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Equal Degree Graphs of Simple Graphs

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Abstract: This paper introduces equal degree graphs of simple existed graphs. These graphs exhibited some properties which are co-related with the older one. We characterize graphs for which their equal degree graphs are connected, completed, disconnected but not totally disconnected. We also obtain several properties of equal degree graphs and specify which graphs are isomorphic to equal degree graphs and complement of equal degree graphs. Furthermore, the relation between equal degree graphs and degree Prime graphs is determined.

Key Words: Parameters of the graph, simple graphs, equal degree graphs, degree prime graphs, degree graph isomorphism, Smarandachely k-degree graph, Smarandachely degree graph.

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§1. Introduction

The evaluations of new graphs are involving sets of objects and binary relations among them. So the construction and preparation of graphs varies author to author, and thus it is difficult to pin point its formulation to a single source. Thus the graphs discovered many times, and each discovery being independent of the other. For this reason, there are various types of graphs each with its own definition.

Many authors, starting from 2003, the parameters vertex degree and degree sequence of graphs were used again in Graph theory, and several types of graphs have been introduced. In this sequel we have introduced equal degree graphs of various simple graphs and characterized their properties.

For any finite group G, the definition and notation of degree graph of a simple group was introduced by Lewis and white [5]. This graph is defined as follows: Let G be a finite group and let $cd(G)$ be the set of irreducible character degrees of G. Then the degree graph $\triangle(G)$ is the graph whose set of vertices is set of primes that divide degrees in $cd(G)$, with an edge between p and q if pq divides a for some $a \in cd(G)$.

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In [6], the authors introduced the degree pattern of a finite group G and denoted by $D(G)$, where $D(G) = (deg(p_1), deg(p_2), \cdots, deg(p_k))$, here $p_1 < p_2 < \cdots < p_k$ are distinct primes in prime decomposition of n. Further, the authors S.F.Kapur, Albert and Curtiss [7] introduced the notation $D(G)$ for degree sets of connected graphs, trees, planner and outer planner graphs. According to these authors, the notation $D(G)$ is a degree set of degrees of vertices of G.

The authors Manoussakis, Patil and Sankar [8] was proved that for any finite non empty set S of non negative integers, there exist a disconnected graph G such that $D(G) = S$, and also the minimum order of such a graph is determined.

There are several ways to produce new graphs from the existing graphs in Graph theory. Recently the authors M.Sattanathan and R.Kala [1] introduced a special way to produce the degree prime graph $DP(G)$ for any finite simple undirected graph G. The $PD(G)$ of a graph G having the same vertex set of G and two vertices are adjacent in $PD(G)$ if and only if their unequal degrees are relatively prime in G. By the motivation of these degree prime graphs, we construct and study the equal degree graphs of simple graphs with usual notation $D(G)$. We suspect that these graphs will be used to solve many computational problems in computer engineering and applied sciences.

Throughout this paper, G and $D(G)$ represent finite simple undirected graphs having without loops and without multiple edges of same order. We have introduced degree graph $D(G)$ of G, which is defined as a graph with same vertex set as G and two vertices of G are adjacent in $D(G)$ if and only if their degrees are equal in G. In this paper we studied interrelations between G and $D(G)$, and hence we obtain several properties and their consequences of $D(G)$ with illustrations and examples. Further we characterize G for which $D(G)$ either is connected, disconnected, totally disconnected or complete.

§2. Basic Definitions and Notations

In this section we consider basic definitions and their graph theoretical notations. Throughout the text, we consider G is an abstract graph structure which is a finite undirected graph without loops and multiple edges. We represent G as $G = G(V, E)$ with vertex set $V = V(G)$ and edge set $E = E(G)$. We are only going to deal with finite graphs, so we define $|V| = n$ to the order of G and $|E| = m$ to be the size of G where n and m are called graph parameters. Further if there is an edge e in G between the vertices u and v, we briefly write $e = uv$ or $e = (u, v)$ and say edge e joins the vertices u and v. A vertex is said to be isolated if it is not adjacent to any other vertex.

The complement of a graph $G(V, E)$ is the graph $G^c(V, E^c)$ having the same vertex set as G, and its edge set E^c is the complement of E, that is, uv is an edge of G^c if and only if uv is not an edge of G. A complete graph of order n is denoted by K_n. A graph of order n with no edges in an empty graph and is denoted by $N_n = K_n^c$ which isomorphic to totally disconnected. A path of length n is denoted by P_n. A cycle of length $n(n \geqslant 3)$ is a cycle of length n and is denoted by C_n.

We now turn to graphs whose vertex sets can be partitioned in special ways. A graph G is a partite graph if $V(G)$ can be partitioned into subsets, called partite sets. A graph G is a $k-$partite graph if $V(G)$ can be partitioned into k subsets $V_1, V_2, \cdots, V_k$ (partite sets) such that uv is an edge of G if u and v belongs to different partite sets. In addition, if every two vertices in different partite sets are joined by an edge, then G is a complete $k-$partite graph. If $|V_i| = n_i$ for $1 \leqslant i \leqslant k$, then we denote this complete $k-$partite graph by $K_{n_1,n_2,\cdots,n_k}$. Thus

(1) $K_{1,1,\cdots,1} \cong K_n$;
(2) $K_{n,n}$ is a complete bipartite;
(3) $K_{1,n}$ is a Star;
(4) K_{n_1,n_2} is a complete bipartite.

In a graph G, the degree of a vertex u is the number of edges of G which are incident to u and denoted by $d(u)$, $deg(u)$ or $deg_G(u)$. A graph is regular if all its vertices are of the same degree and $r-$regular if all of its vertices are of degree r. A $3-$regular graph is a cubic graph Q_8. If $d_i, 1 \leqslant i \leqslant n$, be the degree of the vertices v_i of a graph G then the sequence $d(G) = (d_1, d_2, \cdots, d_n)$ is the degree sequence of G. Usually, we order the vertices so that the degree sequence is monotonically increasing, that is, $\delta(G) = d_1 \leq d_2 \leq \cdots \leq d_n = \triangle(G)$. Also two graphs with the same degree sequence are said to be degree equivalent.

We need the following results for preparing equal degree graphs.

Theorem 2.1([2]) *The sum of the degrees of graphs is even, being twice the number of edges.*

Theorem 2.2([2]) *In any graph there is an even number of vertices of odd degree.*

Theorem 2.3([2]) *If $d_1, d_2, \cdots, d_n$ is the degree sequence of some graph, then, necessarily $\sum_{i=1}^{n} d_i$ is even, and $0 \leq d_i \leq n-1$ for $1 \leq i \leq n$. But converse is not true.*

Theorem 2.5([2]) *Let G be a graph of order $n \geq 2$, then $d(G)$ contains at least two numbers are same.*

Theorem 2.6([2]) *Let G be a r-regular graph of order n. Then $|E(G)| = \dfrac{rn}{2}$.*

Theorem 2.7([3]) *Let G be a r-regular graph of order n. Then G^c is $(n-r-1)$-regular graph of order n.*

Theorem 2.8([3]) *A graph is $1-$regular if and only if it is of even order and is the disjoint union of some $K_2^{'}s$.*

If $n = 1$, then G is called trivial graph, otherwise G is called non-trivial graph. In this paper we consider non-trivial graphs only. For further details and notations we refer [4].

§3. Equal Degree Graphs

We have already mentioned that the best known parameters of a graph are order and size. But another parameter of a graph is degree of a vertex which is a meaningful to have a term for the number of edges meeting at a vertex. By using this parameter we define equal degree graph as follows.

Definition 3.1 *Let G be a simple graph with vertex set $V = \{1, 2, \cdots, n\}, n > 1$, Then the equal degree graph of G is $D(G)$ having the same vertex set as G and two vertices $u, v \in V$ are adjacent in $D(G)$ if and only if $deg_G(u) = deg_G(v)$.*

Generally, we know *Smarandachely k-degree graphs D_kG* of a graph G in which vertices $u, v \in V$ are adjacent if and only if $|deg_G(u) - deg_G(v)| = k$ for integers $k \leq \Delta(G) - \delta(G)$ and *Smarandachely degree graph SG* in which u, v are adjacent if $|deg_G(u) - deg_G(v)| \geq 1$. Clearly, $D_0G = DG$. The definition of equal degree graphs should be noted that the following.

1. $D(G)$ is a non-trivial graph;
2. $D(G)$ has at least one edge;
3. $D(G)$ is also a simple undirected graph having without multiple edges.

The following Fig.1 shows that simple graphs and their equal degree graphs of order 4.

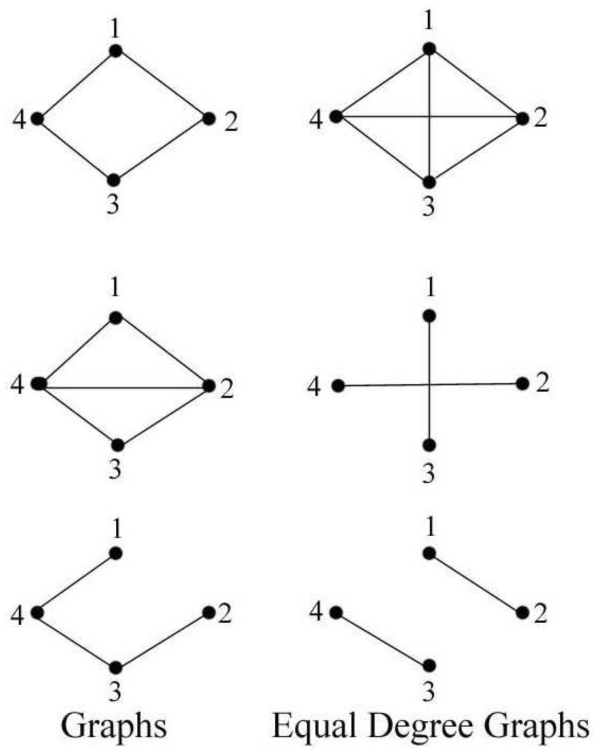

Fig.1 Graphs and their equal degree graphs

We now characterize graphs G for which $D(G)$ is either regular or not. The following propositions are immediate.

Proposition 3.2 *For any graph G, the graph $D(G)$ is never a $0-$regular graph.*

Proof Suppose $D(G)$ is a $0-$regular graph. Then its degree sequence is $d(D(G)) = (0, 0, \cdots, 0)$ for any graph G. This shows that $D(G)$ is isomorphic to empty graph, and thus $D(G)$ is a trivial graph, which is a contradiction to the definition of equal degree graphs. Thus the result follows. □

Remark 3.3 (i) $D(G)$ is 1-regular if and only if $D(G)$ is a graph of order 2. (ii) $D(G)$ is 1-regular if and only if G is a graph of order 4 and $d(G) = (0, 0, 1, 1)$ or $(2, 2, 3, 3)$.

Proposition 3.4 *For any graph G of order $n > 4$, $D(G)$ is never $1-$regular. In particular, $D(G)$ is never $2, 3, \cdots, (n-2)$-regular.*

Proof Suppose G is a graph with size $n > 4$. We show that $D(G)$ is not $1-$regular. Assume that $D(G)$ is $1-$regular. Then the degree sequence of G is $d(D(G)) = (1, 1, \cdots, 1)(n$ times).

$$(1 + 1 + \cdots + 1)(n \text{ times}) = \begin{cases} \text{even}, & \text{if n is even} \\ \text{odd}, & \text{if n is odd} \end{cases}$$

Case 1. If n is odd, then $1 + 1 + \cdots + 1(n$ times) is odd, which contradicts to the Theorem 2.2. So in this case the result is not true.

Case 2. If n is even, then $1 + 1 + \cdots + 1(n$ times) is odd, which is impossible because the graph $D(G)$ does not contain an even number of vertices of same odd degree.

From the above cases our assumption is not true, and hence $D(G)$ is never $1-$regular for any graph G of order $n > 4$. □

Remark 3.5 Similarly we can show that $D(G)$ is never 2-regular, 3-regular,$\cdots$, $(n-2)$-regular but it should be $(n-1)$-regular.

Theorem 3.6(Fundamental Theorem) *For any graph G of order n, the degree graph $D(G)$ is either complete or disconnected but not totally disconnected.*

Proof Suppose $D(G)$ is totally disconnected. Then obviously $D(G)$ is isomorphic to N_n. But by the definition of degree graph, $D(G)$ is never isomorphic to N_n. Hence $D(G)$ is not totally disconnected.

Now we prove that $D(G)$ is either complete or disconnected. Suppose $D(G)$ is disconnected. Then there is nothing to prove. If possible assume that $D(G)$ is connected, then the Proposition 3.4 and Remark 3.5 shows that $D(G)$ is $(n-1)$-regular, and hence $D(G)$ is complete. □

§4. Complete Equal Degree Graphs

In this section we are going to prove that the equal degree graphs are complete. Further we characterize graphs G for which $D(G)$ is complete and also we show that $G \cong D(G)$.

Proposition 4.1 *The degree graph of regular graph is complete.*

Proof Let V be a vertex set of r-regular graph G. Then the degree sequence of G is $d(G) = (r, r, \cdots, r)$, that is, $d(i) = r$ for each $1 \leq i \leq n$. We show that $D(G)$ is complete. For this let $i, j \in V$, $i \neq j$, then $deg(i) = r$ and $deg(j) = r$. Therefore, $deg(i) = deg(i)$, for all $i \neq j$ in $V(G)$. Thus i and j are adjacent in $D(G)$. This shows that every two distinct pair of vertices is joined by an edge in $D(G)$. Hence $D(G)$ is complete. □

This proposition has a number of useful consequences.

Corollary 4.2 *Let G be a connected graph of order $n > 4$. Then $D(G)$ is either complete or disconnected.*

Proof The proof is divided into cases following.

Case 1. Suppose G is a connected regular graph of order $n > 4$. Then, by the Proposition 4.1, $D(G)$ is complete.

Case 2. Suppose G is a connected but not regular. Then the degree sequence of G contains at least two distinct positive integers, say s and t. That is, if $u, v \in G$, then $deg(u) \neq deg(u)$ implies u is not adjacent to v in $D(G)$. Hence $D(G)$ is disconnected. □

Corollary 4.3 *For each $n > 1$, we have*

(*i*) $D(K_n) = K_n$;
(*ii*) $D(C_n) = K_n$;
(*iii*) $D(N_n) = K_n$;
(*iv*) $D(K_{n,n}) = K_{2n}$;
(*v*) $D(Q_8) = K_8$.

Proof (*i*) The complete graph K_n is $(n-1)$-regular, and thus $D(K_n) = K_n$.

(*ii*) For each $n \geq 3$, the cycle C_n is 2-regular. Hence $D(C_n) = K_n$.

(*iii*) For each, the empty graph N_n is 0-regular graph. Hence $D(N_n) = K_n$.

(*iv*) Since the completed bipartite graph $K_{n,n}$ is n-regular and the order of $K_{n,n}$ is $2n$. Thus $D(K_{n,n}) = K_{2n}$.

(*v*) Q_8 is 3-regular of order 8, and thus $D(Q_8) = K_8$. □

Corollary 4.4 *Let G^c be the complement of r-regular graph G of order n, then $D(G) = D(G^c) \cong K_n$.*

Proof We deduces this consequence from the Theorem 2.6 and Proposition 4.1 as follows.

We know that G is r-regular graph of order n if and only if G^c is $(n-r-1)$-regular graph of same order n. Thus the Proposition 4.1 shows that $D(G) \cong K_n$ if and only if $D(G^c) \cong K_n$. Hence $D(G) = D(G^c) \cong K_n$. □

Corollary 4.5 *Let G be any graph of order n. For fixed $m \in Z$, if there exists an integer n such that $deg(v) = mn$ for each vertex v of G. Then, $D(G) = K_n$.*

Proof Obviously follows from Proposition4.1, since G is mn-regular graph of order n.

We now characterize the graphs G which attain bounds for $|E(D(G))|$. We know that

$$0 \leq |E(G)| \leq \frac{n(n-1)}{2}$$

for any simple graph G of order $n \geq 1$. But the following result specifies that the bounds for $|E(D(G))|$. □

Theorem 4.6 *If G is any graph of order $n > 1$, then* $0 < |E(G)| \leq \frac{n(n-1)}{2}$.

Proof From the definition of equal degree graphs, $|V(G)| = |V(D(G))| = n$, and $n > 1$. For any non-trivial graph G, we have $deg_G u \leq (n-1)$ for each $u \in V(G)$. This is also true in $D(G)$, that is, $deg_{D(G)} u \leq (n-1)$ for each $u \in V(D(G))$. From Theorem2.1 we have

$$2|E(D(G))| = \sum_{d \in V(D(G))} deg(u) \Rightarrow 2|E(D(G))| \leqslant n(n-1) \Rightarrow |E(D(G))| \leqslant \frac{n(n-1)}{2}.$$

It is one extreme of the required inequality. At the other extreme, a degree graph $D(G)$ may possess at least one edge at all. That is, $|E(D(G))| \neq 0$. Hence

$$0 < |E(G)| \leq \frac{n(n-1)}{2}.$$ □

Remark 4.7 The above inequality says that the following two specifications for $D(G)$:

(1) $D(G)$ or $D(G^c)$ has at least one edge or at most $\binom{n}{2}$ edges;

(2) $D(G)$ is never totally disconnected. In particular, $D(G) \ncong N_n$ for each G of order $n > 1$.

Corollary 4.8 *Let G be a r-regular graph of order $n > 1$. Then*

$$|E(G)| = \frac{rn}{2} \quad \textit{and} \quad |E(D(G))| = \binom{n}{2}.$$

Proposition 4.9 *Let G be a graph of order n. Then $|E(G)| = |E(D(G))|$ if and only if G and $D(G)$ are $(r-1)$-regular graphs.*

Proof Let G be a r-regular graph of order $n > 1$. By the Corollary 4.7,

$$|E(G)| = \frac{rn}{2} \quad \text{and} \quad |E(D(G))| = \binom{n}{2}.$$

Therefore,

$$|E(G)| = |E(D(G))| \Leftrightarrow \frac{rn}{2} = \binom{n}{2} \Leftrightarrow \frac{rn}{2} = \frac{n(n-2)}{2} \Leftrightarrow = n-1 \Leftrightarrow G$$

is $(n-1)$-regular, and hence $D(G)$ is also $(n-1)$-regular. □

Proposition 4.10 *If G^c is a complement of G, then $D(G^c) = D(G)$.*

Proof Let G^c be the complement of a graph G of order $n > 1$. Then the following cases arise on the regularity of G.

Case 1. Suppose G is a regular graph. Then the result obviously follows from Proposition 4.1.

Case 2. Suppose G is not a regular graph. We show that $D(G^c) = D(G)$. If possible assume that $D(G^c) \neq D(G)$, then the following three subcases arise.

Subcase 2.1 If $V(D(G^c)) \neq V(D(G))$ then obviously $V(G^c) \neq V(G)$, which is a contradiction to the fact that $V(G^c) = V(G)$.

Subcase 2.2 If $E(D(G^c)) \neq E(D(G))$, then $V(D(G^c)) + V(D(G)) = \frac{n(n-1)}{2}$. This shows that either $D(G)$ or $D(G^c)$ has at most $\frac{n(n-1)}{4}$ edges, which is a contradiction to the fact that $D(G)$ or $D(G^c)$ has at most $\frac{n(n-1)}{2}$ edges.

Subcase 2.3 If $V(D(G^c)) \neq V(D(G))$, $E(D(G^c)) \neq E(D(G))$, then trivially it is not true from cases 2.1 and 2.2.

From the above three subcases 2.1, 2.2 and 2.3, we conclude that $D(G^c) \neq D(G)$ is not true. Hence $D(G^c) = D(G)$. □

Remark 4.11 The converse of the above result is not true. For example, $D(N_n) = D(N_n^c)$ but $N_n \neq N_n^c$.

Theorem 4.12 *Let G_1 and G_2 be same regular graphs of order n. Then $D(G_1) = D(G_2)$. But converse is not true.*

Proof Suppose G_1 and G_2 be regular graphs of same order $n > 1$. By the Proposition 4.1, $D(G_1) \cong K_n$ and $D(G_2) \cong K_n$, and thus $D(G_1) = D(G_2)$. But converse of this result is not true. This illustrates the Figure 2. Consider the graphs G_1 and G_2 on four vertices and their degree graphs. □

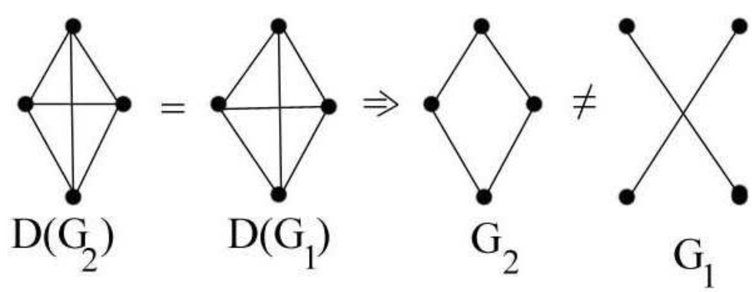

Fig.2 Graphs G_1, G_2, $D(G_1)$ and $D(G_2)$.

Theorem 4.13 *Let G_1 and G_2 be graphs of same order $n > 1$ such that $d(G_1) = d(G_2)$. Then $D(G_1) = D(G_2)$. But converse is not true.*

Proof Suppose G_1 and G_2 be non-regular degree equivalent graphs of same order $n > 1$. Then their degree sequences are equal. That is, $d(G_1) = d(G_1) = (d_1, d_2, \cdots, d_n)$ where $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_n$, $0 \leqslant d_i \leqslant n-1$ for each $1 \leqslant i \leqslant n$. By the definition of degree graphs, G_1 and G_2 ($G_1 \neq G_2$) are both realize same degree graph, that is, $D(G_1) = D(G_2)$.

Converse of the Theorem 4.13 is not true, in general. For the degree sequences $d(G_1) = (2,2,2,2)$ and $d(G_2) = (3,2,3,2)$, we have $D(G_1) = D(G_2)$ implies that $d(G_1) \neq d(G_2)$. □

Theorem 4.14 *If G_1 and G_2 are two graphs such that $G_1 \cong G_2$, then $D(G_1) \cong D(G_2)$. But the converse is not true.*

Proof Suppose $G_1 \cong G_2$. Then there exists an isomorphism φ from G_1 onto G_2. We show that $D(G_1) \cong D(G_2)$. For this let $(u,v) \in E(D(G_1))$, then by the definition of degree graphs, $deg_{G_1}u = deg_{G_1}v \Rightarrow deg_{G_2}\varphi(u) = deg_{G_2}\varphi(v) \Rightarrow (\varphi(u),\varphi(v)) \in E(D(G_2)) \Rightarrow D(G_1) \cong D(G_2)$.

But converse of this result is not true. For example, $D(N_4) \cong D(C_4)$ but $N_4 \ncong C_4$. □

§5. Disconnected Equal Degree Graphs

In this section we characterize the graphs G for which $D(G)$ is disconnected.

Theorem 5.1 *Let G be a graph of order $n+k$. Then $D(G) \cong K_n \cup N_k$ where n is the number of vertices of same degree and k is the number vertices of unequal degree.*

Proof Let $|V(G)| = n+k$.Then V can be partitioned into two subsets S_1 and S_2 such that $S_1 = \{u_1, u_2, \cdots, u_n\}$ and $S_2 = \{v_1, v_2, \cdots, v_k\}$ where $deg_G u_i = deg_G u_j$ for all $1 \leqslant i \neq j \leqslant n$ and $deg_G v_i \neq deg_G v_j$ for all $1 \leqslant i \neq j \leqslant k$. By the definition of Degree graphs, $< S_1 >\cong K_n$ and $< S_2 >\cong N_k$. Hence $D(G) \cong K_n \cup N_k$. □

This theorem gives the following consequences.

Corollary 5.2 $D(P_n) =< S_1 > \cup < S_2 >$ *where* $< S_1 >\cong K_2$ *and* $< S_2 >\cong K_{n-2}$.

Proof Let v_1 and v_{n+1} be the internal and external vertices of the path $P_n : v_1e_1v_2e_2v_3 \cdots v_ne_nv_{n+1}$. Then the vertex set $V = V(P_n)$ can be partitioned two disjoint sets S_1 and S_2 such that $S_1 = \{v_1, v_{n+1}\}$ and $S_2 = \{v_1, v_2, \cdots, v_n\}$.

Case 1. In this case $deg_G v_1 = deg_G v_{n+1} \neq deg_G v_j$ for $j = 2, 3, .., n$. This shows that there exists only one edge between v_1 and v_{n+1} in S_1, and which are not adjacent to the vertices in S_2. Thus the degree graph of S_1 is an induced sub graph $< S_1 >$ which is isomorphic to K_2.

Case 2. Suppose $v_i, v_j \in S_2$ for every $i \neq j$. Then $deg_G v_i = deg_G v_j = 2 \neq 1 = deg_G v_1 = deg_G v_{n+1}$ for each $i \neq j$ such that $2 \leqslant i, j \leqslant n$. Thus the degree graph of S_2 is also an induced subgraph $< S_2 >$ which is isomorphic to K_{n-2}.

Case 3. Suppose $u \in S_1$ and $v \in S_2$. Then there is no edge between u and v in the equal degree graph whose vertex set is $V = S_1 \cup S_2$ since $deg_G u = 1 \neq 2 = deg_G v$.

From the cases 1, 2 and 3 we conclude that $D(P_n)$ is disconnected with two disjoint components $< S_1 >$ and $< S_2 >$. □

The proofs of the following consequences are obvious.

Corollary 5.3 *The following results on $D(G)$ are true:*

(1) $D(K_{1,n}) = K_1 \cup K_{n-1}$;
(2) $D(K_{n_1,n_2,\cdots,n_k}) = K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_k}$;
(3) $D(W_n) = K_1 \cup K_{n-1}$.

Corollary 5.4 *Let G be a graph of order $p+q$. Then $D(G) = K_p \cup K_q$ where p is the number of vertices of same degree and q is the number of vertices of another same degree.*

Corollary 5.5 *Let T be a tree of order $n_1 + n_2 + n_3$. Then $D(T) = K_{n_1} \cup K_{n_2} \cup K_{n_3}$ where n_1 is the number of pendent vertices, n_2 is the number of non-pendent vertices of same degree and n_3 is another non-pendent vertex of another same degree.*

Theorem 5.6 *Let D be a connected Euler graph. Then $D(G)$ is either complete or disjoint union of complete components.*

Proof Consider the two cases on the regularity and non-regularity of a connected Euler graph G.

Case 1. Let G be a regular graph. The Proposition 4.1 shows that $D(G)$ is complete.

Case 2. Let G be a non- regular graph of order $n_1 + n_2 + \cdots + n_k$. Then n_1 is the number of vertices of degree 2, n_2 is the number of vertices of degree 4, and so on n_k is the number of vertices of degree $2k$. The Theorem 5.1 shows that $D(G)$ is isomorphic to the disjoint union of complete components $K_{n_1}, K_{n_2}, \cdots, K_{n_k}$. Hence $D(G) = K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_k}$. □

Example 5.7 For each $n \geqslant 3$, the cycle C_n is a regular Euler graph, and thus $D(C_n) = K_n$.

Example 5.8([3]) The graphs D_{12} and M_{16} are the Davids and Mohammeds graphs of order 12 and 16 respectively, which are non-regular Euler graphs, and their Degree graphs $D(D_6) = K_6 \cup K_6'$ and $D(M_{11}) = K_4 \cup K_7'$, which are disconnected graphs.

Theorem 5.9 *Let G be a simple graph of order $n > 4$. Then $D(D(G)) \cong D(G) \Leftrightarrow G$ is r-regular graph.*

Proof For any simple graph G of order $n > 4$, we have G is r-regular $\Leftrightarrow D(G) \cong K_n$, by the Proposition 4.1 $\Leftrightarrow D(D(G)) \cong D(K_n) \Leftrightarrow D(D(G)) \cong K_n$(since $D(K_n) = K_n$) $\Leftrightarrow D(D(G)) \cong D(G)$. □

§6. Relation Between $D(G)$ and $DP(G)$

In [1], the authors M. Sattanathan and R. Kala introduced Degree prime graphs and studied their characterizations. According to these authors $DP(G)$ is a graph whose vertex set is same as $V(G)$ and $u, v \in V(G)$ are adjacent in $DP(G)$ if and only if

$$deg_G u \neq deg_G v, \quad gcd(deg_G u,\ deg_G v) = 1.$$

In this paper, we are going to study the relation between $D(G)$ and $DP(G)$. Fundamentally we observe that the following:

For any graph G, we have

(1) $D(G) \neq DP(G)$;

(2) $D(DP(G)) \neq DP(D(G))$.

But the following result specifies that the relation between $D(G)$ and $DP(G)$ for some graphs G.

Theorem 6.1 *If G is either totally disconnected, regular or complete, then $D(G) \cong (DP(G))^c$.*

Proofv For any totally disconnected, regular or complete graph G, we know that $D(G) \cong K_n$ and $DP(G) \cong N_n$. But $K_n = N_n^c$. This shows that the required result is obviously true. *Box*

Here, we present an open problem following:

Problem 6.2 *Let G be a graph. Then*

(1) *Find the cardinality of the set $S = \{G : D(G)$ is complete $\}$;*

(2) *Find the cardinality of the set $S = \{G : D(D(G)) = D(G)\}$;*

(3) *For the finite family of graphs $\{G_i\}$, show that*

$$\bigcup_{i=1}^{n} D(G_i) = D\left(\bigcup_{i=1}^{n} G_i\right) \quad \textit{and} \quad \bigcap_{i=1}^{n} D(G_i) = D\left(\bigcap_{i=1}^{n} G_i\right);$$

(4) *Find the graph G such that $D(D(\cdots(D(G))\cdots))$(n times)$= G$.*

References

[1] M.Sattanathan and R.Kala, Degree prime graphs, *Journal of Discrete Mathematical Sciences and Cryptography*, 2009, Vol.12, 167-173.

[2] F.Harary, *Graph Theory*, Addition-Wesley, Reading, Mass., 1972.

[3] G Suresh Singh, *Graph Theory*, PHI New Delhi, 2010.

[4] J.A. Bondy and U.S.R. Mutry, *Graph Theory*, Springer India, 2013.

[5] Mark L. Lewis, Donald L. White, Diameters of degree graphs of nonsolvable groups, *Journal of Algebra*, 283(2005), pp.8092.

[6] Yan Yanxiong, Chen Guiyun, XU Haijing, A new Characterization of certain symmetric and alternating groups. Advances in mathematics. Vol.42, No.5, 2013.pp.601-610.

[7] Kapoor S., Polimeni Albert and Wall Curtiss, Degree sets for graphs, *Fundamenta Mathematicae*, Vol.95, No.3(1977), pp. 189-194.

[8] Y. Manoussakis, H.P. Patil and V. Sankar, *Further results on degree sets for graphs*, *Akce j. Graphs. Combin.*, Vol.1, No. 2(2004), pp. 77-82.