct product of multigroups and its generalization.

Key Words: Multisets, multigroups, direct product of multigroups.

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§1. Introduction

In set theory, repetition of objects are not allowed in a collection. This perspective rendered set almost irrelevant because many real life problems admit repetition. To remedy the handicap in the idea of sets, the concept of multiset was introduced in [10] as a generalization of set wherein objects repeat in a collection. Multiset is very promising in mathematics, computer science, website design, etc. See [14, 15] for details.

Since algebraic structures like groupoids, semigroups, monoids and groups were built from the idea of sets, it is then natural to introduce the algebraic notions of multiset. In [12], the term *multigroup* was proposed as a generalization of group in analogous to some non-classical groups such as fuzzy groups [13], intuitionistic fuzzy groups [3], etc. Although the term *multigroup* was earlier used in [4, 11] as an extension of group theory, it is only the idea of multigroup in [12] that captures multiset and relates to other non-classical groups. In fact, every multigroup is a multiset but the converse is not necessarily true and the concept of classical groups is a specialize multigroup with a unit count [5].

In furtherance of the study of multigroups, some properties of multigroups and the analogous of isomorphism theorems were presented in [2]. Subsequently, in [1], the idea of order of an element with respect to multigroup and some of its related properties were discussed. A complete account on the concept of multigroups from different algebraic perspectives was outlined in [8]. The notions of upper and lower cuts of multigroups were proposed and some of

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their algebraic properties were explicated in [5]. In continuation to the study of homomorphism in multigroup setting (cf. [2, 12]), some homomorphic properties of multigroups were explored in [6]. In [9], the notion of multigroup actions on multiset was proposed and some results were established. An extensive work on normal submultigroups and comultisets of a multigroup were presented in [7].

In this paper, we explicate the notion of direct product of multigroups and its generalization. Some homomorphic properties of direct product of multigroups are also presented. This paper is organized as follows; in Section 2, some preliminary definitions and results are presented to be used in the sequel. Section 3 introduces the concept of direct product between two multigroups and Section 4 considers the case of direct product of k^{th} multigroups. Meanwhile, Section 5 contains some homomorphic properties of direct product of multigroups.

§2. Preliminaries

Definition 2.1([14]) *Let $X = \{x_1, x_2, \cdots, x_n, \cdots\}$ be a set. A multiset A over X is a cardinal-valued function, that is, $C_A : X \to \mathbb{N}$ such that for $x \in Dom(A)$ implies $A(x)$ is a cardinal and $A(x) = C_A(x) > 0$, where $C_A(x)$ denoted the number of times an object x occur in A. Whenever $C_A(x) = 0$, implies $x \notin Dom(A)$.*

The set of all multisets over X is denoted by $MS(X)$.

Definition 2.2([15]) *Let $A, B \in MS(X)$, A is called a submultiset of B written as $A \subseteq B$ if $C_A(x) \leq C_B(x)$ for $\forall x \in X$. Also, if $A \subseteq B$ and $A \neq B$, then A is called a proper submultiset of B and denoted as $A \subset B$. A multiset is called the parent in relation to its submultiset.*

Definition 2.3([12]) *Let X be a group. A multiset G is called a multigroup of X if it satisfies the following conditions:*

(i) $C_G(xy) \geq C_G(x) \wedge C_G(y) \forall x, y \in X$;
(ii) $C_G(x^{-1}) = C_G(x) \forall x \in X$,

where C_G denotes count function of G from X into a natural number $\mathbb{N}$ and $\wedge$ denotes minimum, respectively.

By implication, a multiset G is called a multigroup of a group X if

$$C_G(xy^{-1}) \geq C_G(x) \wedge C_G(y), \quad \forall x, y \in X.$$

It follows immediately from the definition that,

$$C_G(e) \geq C_G(x), \quad \forall x \in X,$$

where e is the identity element of X.

The count of an element in G is the number of occurrence of the element in G. While the order of G is the sum of the count of each of the elements in G, and is given by

$$|G| = \sum_{i=1}^{n} C_G(x_i), \quad \forall x_i \in X.$$

We denote the set of all multigroups of X by $MG(X)$.

Definition 2.4([5]) *Let $A \in MG(X)$. A nonempty submultiset B of A is called a submultigroup of A denoted by $B \sqsubseteq A$ if B form a multigroup. A submultigroup B of A is a proper submultigroup denoted by $B \sqsubset A$, if $B \sqsubseteq A$ and $A \neq B$.*

Definition 2.5([5]) *Let $A \in MG(X)$. Then the sets $A_{[n]}$ and $A_{(n)}$ defined as*

(*i*) $A_{[n]} = \{x \in X \mid C_A(x) \geq n, n \in \mathbb{N}\}$ *and*
(*ii*) $A_{(n)} = \{x \in X \mid C_A(x) > n, n \in \mathbb{N}\}$

are called strong upper cut and weak upper cut of A.

Definition 2.6([5]) *Let $A \in MG(X)$. Then the sets $A^{[n]}$ and $A^{(n)}$ defined as*

(*i*) $A^{[n]} = \{x \in X \mid C_A(x) \leq n, n \in \mathbb{N}\}$ *and*
(*ii*) $A^{(n)} = \{x \in X \mid C_A(x) < n, n \in \mathbb{N}\}$

are called strong lower cut and weak lower cut of A.

Definition 2.7([12]) *Let $A \in MG(X)$. Then the sets A_* and A^* are defined as*

(*i*) $A_* = \{x \in X \mid C_A(x) > 0\}$ *and*
(*ii*) $A^* = \{x \in X \mid C_A(x) = C_A(e)\}$, *where e is the identity element of X.*

Proposition 2.8([12]) *Let $A \in MG(X)$. Then A_* and A^* are subgroups of X.*

Theorem 2.9([5]) *Let $A \in MG(X)$. Then $A_{[n]}$ is a subgroup of X $\forall\, n \leq C_A(e)$ and $A^{[n]}$ is a subgroup of X $\forall\, n \geq C_A(e)$, where e is the identity element of X and $n \in \mathbb{N}$.*

Definition 2.10([7]) *Let $A, B \in MG(X)$ such that $A \subseteq B$. Then A is called a normal submultigroup of B if for all $x, y \in X$, it satisfies $C_A(xyx^{-1}) \geq C_A(y)$.*

Proposition 2.11([7]) *Let $A, B \in MG(X)$. Then the following statements are equivalent:*

(*i*) *A is a normal submultigroup of B;*
(*ii*) $C_A(xyx^{-1}) = C_A(y) \forall x, y \in X$;
(*iii*) $C_A(xy) = C_A(yx) \forall x, y \in X$.

Definition 2.12([7]) *Two multigroups A and B of X are conjugate to each other if for all $x, y \in X$, $C_A(x) = C_B(yxy^{-1})$ and $C_B(y) = C_A(xyx^{-1})$.*

Definition 2.13([6]) *Let X and Y be groups and let $f : X \to Y$ be a homomorphism. Suppose A and B are multigroups of X and Y, respectively. Then f induces a homomorphism from A to B which satisfies* *(i)* $C_A(f^{-1}(y_1y_2)) \geq C_A(f^{-1}(y_1)) \wedge C_A(f^{-1}(y_2))\ \forall y_1, y_2 \in Y$;

(ii) $C_B(f(x_1x_2)) \geq C_B(f(x_1)) \wedge C_B(f(x_2))\ \forall x_1, x_2 \in X$,

where

(i) the image of A under f, denoted by $f(A)$, is a multiset of Y defined by

$$C_{f(A)}(y) = \begin{cases} \bigvee_{x \in f^{-1}(y)} C_A(x), & f^{-1}(y) \neq \emptyset \\ 0, & otherwise \end{cases}$$

for each $y \in Y$ and

(ii) the inverse image of B under f, denoted by $f^{-1}(B)$, is a multiset of X defined by

$$C_{f^{-1}(B)}(x) = C_B(f(x))\ \forall x \in X.$$

Proposition 2.14([12]) *Let X and Y be groups and $f : X \to Y$ be a homomorphism. If $A \in MG(X)$, then $f(A) \in MG(Y)$.*

Corollary 2.15([12]) *Let X and Y be groups and $f : X \to Y$ be a homomorphism. If $B \in MG(Y)$, then $f^{-1}(B) \in MG(X)$.*

§3. Direct Product of Multigroups

Given two groups X and Y, the direct product, $X \times Y$ is the Cartesian product of ordered pair (x, y) such that $x \in X$ and $y \in Y$, and the group operation is component-wise, so

$$(x_1, y_1) \times (x_2, y_2) = (x_1x_2, y_1y_2).$$

The resulting algebraic structure satisfies the axioms for a group. Since the ordered pair (x, y) such that $x \in X$ and $y \in Y$ is an element of $X \times Y$, we simply write $(x, y) \in X \times Y$. In this section, we discuss the notion of direct product of two multigroups defined over X and Y, respectively.

Definition 3.1 *Let X and Y be groups, $A \in MG(X)$ and $B \in MG(Y)$, respectively. The direct product of A and B depicted by $A \times B$ is a function*

$$C_{A\times B} : X \times Y \to \mathbb{N}$$

defined by

$$C_{A\times B}((x, y)) = C_A(x) \wedge C_B(y) \forall x \in X, \forall y \in Y.$$

Example 3.2 Let $X = \{e, a\}$ be a group, where $a^2 = e$ and $Y = \{e', x, y, z\}$ be a Klein 4-group, where $x^2 = y^2 = z^2 = e'$. Then

$$A = [e^5, a]$$

and

$$B = [(e')^6, x^4, y^5, z^4]$$

are multigroups of X and Y by Definition 2.3. Now

$$X \times Y = \{(e, e'), (e, x), (e, y), (e, z), (a, e'), (a, x), (a, y), (a, z)\}$$

is a group such that

$$(e, x)^2 = (e, y)^2 = (e, z)^2 = (a, e')^2 = (a, x)^2 = (a, y)^2 = (a, z)^2 = (e, e')$$

is the identity element of $X \times Y$. Then using Definition 3.1,

$$A \times B = [(e, e')^5, (e, x)^4, (e, y)^5, (e, z)^4, (a, e'), (a, x), (a, y), (a, z)]$$

is a multigroup of $X \times Y$ satisfying the conditions in Definition 2.3.

Example 3.3 Let X and Y be groups as in Example 3.2. Let

$$A = [e^5, a^4]$$

and

$$B = [(e')^7, x^9, y^6, z^5]$$

be multisets of X and Y, respectively. Then

$$A \times B = [(e, e')^5, (e, x)^5, (e, y)^5, (e, z)^5, (a, e')^4, (a, x)^4, (a, y)^4, (a, z)^4].$$

By Definition 2.3, it follows that $A \times B$ is a multigroup of $X \times Y$ although B is not a multigroup of Y while A is a multigroup of X.

From the notion of direct product in multigroup context, we observe that

$$|A \times B| < |A||B|$$

unlike in classical group where $|X \times Y| = |X||Y|$.

Theorem 3.4 *Let $A \in MG(X)$ and $B \in MG(Y)$, respectively. Then for all $n \in \mathbb{N}$, $(A \times B)_{[n]} = A_{[n]} \times B_{[n]}$.*

Proof Let $(x, y) \in (A \times B)_{[n]}$. Using Definition 2.5, we have

$$C_{A \times B}((x, y)) = (C_A(x) \wedge C_B(y)) \geq n.$$

This implies that $C_A(x) \geq n$ and $C_B(y) \geq n$, then $x \in A_{[n]}$ and $y \in B_{[n]}$. Thus,

$$(x, y) \in A_{[n]} \times B_{[n]}.$$

Also, let $(x,y) \in A_{[n]} \times B_{[n]}$. Then $C_A(x) \geq n$ and $C_B(y) \geq n$. That is,

$$(C_A(x) \wedge C_B(y)) \geq n.$$

This yields us $(x,y) \in (A \times B)_{[n]}$. Therefore, $(A \times B)_{[n]} = A_{[n]} \times B_{[n]}$ $\forall n \in \mathbb{N}$. □

Corollary 3.5 *Let $A \in MG(X)$ and $B \in MG(Y)$, respectively. Then for all $n \in \mathbb{N}$, $(A \times B)^{[n]} = A^{[n]} \times B^{[n]}$.*

Proof Straightforward from Theorem 3.4. □

Corollary 3.6 *Let $A \in MG(X)$ and $B \in MG(Y)$, respectively. Then*

(i) $(A \times B)_* = A_* \times B_*$;
(ii) $(A \times B)^* = A^* \times B^*$.

Proof Straightforward from Theorem 3.4. □

Theorem 3.7 *Let A and B be multigroups of X and Y, respectively, then $A \times B$ is a multigroup of $X \times Y$.*

Proof Let $(x,y) \in X \times Y$ and let $x = (x_1, x_2)$ and $y = (y_1, y_2)$. We have

$$\begin{aligned}
C_{A\times B}(xy) &= C_{A\times B}((x_1,x_2)(y_1,y_2)) \\
&= C_{A\times B}((x_1y_1, x_2y_2)) \\
&= C_A(x_1y_1) \wedge C_B(x_2y_2) \\
&\geq \wedge(C_A(x_1) \wedge C_A(y_1), C_B(x_2) \wedge C_B(y_2)) \\
&= \wedge(C_A(x_1) \wedge C_B(x_2), C_A(y_1) \wedge C_B(y_2)) \\
&= C_{A\times B}((x_1,x_2)) \wedge C_{A\times B}((y_1,y_2)) \\
&= C_{A\times B}(x) \wedge C_{A\times B}(y).
\end{aligned}$$

Also,

$$\begin{aligned}
C_{A\times B}(x^{-1}) &= C_{A\times B}((x_1,x_2)^{-1}) = C_{A\times B}((x_1^{-1}, x_2^{-1})) \\
&= C_A(x_1^{-1}) \wedge C_B(x_2^{-1}) = C_A(x_1) \wedge C_B(x_2) \\
&= C_{A\times B}((x_1,x_2)) = C_{A\times B}(x).
\end{aligned}$$

Hence, $A \times B \in MG(X \times Y)$. □

Corollary 3.8 *Let $A_1, B_1 \in MG(X_1)$ and $A_2, B_2 \in MG(X_2)$, respectively such that $A_1 \subseteq B_1$ and $A_2 \subseteq B_2$. If A_1 and A_2 are normal submultigroups of B_1 and B_2, then $A_1 \times A_2$ is a normal submultigroup of $B_1 \times B_2$.*

Proof By Theorem 3.7, $A_1 \times A_2$ is a multigroup of $X_1 \times X_2$. Also, $B_1 \times B_2$ is a multigroup of $X_1 \times X_2$. We show that $A_1 \times A_2$ is a normal submultigroup of $B_1 \times B_2$. Let $(x,y) \in X_1 \times X_2$

such that $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Then we get

$$\begin{aligned}
C_{A_1 \times A_2}(xy) &= C_{A_1 \times A_2}((x_1, x_2)(y_1, y_2)) \\
&= C_{A_1 \times A_2}((x_1 y_1, x_2 y_2)) \\
&= C_{A_1}(x_1 y_1) \wedge C_{A_2}(x_2 y_2) \\
&= C_{A_1}(y_1 x_1) \wedge C_{A_2}(y_2 x_2) \\
&= C_{A_1 \times A_2}((y_1 x_1, y_2 x_2)) \\
&= C_{A_1 \times A_2}((y_1, y_2)(x_1, x_2)) \\
&= C_{A_1 \times A_2}(yx).
\end{aligned}$$

Hence $A_1 \times A_2$ is a normal submultigroup of $B_1 \times B_2$ by Proposition 2.11. □

Theorem 3.9 *Let A and B be multigroups of X and Y, respectively. Then*

(i) $(A \times B)_*$ *is a subgroup of* $X \times Y$*;*
(ii) $(A \times B)^*$ *is a subgroup of* $X \times Y$*;*
(iii) $(A \times B)_{[n]}, n \in \mathbb{N}$ *is a subgroup of* $X \times Y$*,* $\forall\, n \leq C_{A\times B}(e, e')$*;*
(iv) $(A \times B)^{[n]}, n \in \mathbb{N}$ *is a subgroup of* $X \times Y$*,* $\forall\, n \geq C_{A\times B}(e, e')$*.*

Proof Combining Proposition 2.8, Theorem 2.9 and Theorem 3.7, the results follow. □

Corollary 3.10 *Let $A, C \in MG(X)$ such that $A \subseteq C$ and $B, D \in MG(Y)$ such that $B \subseteq D$, respectively. If A and B are normal, then*

(i) $(A \times B)_*$ *is a normal subgroup of* $(C \times D)_*$*;*
(ii) $(A \times B)^*$ *is a normal subgroup of* $(C \times D)^*$*;*
(iii) $(A \times B)_{[n]}, n \in \mathbb{N}$ *is a normal subgroup of* $(C \times D)_{[n]}$*,* $\forall\, n \leq C_{A\times B}(e, e')$*;*
(iv) $(A \times B)^{[n]}, n \in \mathbb{N}$ *is a normal subgroup of* $(C \times D)^{[n]}$*,* $\forall\, n \geq C_{A\times B}(e, e')$*.*

Proof Combining Proposition 2.8, Theorem 2.9, Theorem 3.7 and Corollary 3.8, the results follow. □

Proposition 3.11 *Let $A \in MG(X)$, $B \in MG(Y)$ and $A \times B \in MG(X \times Y)$. Then $\forall (x, y) \in X \times Y$, we have*

(i) $C_{A\times B}((x^{-1}, y^{-1})) = C_{A\times B}((x, y))$*;*
(ii) $C_{A\times B}((e, e')) \geq C_{A\times B}((x, y))$*;*
(iii) $C_{A\times B}((x, y)^n) \geq C_{A\times B}((x, y))$*, where e and e' are the identity elements of X and Y, respectively and $n \in \mathbb{N}$.*

Proof For $x \in X,\ y \in Y$ and $(x, y) \in X \times Y$, we get

(i) $C_{A\times B}((x^{-1}, y^{-1})) = C_A(x^{-1}) \wedge C_B(y^{-1}) = C_A(x) \wedge C_B(y) = C_{A\times B}((x, y))$.
Clearly, $C_{A\times B}((x^{-1}, y^{-1})) = C_{A\times B}((x, y))\ \forall (x, y) \in X \times Y$.

(*ii*)

$$\begin{aligned} C_{A\times B}((e,e')) &= C_{A\times B}((x,y)(x^{-1},y^{-1})) \\ &\geq C_{A\times B}((x,y)) \wedge C_{A\times B}((x^{-1},y^{-1})) \\ &= C_{A\times B}((x,y)) \wedge C_{A\times B}((x,y)) \\ &= C_{A\times B}((x,y)) \, \forall (x,y) \in X \times Y. \end{aligned}$$

Hence, $C_{A\times B}((e,e')) \geq C_{A\times B}((x,y))$.

(*iii*)

$$\begin{aligned} C_{A\times B}((x,y)^n) &= C_{A\times B}((x^n,y^n)) \\ &= C_{A\times B}((x^{n-1},y^{n-1})(x,y)) \\ &\geq C_{A\times B}((x^{n-1},y^{n-1})) \wedge C_{A\times B}((x,y)) \\ &\geq C_{A\times B}((x^{n-2},y^{n-2})) \wedge C_{A\times B}((x,y)) \wedge C_{A\times B}((x,y)) \\ &\geq C_{A\times B}((x,y)) \wedge C_{A\times B}((x,y)) \wedge ... \wedge C_{A\times B}((x,y)) \\ &= C_{A\times B}((x,y)), \end{aligned}$$

which implies that $C_{A\times B}((x,y)^n) = C_{A\times B}((x^n,y^n)) \geq C_{A\times B}((x,y)) \, \forall (x,y) \in X \times Y$. □

Theorem 3.12 *Let A and B be multisets of groups X and Y, respectively. Suppose that e and e' are the identity elements of X and Y, respectively. If $A \times B$ is a multigroup of $X \times Y$, then at least one of the following statements hold.*

(*i*) $C_B(e') \geq C_A(x) \, \forall x \in X$;

(*ii*) $C_A(e) \geq C_B(y) \, \forall y \in Y$.

Proof Let $A \times B \in MG(X \times Y)$. By contrapositive, suppose that none of the statements holds. Then suppose we can find a in X and b in Y such that

$$C_A(a) > C_B(e') \text{ and } C_B(b) > C_A(e).$$

From these we have

$$\begin{aligned} C_{A\times B}((a,b)) &= C_A(a) \wedge C_B(b) \\ &> C_A(e) \wedge C_B(e') \\ &= C_{A\times B}((e,e')). \end{aligned}$$

Thus, $A \times B$ is not a multigroup of $X \times Y$ by Proposition 3.11. Hence, either $C_B(e') \geq C_A(x) \, \forall x \in X$ or $C_A(e) \geq C_B(y) \, \forall y \in Y$. This completes the proof. □

Theorem 3.13 *Let A and B be multisets of groups X and Y, respectively, such that $C_A(x) \leq C_B(e') \, \forall x \in X$, e' being the identity element of Y. If $A \times B$ is a multigroup of $X \times Y$, then A is a multigroup of X.*

Proof Let $A \times B$ be a multigroup of $X \times Y$ and $x, y \in X$. Then $(x, e'), (y, e') \in X \times Y$. Now, using the property $C_A(x) \leq C_B(e')\, \forall x \in X$, we get

$$\begin{aligned} C_A(xy) &= C_A(xy) \wedge C_B(e'e') \\ &= C_{A\times B}((x, e')(y, e')) \\ &\geq C_{A\times B}((x, e')) \wedge C_{A\times B}((y, e')) \\ &= \wedge(C_A(x) \wedge C_B(e'), C_A(y) \wedge C_B(e')) \\ &= C_A(x) \wedge C_A(y). \end{aligned}$$

Also,

$$\begin{aligned} C_A(x^{-1}) &= C_A(x^{-1}) \wedge C_B(e'^{-1}) = C_{A\times B}((x^{-1}, e'^{-1})) \\ &= C_{A\times B}((x, e')^{-1}) = C_{A\times B}((x, e')) \\ &= C_A(x) \wedge C_B(e') = C_A(x). \end{aligned}$$

Hence, A is a multigroup of X. This completes the proof. □

Theorem 3.14 *Let A and B be multisets of groups X and Y, respectively, such that $C_B(x) \leq C_A(e)\, \forall x \in Y$, e being the identity element of X. If $A \times B$ is a multigroup of $X \times Y$, then B is a multigroup of Y.*

Proof Similar to Theorem 3.13. □

Corollary 3.15 *Let A and B be multisets of groups X and Y, respectively. If $A \times B$ is a multigroup of $X \times Y$, then either A is a multigroup of X or B is a multigroup of Y.*

Proof Combining Theorems 3.12 − 3.14, the result follows. □

Theorem 3.16 *If A and C are conjugate multigroups of a group X, and B and D are conjugate multigroups of a group Y. Then $A \times B \in MG(X \times Y)$ is a conjugate of $C \times D \in MG(X \times Y)$.*

Proof Since A and C are conjugate, it implies that for $g_1 \in X$, we have

$$C_A(x) = C_C(g_1^{-1} x g_1)\, \forall x \in X.$$

Also, since B and D are conjugate, for $g_2 \in Y$, we get

$$C_B(y) = C_D(g_2^{-1} y g_2)\, \forall y \in Y.$$

Now,

$$\begin{aligned} C_{A\times B}((x,y)) = C_A(x) \wedge C_B(y) &= C_C(g_1^{-1}xg_1) \wedge C_D(g_2^{-1}yg_2) \\ &= C_{C\times D}((g_1^{-1}xg_1),(g_2^{-1}yg_2)) \\ &= C_{C\times D}((g_1^{-1},g_2^{-1})(x,y)(g_1,g_2)) \\ &= C_{C\times D}((g_1,g_2)^{-1}(x,y)(g_1,g_2)). \end{aligned}$$

Hence, $C_{A\times B}((x,y)) = C_{C\times D}((g_1,g_2)^{-1}(x,y)(g_1,g_2))$. This completes the proof. □

§4. Generalized Direct Product of Multigroups

In this section, we defined direct product of k^{th} multigroups and obtain some results which generalized the results in Section 3.

Definition 4.1 *Let $A_1, A_2, \cdots, A_k$ be multigroups of $X_1, X_2, \cdots, X_k$, respectively. Then the direct product of $A_1, A_2, \cdots, A_k$ is a function*

$$C_{A_1\times A_2\times\cdots\times A_k} : X_1 \times X_2 \times \cdots \times X_k \to \mathbb{N}$$

defined by

$$C_{A_1\times A_2\times\cdots\times A_k}(x) = C_{A_1}(x_1) \wedge C_{A_2}(x_2) \wedge \cdots \wedge C_{A_{k-1}}(x_{k-1}) \wedge C_{A_k}(x_k)$$

where $x = (x_1, x_2, \cdots, x_{k-1}, x_k)$, $\forall x_1 \in X_1, \forall x_2 \in X_2, \cdots, \forall x_k \in X_k$. If we denote $A_1, A_2, \cdots, A_k$ by $A_i, (i \in I), X_1, X_2, \cdots, X_k$ by $X_i, (i \in I), A_1\times A_2\times\cdots\times A_k$ by $\prod_{i=1}^k A_i$ and $X_1\times X_2\times\cdots\times X_k$ by $\prod_{i=1}^k X_i$. Then the direct product of A_i is a function

$$C_{\prod_{i=1}^k A_i} : \prod_{i=1}^k X_i \to \mathbb{N}$$

defined by

$$C_{\prod_{i=1}^k A_i}((x_i)_{i\in I}) = \wedge_{i\in I} C_{A_i}((x_i))\ \forall x_i \in X_i,\ I = 1, \cdots, k.$$

Unless otherwise specified, it is assumed that X_i is a group with identity e_i for all $i \in I$, $X = \prod_{i\in I}^k X_i$, and so $e = (e_i)_{i\in I}$.

Theorem 4.2 *Let $A_1, A_2, \cdots, A_k$ be multisets of the sets $X_1, X_2, \cdots, X_k$, respectively and let $n \in \mathbb{N}$. Then*

$$(A_1 \times A_2 \times \cdots \times A_k)_{[n]} = A_{1[n]} \times A_{2[n]} \times \cdots \times A_{k[n]}.$$

Proof Let $(x_1, x_2, \cdots, x_k) \in (A_1 \times A_2 \times \cdots \times A_k)_{[n]}$. From Definition 2.5, we have

$$C_{A_1\times A_2\times\cdots\times A_k}((x_1, x_2, \cdots, x_k)) = (C_{A_1}(x_1) \wedge C_{A_2}(x_2) \wedge \cdots \wedge C_{A_k}(x_k)) \geq n.$$

This implies that $C_{A_1}(x_1) \geq n$, $C_{A_2}(x_2) \geq n, \cdots, C_{A_k}(x_k) \geq n$ and $x_1 \in A_{1[n]}$, $x_2 \in A_{2[n]}, \cdots, x_k \in A_{k[n]}$. Thus, $(x_1, x_2, \cdots, x_k) \in A_{1[n]} \times A_{2[n]} \times \cdots \times A_{k[n]}$.

Again, let $(x_1, x_2, \cdots, x_k) \in A_{1[n]} \times A_{2[n]} \times \cdots \times A_{k[n]}$. Then $x_i \in A_{i[n]}$, for $i = 1, 2, \cdots, k$, $C_{A_1}(x_1) \geq n$, $C_{A_2}(x_2) \geq n, \cdots, C_{A_k}(x_k) \geq n$. That is,

$$(C_{A_1}(x_1) \wedge C_{A_2}(x_2) \wedge \cdots \wedge C_{A_k}(x_k)) \geq n.$$

Implies that

$$(x_1, x_2, \cdots, x_k) \in (A_1 \times A_2 \times ... \times A_k)_{[n]}.$$

Hence, $(A_1 \times A_2 \times \cdots \times A_k)_{[n]} = A_{1[n]} \times A_{2[n]} \times \cdots \times A_{k[n]}$. □

Corollary 4.3 *Let $A_1, A_2, \cdots, A_k$ be multisets of the sets $X_1, X_2, \cdots, X_k$, respectively and let $n \in \mathbb{N}$. Then*

(i) $(A_1 \times A_2 \times \cdots \times A_k)^{[n]} = A_1^{[n]} \times A_2^{[n]} \times \cdots \times A_k^{[n]}$;
(ii) $(A_1 \times A_2 \times \cdots \times A_k)^* = A_1^* \times A_2^* \times \cdots \times A_k^*$;
(iii) $(A_1 \times A_2 \times \cdots \times A_k)_* = A_{1*} \times A_{2*} \times \cdots \times A_{k*}$.

Proof Straightforward from Theorem 4.2. □

Theorem 4.4 *Let $A_1, A_2, \cdots, A_k$ be multigroups of the groups $X_1, X_2, \cdots, X_k$, respectively. Then $A_1 \times A_2 \times \cdots \times A_k$ is a multigroup of $X_1 \times X_2 \times \cdots \times X_k$.*

Proof Let $(x_1, x_2, \cdots, x_k), (y_1, y_2, \cdots, y_k) \in X_1 \times X_2 \times \cdots \times X_k$. We get

$$\begin{aligned}
&C_{A_1 \times \cdots \times A_k}((x_1, \cdots, x_k)(y_1, \cdots, y_k)) \\
&= C_{A_1 \times \cdots \times A_k}((x_1 y_1, \cdots, x_k y_k)) \\
&= C_{A_1}(x_1 y_1) \wedge \cdots \wedge C_{A_k}(x_k y_k) \\
&\geq (C_{A_1}(x_1) \wedge C_{A_1}(y_1)) \wedge \cdots \wedge (C_{A_k}(x_k) \wedge C_{A_k}(y_k)) \\
&= \wedge(\wedge(C_{A_1}(x_1), C_{A_1}(y_1)), \cdots, \wedge(C_{A_k}(x_k), C_{A_k}(y_k)) \\
&= \wedge(\wedge(C_{A_1}(x_1), \cdots, C_{A_k}(x_k)), \wedge(C_{A_1}(y_1), \cdots, C_{A_k}(y_k))) \\
&= C_{A_1 \times \cdots \times A_k}((x_1, \cdots, x_k)) \wedge C_{A_1 \times \cdots \times A_k}((y_1, \cdots, y_k)).
\end{aligned}$$

Also,

$$\begin{aligned}
C_{A_1 \times \cdots \times A_k}((x_1, \cdots, x_k)^{-1}) &= C_{A_1 \times \cdots \times A_k}((x_1^{-1}, \cdots, x_k^{-1})) \\
&= C_{A_1}(x_1^{-1}) \wedge \cdots \wedge C_{A_k}(x_k^{-1}) \\
&= C_{A_1}(x_1) \wedge \cdots \wedge C_{A_k}(x_k) \\
&= C_{A_1 \times \cdots \times A_k}((x_1, \cdots, x_k))
\end{aligned}$$

Hence, $A_1 \times A_2 \times \cdots \times A_k$ is a multigroup of $X_1 \times X_2 \times \cdots \times X_k$. □

Corollary 4.5 *Let $A_1, A_2, \cdots, A_k$ and $B_1, B_2, \cdots, B_k$ be multigroups of $X_1, X_2, \cdots, X_k$, re-*

spectively, such that $A_1, A_2, \cdots, A_k \subseteq B_1, B_2, \cdots, B_k$. *If* $A_1, A_2, \cdots, A_k$ *are normal submultigroups of* $B_1, B_2, \cdots, B_k$, *then* $A_1 \times A_2 \times \cdots \times A_k$ *is a normal submultigroup of* $B_1 \times B_2 \times \cdots \times B_k$.

Proof By Theorem 4.4, $A_1 \times A_2 \times \cdots \times A_k$ is a multigroup of $X_1, X_2, \cdots, X_k$. Also, $B_1 \times B_2 \times \cdots \times B_k$ is a multigroup of $X_1, X_2, \cdots, X_k$.

Let $(x_1, x_2, \cdots, x_k), (y_1, y_2, \cdots, y_k) \in X_1 \times X_2 \times \cdots \times X_k$. Then we get

$$\begin{aligned} C_{A_1 \times \cdots \times A_k}((x_1, \cdots, x_k)(y_1, \cdots, y_k)) &= C_{A_1 \times \cdots \times A_k}((x_1 y_1, \cdots, x_k y_k)) \\ &= C_{A_1}(x_1 y_1) \wedge \cdots \wedge C_{A_k}(x_k y_k) \\ &= C_{A_1}(y_1 x_1) \wedge \cdots \wedge C_{A_k}(y_k x_k) \\ &= C_{A_1 \times \cdots \times A_k}((y_1 x_1, \cdots, y_k x_k)) \\ &= C_{A_1 \times \cdots \times A_k}((y_1, \cdots, y_k)(x_1, \cdots, x_k)). \end{aligned}$$

Thus, $A_1 \times \cdots \times A_k$ is a normal submultigroup of $B_1 \times \cdots \times B_k$ by Proposition 2.11. □

Theorem 4.6 *If* $A_1, A_2, \cdots, A_k$ *are multigroups of* $X_1, X_2, \cdots, X_k$, *respectively, then*

(*i*) $(A_1 \times A_2 \times \cdots \times A_k)_*$ *is a subgroup of* $X_1 \times X_2 \times \cdots \times X_k$;

(*ii*) $(A_1 \times A_2 \times \cdots \times A_k)^*$ *is a subgroup of* $X_1 \times X_2 \times \cdots \times X_k$;

(*iii*) $(A_1 \times A_2 \times \cdots \times A_k)_{[n]}, n \in \mathbb{N}$ *is a subgroup of* $X_1 \times X_2 \times \cdots \times X_k$, $\forall\, n \leq C_{A_1}(e_1) \wedge C_{A_2}(e_2) \wedge \cdots \wedge C_{A_k}(e_k)$;

(*iv*) $(A_1 \times A_2 \times \cdots \times A_k)^{[n]}, n \in \mathbb{N}$ *is a subgroup of* $X_1 \times X_2 \times \cdots \times X_k$, $\forall\, n \geq C_{A_1}(e_1) \wedge C_{A_2}(e_2) \wedge \cdots \wedge C_{A_k}(e_k)$.

Proof Combining Proposition 2.8, Theorem 2.9 and Theorem 4.4, the results follow. □

Corollary 4.7 *Let* $A_1, A_2, \cdots, A_k$ *and* $B_1, B_2, \cdots, B_k$ *be multigroups of* $X_1, X_2, \cdots, X_k$ *such that* $A_1, A_2, \cdots, A_k \subseteq B_1, B_2, \cdots, B_k$. *If* $A_1, A_2, \cdots, A_k$ *are normal submultigroups of* $B_1, B_2, \cdots, B_k$, *then*

(*i*) $(A_1 \times A_2 \times \cdots \times A_k)_*$ *is a normal subgroup of* $(B_1 \times B_2 \times \cdots \times B_k)_*$;

(*ii*) $(A_1 \times A_2 \times \cdots \times A_k)^*$ *is a normal subgroup of* $(B_1 \times B_2 \times \cdots \times B_k)^*$;

(*iii*) $(A_1 \times A_2 \times \cdots \times A_k)_{[n]}, n \in \mathbb{N}$ *is a normal subgroup of* $(B_1 \times B_2 \times \cdots \times B_k)_{[n]}$, $\forall\, n \leq C_{A_1}(e_1) \wedge C_{A_2}(e_2) \wedge \cdots \wedge C_{A_k}(e_k)$;

(*iv*) $(A_1 \times A_2 \times \cdots \times A_k)^{[n]}, n \in \mathbb{N}$ *is a normal subgroup of* $(B_1 \times B_2 \times \cdots \times B_k)^{[n]}$, $\forall\, n \geq C_{A_1}(e_1) \wedge C_{A_2}(e_2) \wedge \cdots \wedge C_{A_k}(e_k)$.

Proof Combining Proposition 2.8, Theorem 2.9, Theorem 4.4 and Corollary 4.5, the results follow. □

Theorem 4.8 *Let* $A_1, A_2, \cdots, A_k$ *and* $B_1, B_2, \cdots, B_k$ *be multigroups of groups* $X_1, X_2, \cdots, X_k$, *respectively. If* $A_1, A_2, \cdots, A_k$ *are conjugate to* $B_1, B_2, \cdots, B_k$, *then the multigroup* $A_1 \times A_2 \times \cdots \times A_k$ *of* $X_1 \times X_2 \times \cdots \times X_k$ *is conjugate to the multigroup* $B_1 \times B_2 \times \cdots \times B_k$ *of* $X_1 \times X_2 \times \cdots \times X_k$.

Proof By Definition 2.12, if multigroup A_i of X_i conjugates to multigroup B_i of X_i, then exist $x_i \in X_i$ such that for all $y_i \in X_i$,

$$C_{A_i}(y_i) = C_{B_i}(x_i^{-1} y_i x_i), i = 1, 2, \cdots, k.$$

Then we have

$$\begin{aligned} C_{A_1 \times \cdots \times A_k}((y_1, \cdots, y_k)) &= C_{A_1}(y_1) \wedge \cdots \wedge C_{A_k}(y_k) \\ &= C_{B_1}(x_1^{-1} y_1 x_1) \wedge \cdots \wedge C_{B_k}(x_k^{-1} y_k x_k) \\ &= C_{B_1 \times \cdots \times B_k}((x_1^{-1} y_1 x_1, \cdots, x_k^{-1} y_k x_k)). \end{aligned}$$

This completes the proof. □

Theorem 4.9 *Let $A_1, A_2, \cdots, A_k$ be multisets of the groups $X_1, X_2, \cdots, X_k$, respectively. Suppose that $e_1, e_2, \cdots, e_k$ are identities elements of $X_1, X_2, \cdots, X_k$, respectively. If $A_1 \times A_2 \times \cdots \times A_k$ is a multigroup of $X_1 \times X_2 \times \cdots \times X_k$, then for at least one $i = 1, 2, \cdots, k$, the statement*

$$C_{A_1 \times A_2 \times \cdots \times A_{i-1} \times A_{i+1} \times \cdots \times A_k}((e_1, e_2, \cdots, e_{i-1}, e_{i+1}, \cdots, e_k)) \geq C_{A_i}((x_i)), \quad \forall x_i \in X_i$$

holds.

Proof Let $A_1 \times A_2 \times \cdots \times A_k$ be a multigroup of $X_1 \times X_2 \times \cdots \times X_k$. By contraposition, suppose that for none of $i = 1, 2, \cdots, k$, the statement holds. Then we can find $(a_1, a_2, \cdots, a_k) \in X_1 \times X_2 \times \cdots \times X_k$, respectively, such that

$$C_{A_i}((a_i)) > C_{A_1 \times A_2 \times \cdots \times A_{i-1} \times A_{i+1} \times \cdots \times A_k}((e_1, e_2, \cdots, e_{i-1}, e_{i+1}, \cdots, e_k)).$$

Then we have

$$\begin{aligned} C_{A_1 \times \cdots \times A_k}((a_1, \cdots, a_k)) &= C_{A_1}(a_1) \wedge \cdots \wedge C_{A_k}(a_k) \\ &> C_{A_1 \times \cdots \times A_{i-1} \times A_{i+1} \times \cdots \times A_k}((e_1, \cdots, e_{i-1}, e_{i+1}, \cdots, e_k)) \\ &= C_{A_1}(e_1) \wedge \cdots \wedge C_{A_{i-1}}(e_{i-1}) \wedge C_{A_{i+1}}(e_{i+1}) \wedge \cdots \wedge C_{A_k}(e_k) \\ &= C_{A_1}(e_1) \wedge \cdots \wedge C_{A_k}(e_k) \\ &= C_{A_1 \times \cdots \times A_k}((e_1, ..., e_k)). \end{aligned}$$

So, $A_1 \times A_2 \times ... \times A_k$ is not a multigroup of $X_1 \times X_2 \times \cdots \times X_k$. Hence, for at least one $i = 1, 2, \cdots, k$, the inequality

$$C_{A_1 \times \cdots \times A_{i-1} \times A_{i+1} \times \cdots \times A_k}((e_1, \cdots, e_{i-1}, e_{i+1}, \cdots, e_k)) \geq C_{A_i}((x_i))$$

is satisfied for all $x_i \in X_i$. □

Theorem 4.10 *Let $A_1, A_2, \cdots, A_k$ be multisets of the groups $X_1, X_2, \cdots, X_k$, respectively, such that*

$$C_{A_i}((x_i)) \leq C_{A_1 \times A_2 \times \cdots \times A_{i-1} \times A_{i+1} \times \cdots \times A_k}((e_1, e_2, \cdots, e_{i-1}, e_{i+1}, \cdots, e_k))$$

$\forall x_i \in X_i$, e_i being the identity element of X_i. If $A_1 \times A_2 \times \cdots \times A_k$ is a multigroup of $X_1 \times X_2 \times \cdots \times X_k$, then A_i is a multigroup of X_i.

Proof Let $A_1 \times A_2 \times \cdots \times A_k$ be a multigroup of $X_1 \times X_2 \times \cdots \times X_k$ and $x_i, y_i \in X_i$. Then

$$(e_1, \cdots, e_{i-1}, x_i, e_{i+1}, \cdots, e_k), (e_1, \cdots, e_{i-1}, y_i, e_{i+1}, \cdots, e_k) \in X_1 \times X_2 \times \cdots \times X_k.$$

Now, using the given inequality, we have

$$\begin{aligned}
C_{A_i}((x_i y_i)) &= C_{A_i}((x_i y_i)) \wedge C_{A_1 \times \cdots \times A_{i-1} \times A_{i+1} \times \cdots \times A_k}((e_1, \cdots, e_{i-1}, e_{i+1}, \cdots, e_k) \\
&\quad (e_1, \cdots, e_{i-1}, e_{i+1}, \cdots, e_k)) \\
&= C_{A_1 \times \cdots \times A_i \times \cdots \times A_k}((e_1, \cdots, x_i, \cdots, e_k)(e_1, \cdots, y_i, \cdots, e_k)) \\
&\geq C_{A_1 \times \cdots \times A_i \times \cdots \times A_k}((e_1, \cdots, x_i, \cdots, e_k)) \wedge C_{A_1 \times \cdots \times A_i \times \cdots \times A_k}((e_1, \cdots, y_i, \cdots, e_k)) \\
&= \wedge (C_{A_i}((x_i)) \wedge C_{A_1 \times \cdots \times A_{i-1} \times A_{i+1} \times \cdots \times A_k}((e_1, \cdots, e_{i-1}, e_{i+1}, \cdots, e_k)), C_{A_i}((y_i)) \\
&\quad \wedge C_{A_1 \times \cdots \times A_{i-1} \times A_{i+1} \times \cdots \times A_k}((e_1, \cdots, e_{i-1}, e_{i+1}, \cdots, e_k)) \\
&= C_{A_i}((x_i)) \wedge C_{A_i}((y_i)).
\end{aligned}$$

Also,

$$\begin{aligned}
C_{A_i}((x_i^{-1})) &= C_{A_i}((x_i^{-1})) \wedge C_{A_1 \times \cdots \times A_{i-1} \times A_{i+1} \times \cdots \times A_k}((e_1^{-1}, \cdots, e_{i-1}^{-1}, e_{i+1}^{-1}, \cdots, e_k^{-1})) \\
&= C_{A_1 \times \cdots \times A_i \times \cdots \times A_k}((e_1^{-1}, \cdots, x_i^{-1}, \cdots, e_k^{-1})) \\
&= C_{A_1 \times \cdots \times A_i \times \cdots \times A_k}((e_1, \cdots, x_i, \cdots, e_k)^{-1}) \\
&= C_{A_1 \times \cdots \times A_i \times \cdots \times A_k}((e_1, \cdots, x_i, \cdots, e_k)) \\
&= C_{A_i}((x_i)) \wedge C_{A_1 \times \cdots \times A_{i-1} \times A_{i+1} \times \cdots \times A_k}((e_1, \cdots, e_{i-1}, e_{i+1}, \cdots, e_k)) \\
&= C_{A_i}((x_i)).
\end{aligned}$$

Hence, $A_i \in MG(X_i)$. □

Theorem 4.11 *Let $A_1, A_2, \cdots, A_k$ be multisets of the groups $X_1, X_2, \cdots, X_k$, respectively, such that*

$$C_{A_1 \times A_2 \times \cdots \times A_{i-1} \times A_{i+1} \times \cdots \times A_k}((x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_k)) \leq C_{A_i}((e_i))$$

for $\forall (x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_k) \in X_1 \times X_2 \times \cdots \times X_{i-1} \times X_{i+1} \times \cdots \times X_k$, e_i being the identity element of X_i. If $A_1 \times A_2 \times \cdots \times A_k$ is a multigroup of $X_1 \times X_2 \times \cdots \times X_k$, then $A_1 \times A_2 \times \cdots \times A_{i-1} \times A_{i+1} \times \cdots \times A_k$ is a multigroup of $X_1 \times X_2 \times \cdots \times X_{i-1} \times X_{i+1} \times \cdots \times X_k$.

Proof Let $A_1 \times A_2 \times \cdots \times A_k$ be a multigroup of $X_1 \times X_2 \times \cdots \times X_k$ and $(x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_k), (y_1, y_2, \cdots, y_{i-1}, y_{i+1}, \cdots, y_k) \in X_1 \times X_2 \times \cdots \times X_{i-1} \times X_{i+1} \times \cdots \times X_k$. Then

$$(x_1, \cdots, x_{i-1}, e_i, x_{i+1}, \cdots, x_k), (y_1, \cdots, y_{i-1}, e_i, y_{i+1}, \cdots, y_k) \in X_i.$$

Using the given inequality, we arrive at

$$\begin{aligned}
&C_{A_1\times\cdots\times A_{i-1}\times A_{i+1}\times\cdots\times A_k}((x_1,\cdots,x_{i-1},x_{i+1},\cdots,x_k)(y_1,\cdots,y_{i-1},y_{i+1},\cdots,y_k))\\
&= C_{A_1\times\cdots\times A_{i-1}\times A_{i+1}\times\cdots\times A_k}((x_1,\cdots,x_{i-1},x_{i+1},\cdots,x_k)(y_1,\cdots,y_{i-1},y_{i+1},\cdots,y_k))\\
&\quad \wedge C_{A_i}((e_i)) = C_{A_1\times\cdots\times A_i\times\cdots\times A_k}((x_1,\cdots,e_i,\cdots,x_k)(y_1,\cdots,e_i,\cdots,y_k))\\
&\geq C_{A_1\times\cdots\times A_i\times\cdots\times A_k}((x_1,\cdots,e_i,\cdots,x_k))\wedge C_{A_1\times\cdots\times A_i\times\cdots\times A_k}((y_1,\cdots,e_i,\cdots,y_k))\\
&= \wedge(C_{A_i}((e_i))\wedge C_{A_1\times\cdots\times A_{i-1}\times A_{i+1}\times\cdots\times A_k}((x_1,\cdots,x_{i-1},x_{i+1},\cdots,x_k)),C_{A_i}((e_i))\\
&\quad \wedge C_{A_1\times\cdots\times A_{i-1}\times A_{i+1}\times\cdots\times A_k}((y_1,\cdots,y_{i-1},y_{i+1},\cdots,y_k))) = C_{A_1\times\cdots\times A_{i-1}\times A_{i+1}\times\cdots\times A_k}\\
&\quad ((x_1,\cdots,x_{i-1},x_{i+1},\cdots,x_k))\wedge C_{A_1\times\cdots\times A_{i-1}\times A_{i+1}\times\cdots\times A_k}((y_1,y_2,\cdots,y_{i-1},y_{i+1},\cdots,y_k)).
\end{aligned}$$

Again,

$$\begin{aligned}
&C_{A_1\times\cdots\times A_{i-1}\times A_{i+1}\times\cdots\times A_k}((x_1^{-1},\cdots,x_{i-1}^{-1},x_{i+1}^{-1},\cdots,x_k^{-1}))\\
&= C_{A_1\times\cdots\times A_{i-1}\times A_{i+1}\times\cdots\times A_k}((x_1^{-1},\cdots,x_{i-1}^{-1},x_{i+1}^{-1},\cdots,x_k^{-1}))\wedge C_{A_i}((e_i^{-1}))\\
&= C_{A_1\times\cdots\times A_i\times\cdots\times A_k}((x_1^{-1},\cdots,e_i^{-1},\cdots,x_k^{-1}))\\
&= C_{A_1\times\cdots\times A_i\times\cdots\times A_k}((x_1,\cdots,e_i,\cdots,x_k)^{-1}) = C_{A_1\times\cdots\times A_i\times\cdots\times A_k}((x_1,\cdots,e_i,\cdots,x_k))\\
&= C_{A_1\times\cdots\times A_{i-1}\times A_{i+1}\times\cdots\times A_k}((x_1,\cdots,x_{i-1},x_{i+1},\cdots,x_k))\wedge C_{A_i}((e_i))\\
&= C_{A_1\times\cdots\times A_{i-1}\times A_{i+1}\times\cdots\times A_k}((x_1,\cdots,x_{i-1},x_{i+1},\cdots,x_k)).
\end{aligned}$$

Hence, $A_1\times A_2\times\cdots\times A_{i-1}\times A_{i+1}\times\cdots\times A_k$ is the multigroup of $X_1\times X_2\times\cdots\times X_{i-1}\times X_{i+1}\times\cdots\times X_k$. □

§5. Homomorphism of Direct Product of Multigroups

In this section, we present some homomorphic properties of direct product of multigroups. This is an extension of the notion of homomorphism in multigroup setting (cf. [6, 12]) to direct product of multigroups.

Definition 5.1 *Let $W\times X$ and $Y\times Z$ be groups and let $f: W\times X\to Y\times Z$ be a homomorphism. Suppose $A\times B\in MS(W\times X)$ and $C\times D\in MS(Y\times Z)$, respectively. Then*

(i) the image of $A\times B$ under f, denoted by $f(A\times B)$, is a multiset of $Y\times Z$ defined by

$$C_{f(A\times B)}((y,z)) = \begin{cases} \bigvee_{(w,x)\in f^{-1}((y,z))} C_{A\times B}((w,x)), & f^{-1}((y,z))\neq\emptyset\\ 0, & \textit{otherwise},\end{cases}$$

for each $(y,z)\in Y\times Z$;

(ii) the inverse image of $C\times D$ under f, denoted by $f^{-1}(C\times D)$, is a multiset of $W\times X$ defined by

$$C_{f^{-1}(C\times D)}((w,x)) = C_{C\times D}(f((w,x)))\ \forall (w,x)\in W\times X.$$

Theorem 5.2 *Let W, X, Y, Z be groups, $A \in MS(W), B \in MS(X), C \in MS(Y)$ and $D \in MS(Z)$. If $f : W \times X \to Y \times Z$ is a homomorphism, then*

(i) $f(A \times B) \subseteq f(A) \times f(B)$;
(ii) $f^{-1}(C \times D) = f^{-1}(C) \times f^{-1}(D)$.

Proof (i) Let $(w, x) \in W \times X$. Suppose $\exists\,(y, z) \in Y \times Z$ such that

$$f((w, x)) = (f(w), f(x)) = (y, z).$$

Then we get

$$\begin{aligned} C_{f(A\times B)}((y,z)) &= C_{A\times B}(f^{-1}((y,z))) \\ &= C_{A\times B}((f^{-1}(y), f^{-1}(z))) \\ &= C_A(f^{-1}(y)) \wedge C_B(f^{-1}(z)) \\ &= C_{f(A)}(y) \wedge C_{f(B)}(z) \\ &= C_{f(A)\times f(B)}((y,z)) \end{aligned}$$

Hence, we conclude that, $f(A \times B) \subseteq f(A) \times f(B)$.

(ii) For $(w, x) \in W \times X$, we have

$$\begin{aligned} C_{f^{-1}(C\times D)}((w,x)) &= C_{C\times D}(f((w,x))) \\ &= C_{C\times D}((f(w), f(x))) \\ &= C_C(f(w)) \wedge C_D(f(x)) \\ &= C_{f^{-1}(C)}(w) \wedge C_{f^{-1}(D)}(x) \\ &= C_{f^{-1}(C)\times f^{-1}(D)}((w,x)). \end{aligned}$$

Hence, $f^{-1}(C \times D) \subseteq f^{-1}(C) \times f^{-1}(D)$.

Similarly,

$$\begin{aligned} C_{f^{-1}(C)\times f^{-1}(D)}((w,x)) &= C_{f^{-1}(C)}(w) \wedge C_{f^{-1}(D)}(x) \\ &= C_C(f(w)) \wedge C_D(f(x)) \\ &= C_{C\times D}((f(w), f(x))) \\ &= C_{C\times D}(f((w,x))) \\ &= C_{f^{-1}(C\times D)}((w,x)). \end{aligned}$$

Again, $f^{-1}(C) \times f^{-1}(D) \subseteq f^{-1}(C \times D)$. Therefore, the result follows. □

Theorem 5.3 *Let $f : W \times X \to Y \times Z$ be an isomorphism, A, B, C and D be multigroups of W, X, Y and Z, respectively. Then the following statements hold:*

(i) $f(A \times B) \in MG(Y \times Z)$;
(ii) $f^{-1}(C) \times f^{-1}(D) \in MG(W \times X)$.

Proof (i) Since $A \in MG(W)$ and $B \in MG(X)$, then $A \times B \in MG(W \times X)$ by Theorem 3.7. From Proposition 2.14 and Definition 5.1, it follows that, $f(A \times B) \in MG(Y \times Z)$.

(ii) Combining Corollary 2.15, Theorem 3.7, Definition 5.1 and Theorem 5.2, the result follows. □

Corollary 5.4 *Let X and Y be groups, $A \in MG(X)$ and $B \in MG(Y)$. If*

$$f : X \times X \to Y \times Y$$

be homomorphism, then

(i) $f(A \times A) \in MG(Y \times Y)$;
(ii) $f^{-1}(B \times B) \in MG(X \times X)$.

Proof Straightforward from Theorem 5.3. □

Proposition 5.5 *Let $X_1, X_2, \cdots, X_k$ and $Y_1, Y_2, \cdots, Y_k$ be groups, and*

$$f : X_1 \times X_2 \times \cdots \times X_k \to Y_1 \times Y_2 \times \cdots \times Y_k$$

be homomorphism. If $A_1 \times A_2 \times \cdots \times A_k \in MG(X_1 \times X_2 \times \cdots \times X_k)$ and $B_1 \times B_2 \times \cdots \times B_k \in MG(Y_1 \times Y_2 \times \cdots \times Y_k)$, then

(i) $f(A_1 \times A_2 \times \cdots \times A_k) \in MG(Y_1 \times Y_2 \times \cdots \times Y_k)$;
(ii) $f^{-1}(B_1 \times B_2 \times \cdots \times B_k) \in MG(X_1 \times X_2 \times \cdots \times X_k)$.

Proof Straightforward from Corollary 5.4. □

§6. Conclusions

The concept of direct product in groups setting has been extended to multigroups. We lucidly exemplified direct product of multigroups and deduced several results. The notion of generalized direct product of multigroups was also introduced in the case of finitely k^{th} multigroups. Finally, homomorphism and some of its properties were proposed in the context of direct product of multigroups.

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