International J.Math. Combin. Vol.1(2018), 41-50

Extended Quasi Conformal Curvature Tensor on $N(k)$-Contact Metric Manifold

Venkatesha, H.Aruna Kumara

(Department of Mathematics, Kuvempu University, Shankaraghatta - 577 451, Shimoga, Karnataka, India)

R.T.Naveen Kumar

(Department of Mathematics, Siddaganga Institute of Technology, Tumakuru - 572 103, Karnataka, India)

vensmath@gmail.com, arunkumarh4665@gmail.com,rtnaveenkumar@gmail.com

Abstract: In this paper certain results on $N(k)$-contact metric manifold endowed with a extended quasi conformal curvature tensor are formulated. First, we consider ξ-extended quasi conformally flat $N(k)$-contact metric manifold. Next we describe extended quasi-conformally semi-symmetric and extended quasi conformal pseudo-symmetric $N(k)$-contact metric manifold. Finally, we study the conditions $\widetilde{C}_e(\xi, X) \cdot R = 0$ and $\widetilde{C}_e(\xi, X) \cdot S = 0$ on $N(k)$-contact metric manifold.

Key Words: $N(k)$-contact metric manifold, quasi conformal curvature tensor, extended quasi conformal curvature tensor, η-Einstein.

AMS(2010): 53C10, 53C15, 53C25.

§1. Introduction

In 1968, Yano and Sawaki [20] introduced the notion of quasi conformal curvature tensor $\tilde{C}$ on a Riemannian manifold M and is given by

$$\begin{aligned}\widetilde{C}(X,Y)Z &= aR(X,Y)Z + b\left[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\right] \\ &- \frac{r}{2n+1}\left[\frac{a}{2n} + 2b\right]\{g(Y,Z)X - g(X,Z)Y\} \qquad (1.1)\end{aligned}$$

for all $X,\ Y\ \in TM$, where a and b are constants and r is a scalar curvature. If $a = 1$ and $b = -\frac{1}{2n-1}$, then quasi conformal curvature tensor reduces to conformal curvature tensor.

The extended form of quasi conformal curvature tensor [8] is given by

$$\widetilde{C}_e(X,Y)Z = \widetilde{C}(X,Y)Z - \eta(X)\widetilde{C}(\xi,Y)Z - \eta(Y)\widetilde{C}(X,\xi)Z - \eta(Z)\widetilde{C}(X,Y)\xi. \qquad (1.2)$$

On the other hand Tanno [19] introduced a class of contact metric manifolds for which the characteristic vector field ξ belongs to the k-nullity distribution for some real number k. Such manifolds are known as $N(k)$-contact metric manifolds. The authors Blair, Kim and Tripathi [2]

[1]Received August 20, 2017, Accepted February 15, 2018.

gave the classification of $N(k)$-contact metric manifold satisfying the condition $Z(\xi, X) \cdot Z = 0$. Also quasi conformal curvature tensor on a sasakian manifold has been studied by De et al., [11]. Recently in [10], the authors study certain properties of $N(k)$-contact metric manifold endowed with a concircular curvature tensor.

Motivated by these studies the present paper is organized as follows: After giving preliminaries and basic formulas in Section 2, we study ξ-extended quasi conformally flat $N(k)$-contact metric manifolds in Section 3 and we found that the manifold is η-Einstein and also it admits a η-parallel Ricci tensor. In fact Section 4 is devoted to the study of extended quasi-conformally semi-symmetric $N(k)$-contact metric manifold and proved that the manifold is either locally isometric to $E^{n+1} \times S^n(4)$ or it is extended quasi-conformally flat. Then, in Section 5, we consider extended quasi conformal pseudo-symmetric $N(k)$-contact metric manifold and we found that the manifold reduces to η-Einstein. Finally in the last section, we discuss $N(k)$-contact metric manifolds satisfying conditions $\widetilde{C}_e(\xi, X) \cdot R = 0$ and $\widetilde{C}_e(\xi, X) \cdot S = 0$

§2. Preliminaries

A $(2n+1)$-dimensional smooth manifold M is said to be a contact manifold if it carries a global differentiable 1-form η which satisfies the condition $\eta \wedge (d\eta)^n \neq 0$ everywhere on M. Also a contact manifold admits an almost contact structure (ϕ, ξ, η), where ϕ is $(1,1)$-tensor field, ξ is a characteristic vector field and η is a global 1-form such that

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi(\xi) = 0, \quad \eta \cdot \phi = 0. \tag{2.1}$$

An almost contact metric structure is said to be be normal if the induced complex structure J on the product manifold $M \times R$ is defined by,

$$J\left(X, f\frac{d}{dt}\right) = \left(\phi X - f\xi, \eta(X)\frac{d}{dt}\right),$$

is integrable, where X is tangent to M, t is the coordinate of R and f is a smooth function on $M \times R$. Let g be the Riemannian metric with almost contact structure (ϕ, ξ, η) i.e.,

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).$$

From (2.1), it can be easily seen that

$$g(X, \phi Y) = -g(\phi X, Y), \qquad g(X, \xi) = \eta(X), \tag{2.2}$$

for all $X, Y \in TM$. An almost contact metric structure is called contact metric structure if $g(X, \phi Y) = d\eta(X, Y)$. Moreover, if ∇ denotes the Riemannian connection of g, then the following relation holds;

$$\nabla_X \xi = -\phi X - \phi h X. \tag{2.3}$$

A normal contact metric manifold is a Sasakian manifold. An almost metric manifold is

Sasakian if and only if

$$(\nabla_X \phi)Y = g(X,Y)\xi - \eta(Y)X,$$

where ∇ is the Levi-Civita connection of the Riemannian metric g.

As a generalization of both $R(X,Y)\xi = 0$ and Sasakian case, the authors Blair, Koufogiorgos and Papantoniou [4] introduced the idea of (k,μ)-nullity distribution on a contact metric manifold and gave several reasons for studying it. The (k,μ)-nullity distribution $N(k,\mu)$ of a contact metric manifold M is defined by

$$N(k,\mu): \;\; p \rightarrow N_p(k,\mu) = \{Z \in T_pM : R(X,Y)Z = (kI + \mu h)(g(Y,Z)X - g(X,Z)Y)\},$$

where $(k,\mu) \in R^2$. A contact metric manifold with $\xi \in N(k,\mu)$ is called a (k,μ)- contact metric manifold. If $\mu = 0$, the (k,μ)- nullity distribution reduces to k- nullity distribution [19]. The k- nullity distribution $N(k)$ of a Riemannian manifold is defined by

$$N(k): \;\; p \rightarrow N_p(k) = \{Z \in T_pM : R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y]\}, \tag{2.4}$$

k being a constant. If the characteristic vector field $\xi \in N(k)$, then we call a contact metric manifold as $N(k)$- contact metric manifold [2]. If $k = 1$, then the manifold is Sasakian and if $k = 0$, then the manifold is locally isometric to the product $E^{n+1} \times S^n(4)$ for $n > 1$ and flat for $n = 1$ [3]. In an $N(k)$-contact metric manifold, the following relations holds:

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y], \tag{2.5}$$

$$R(\xi,X)Y = k[g(X,Y)\xi - \eta(Y)X], \tag{2.6}$$

$$\begin{aligned} S(X,Y) &= 2(n-1)g(X,Y) + 2(n-1)g(hX,Y) \\ &\quad +[2nk - 2(n-1)]\eta(X)\eta(Y), \end{aligned} \tag{2.7}$$

$$S(X,\xi) = 2nk\eta(X), \quad Q\xi = 2nk\xi. \tag{2.8}$$

Also in an $N(k)$-contact metric manifold, extended quasi conformal curvature tensor satisfies the following:

$$\begin{aligned} \widetilde{C}_e(X,Y)\xi &= \left[a\left(\frac{r}{2n(2n+1)} - k\right) + 2b\left(\frac{r}{2n+1} - nk\right)\right](\eta(Y)X - \eta(X)Y) \\ &\quad -b[\eta(Y)QX - \eta(X)QY], \end{aligned} \tag{2.9}$$

$$\begin{aligned} \widetilde{C}_e(\xi,X)Y &= \left[a\left(k - \frac{r}{2n(2n+1)}\right) + 2b\left(nk - \frac{r}{2n+1}\right)\right](\eta(Y)X \\ &\quad -\eta(X)\eta(Y)\xi) + b[\eta(Y)QX - 2nk\eta(X)\eta(Y)\xi] = -\widetilde{C}_e(X,\xi)Y, \end{aligned} \tag{2.10}$$

$$\widetilde{C}_e(\xi,\xi)X = 0, \tag{2.11}$$

$$\eta(\widetilde{C}_e(X,Y)\xi) = \eta(\widetilde{C}_e(\xi,X)Y) = \eta(\widetilde{C}_e(X,\xi)Y) = \eta(\widetilde{C}_e(X,Y)Z) = 0. \tag{2.12}$$

By virtue of (1.2), let $\{e_i\}$ be an orthonormal basis of the tangent space at each point of the manifold and using (1.1) and (2.8), we get

$$\sum_{i=1}^{2n} g(\widetilde{C}_e(e_i, Y)Z, e_i) = Lg(Y,Z) + MS(Y,Z) + N\eta(Y)\eta(W), \tag{2.13}$$

where,

$$\begin{aligned} L &= b(r-2nk) - \left[\frac{r(2n-1)}{2n+1}\left(\frac{a}{2n}+2b\right)\right] - \left[ka + 2nkb - \frac{r}{2n+1}\left(\frac{a}{2n}+2b\right)\right], \\ M &= a + b(2n-3) \end{aligned}$$

and

$$N = 4nkb - (4n-3)\left[ak + 2nkb - \frac{r}{2n+1}\left(\frac{a}{2n}+2b\right)\right] - 2b(r-2nk).$$

Definition 2.1 *A* $(2n+1)$*-dimensional* $N(k)$*-contact metric manifold* M *is said to be* η*-Einstein if its Ricci tensor* S *is of the form*

$$S(X,Y) = \alpha g(X,Y) + \beta\eta(X)\eta(Y),$$

for any vector fields X *and* Y*, where* α *and* β *are constants. If* β*=0, then the manifold* M *is an Einstein manifold.*

Definition 2.2 *In a* $(2n+1)$*-dimensional* $N(k)$*-contact metric manifold, if the Ricci tensor* S *satisfies* $(\nabla_W S)(\phi X, \phi Y) = 0$*, then the Ricci tensor is said to be* η*-parallel.*

In [1], Baikoussis and Koufogiorgos proved the following lemma.

Lemma 2.1 *Let* M *be an* η*-Einstein manifold of dimension* $(2n+1)(n \geq 1)$*. If* ξ *belongs to the* k*-nullity distribution, then* $k=1$ *and the structure is Sasakian.*

§3. ξ-Extended Quasi Conformally Flat $N(k)$-Contact Metric Manifolds

Definition 3.1 *A* $(2n+1)$*-dimensional* $N(k)$*-contact metric manifold is said to be* ξ*-extended quasi conformally flat if*

$$\widetilde{C}_e(X,Y)\xi = 0 \quad \text{for all} \quad X, Y \in TM. \tag{3.1}$$

Let us consider ξ-extended quasi conformally flat $N(k)$-contact metric manifold. Then from (3.1) and (2.9), it can be easily seen that

$$\begin{aligned} 0 &= \left[a\left(\frac{r}{2n(2n+1)} - k\right) + 2b\left(\frac{r}{2n+1} - nk\right)\right](\eta(Y)X - \eta(X)Y) \\ &- b[\eta(Y)QX - \eta(X)QY]. \end{aligned} \tag{3.2}$$

Taking inner product of (3.2) with respect to W, we get

$$\begin{aligned} 0 &= \left[a\left(\frac{r}{2n(2n+1)}-k\right)+2b\left(\frac{r}{2n+1}-nk\right)\right](\eta(Y)g(X,W)-\eta(X)g(Y,W)) \\ &- b[\eta(Y)S(X,W)-\eta(X)S(Y,W)]. \end{aligned}$$

On plugging $Y=\xi$ in above equation, gives

$$S(X,W)=Ag(X,W)+B\eta(X)\eta(W), \tag{3.3}$$

where

$$A=\left[\frac{r}{b(2n+1)}\left(\frac{a}{2n}+2b\right)-2nk-\frac{ka}{b}\right] \text{ and } B=\left[4nk+\frac{ka}{b}-\frac{r}{b(2n+1)}\left(\frac{a}{2n}+2b\right)\right].$$

Hence we can state the following:

Theorem 3.1 *A (2n+1)-dimensional ξ-extended quasi conformally flat $N(k)$-contact metric manifold is an η-Einstein manifold.*

Hence in view of Lemma 2.1 and above result, we can state the following result:

Corollary 3.1 *Let M be a $(2n+1)$-dimensional ξ-extended quasi conformally flat $N(k)$-contact metric manifold, then $k=1$ and the structure is Sasakian.*

Replacing X and W by ϕX and ϕW in (3.3) and using (2.1), we obtain

$$S(\phi X,\phi W)=M'g(\phi X,\phi W). \tag{3.4}$$

Now taking the covariant derivative of (3.4) with respect to U, yields

$$(\nabla_U S)(\phi X,\phi W)=\frac{dr(U)}{b(2n+1)}\left(\frac{a}{2n}+2b\right)g(\phi X,\phi W).$$

If we consider $N(k)$-contact metric manifold with constant scalar curvature, then above equation becomes

$$(\nabla_U S)(\phi X,\phi W)=0.$$

Hence this leads us to the following result:

Corollary 3.2 *A $(2n+1)$-dimensional ξ-extended quasi conformally flat $N(k)$-contact metric manifold with constant scalar curvature admits a η-parallel Ricci tensor.*

§4. Extended Quasi-Conformally Semi-Symmetric $N(k)$-Contact Metric Manifold

Let us consider an extended quasi-conformally semi-symmetric $N(k)$-contact metric manifold i.e.,

$$R(\xi, X)\cdot \widetilde{C}_e = 0.$$

Then the above condition turns into,

$$0 = R(\xi, X)\widetilde{C}_e(U,V)W - \widetilde{C}_e(R(\xi, X)U, V)W - \widetilde{C}_e(U, R(\xi, X)V)W - \widetilde{C}_e(U,V)R(\xi, X)W. \quad (4.1)$$

In view of (2.6), equation (4.1) can be written as

$$0 = k\Big[g(X, \widetilde{C}_e(U,V)W)\xi - \eta(\widetilde{C}_e(U,V)W)X - g(X,U)\widetilde{C}_e(\xi, V)W + \eta(U)\widetilde{C}_e(X,V)W - g(X,V)\widetilde{C}_e(U,\xi)W + \eta(V)\widetilde{C}_e(U,X)W - g(X,W)\widetilde{C}_e(U,V)\xi + \eta(W)\widetilde{C}_e(U,V)X\Big]. \quad (4.2)$$

Which implies that either $k = 0$ or

$$\Big[g(X, \widetilde{C}_e(U,V)W)\xi - \eta(\widetilde{C}_e(U,V)W)X - g(X,U)\widetilde{C}_e(\xi, V)W + \eta(U)\widetilde{C}_e(X,V)W - g(X,V)\widetilde{C}_e(U,\xi)W + \eta(V)\widetilde{C}_e(U,X)W - g(X,W)\widetilde{C}_e(U,V)\xi + \eta(W)\widetilde{C}_e(U,V)X\Big] = 0.$$

Now taking inner product of above equation with ξ and then using (2.12), we get

$$g(X, \widetilde{C}_e(U,V)W) = 0.$$

Which implies that $\widetilde{C}_e(U,V)W = 0$. Hence we can state the following:

Theorem 4.1 *An extended quasi-conformally semi-symmetric $N(k)$-contact metric manifold is either locally isometric to $E^{n+1}\times S^n(4)$ for $n > 1$ and flat for $n = 1$ or the manifold is extended quasi-conformally flat.*

§5. Extended Quasi Conformal Pseudo-Symmetric $N(k)$-Contact Metric Manifold

Definition 5.1 *A $(2n+1)$-dimensional $N(k)$-contact metric manifold M is said to be extended quasi conformal pseudo-symmetric if*

$$(R(X,Y)\cdot \widetilde{C}_e)(U,V)W = L_{\widetilde{C}_e}[((X\wedge Y)\cdot \widetilde{C}_e)(U,V)W], \quad (5.1)$$

holds for any vector fields X, Y, U, V, $W \in TM$, where $L_{\widetilde{C}_e}$ is function of M. The endomorphism $X\wedge Y$ is defined by

$$(X\wedge Y)Z = g(Y,Z)X - g(X,Z)Y. \quad (5.2)$$

Now we prove the following result:

Theorem 5.1 *Let M be a $(2n+1)$-dimensional extended quasi conformal pseudo-symmetric*

$N(k)$-contact metric manifold. Then either $L_{\widetilde{C}_e} = k$ or the the manifold is η-Einstein.

Proof Let us consider a $(2n+1)$-dimensional extended quasi conformal pseudo symmetric $N(k)$-contact metric manifold. Taking $Y = \xi$ in (5.1), we get

$$(R(X,\xi)\cdot\widetilde{C}_e)(U,V)W = L_{\widetilde{C}_e}[((X\wedge\xi)\cdot\widetilde{C}_e)(U,V)W]. \tag{5.3}$$

By virtue of (5.2) and (2.12), right hand side of above equation becomes

$$\begin{aligned}&L_{\widetilde{C}_e}[-g(X,\widetilde{C}_e(U,V)W)\xi-\eta(U)\widetilde{C}_e(X.V)W+g(X,U)\widetilde{C}_e(\xi,V)W-\eta(V)\widetilde{C}_e(U,X)W\\&+g(X,V)\widetilde{C}_e(U,\xi)W-\eta(W)\widetilde{C}_e(U,V)X+g(X,W)\widetilde{C}_e(U,V)\xi].\end{aligned} \tag{5.4}$$

In view of (2.5), left hand side of (5.3) gives

$$\begin{aligned}&k[-g(X,\widetilde{C}_e(U,V)W)\xi-\eta(U)\widetilde{C}_e(X.V)W+g(X,U)\widetilde{C}_e(\xi,V)W-\eta(V)\widetilde{C}_e(U,X)W\\&+g(X,V)\widetilde{C}_e(U,\xi)W-\eta(W)\widetilde{C}_e(U,V)X+g(X,W)\widetilde{C}_e(U,V)\xi].\end{aligned} \tag{5.5}$$

By considering (5.5) and (5.4) in (5.3) with $V = \xi$, we get

$$\begin{aligned}&(L_{\widetilde{C}_e}-k)[-g(X,\widetilde{C}_e(U,\xi)W)\xi-\eta(U)\widetilde{C}_e(X.\xi)W+g(X,U)\widetilde{C}_e(\xi,\xi)W\\&-\eta(\xi)\widetilde{C}_e(U,X)W+g(X,\xi)\widetilde{C}_e(U,\xi)W-\eta(W)\widetilde{C}_e(U,\xi)X+g(X,W)\widetilde{C}_e(U,\xi)\xi]=0.\end{aligned} \tag{5.6}$$

By using (2.9)-(2.11) in (5.6), we have either $(L_{\widetilde{C}_e} - k) = 0$ or

$$\begin{aligned}\widetilde{C}_e(U,X)W &= \left[a\left(k-\frac{r}{2n(2n+1)}\right)+2b\left(nk-\frac{r}{2n+1}\right)\right]\{\eta(W)g(X,U)\xi\\&- 2\eta(U)\eta(X)\eta(W)\xi+\eta(U)\eta(W)X+g(X,W)\eta(U)\xi-g(X,W)U\}\\&+ b[\eta(W)S(X,U)\xi-4nk\eta(U)\eta(X)\eta(W)\xi+\eta(U)\eta(W)QX\\&+ 2nkg(X,W)\eta(U)\xi-g(X,W)QU].\end{aligned} \tag{5.7}$$

On contracting (5.7) with respect to U and then using (2.13), we have

$$S(X,W) = A'g(X,W) + B'\eta(X)\eta(W),$$

where,

$$\begin{aligned}A' &= \frac{1}{a+b(2n-3)}\left[(2-2n)ka+(6-2n)2nkb-\frac{r(3-4n)}{2n+1}\left(\frac{a}{2n}+2b\right)-2rb\right],\\B' &= \frac{1}{a+b(2n-3)}\left[(4n-3)\left(ka+2nkb-\frac{r}{2n+1}\left(\frac{a}{2n}+2b\right)\right)+2br-12nkb\right].\end{aligned}$$

Thus M is a η-Einstein manifold. □

§6. $N(k)$-Contact Metric Manifold Satisfying $\widetilde{C}_e(\xi,X)\cdot R=0$ and $\widetilde{C}_e(\xi,X)\cdot S=0$

First we consider an $N(k)$-contact metric manifold satisfying $\widetilde{C}_e(\xi,X)\cdot R=0$. Now it follows from above condition that

$$0=\widetilde{C}_e(\xi,X)R(U,V)Y-R(\widetilde{C}_eU,V)Y-R(U,\widetilde{C}_e(\xi,X)V)Y-R(U,V)\widetilde{C}_e(\xi,X)Y. \tag{6.1}$$

By virtue of (2.10) in (6.1), gives

$$\begin{aligned}0 &= \left[a\left(k-\frac{r}{2n(2n+1)}\right)+2b\left(nk-\frac{r}{2n+1}\right)\right]\{\eta(R(U,V)Y)[X-\eta(X)\xi]\\ &- \eta(U)[R(X,V)Y-\eta(X)R(\xi,V)Y]-\eta(V)[R(U,X)Y-\eta(X)R(U,\xi)Y]\\ &- \eta(Y)[R(U,V)X-\eta(X)R(U,V)\xi]\}+b\{\eta(R(U,V)Y)[QX-2nk\eta(X)\xi]\\ &- \eta(U)[R(QX,V)Y-2nk\eta(X)R(\xi,V)Y]-\eta(V)[R(U,QX)Y\\ &- 2nk\eta(X)R(U,\xi)Y]-\eta(Y)[R(U,V)QX-2nk\eta(X)R(U,V)\xi]\}.\end{aligned} \tag{6.2}$$

Considering $U=\xi$ in (6.2), gives

$$\begin{aligned}0 &= \left[a\left(k-\frac{r}{2n(2n+1)}\right)+2b\left(nk-\frac{r}{2n+1}\right)\right]\{\eta(R(\xi,V)Y)[X-\eta(X)\xi]\\ &- [R(X,V)Y-\eta(X)R(\xi,V)Y]-\eta(V)[R(\xi,X)Y-\eta(X)R(\xi,\xi)Y]\\ &- \eta(Y)[R(\xi,V)X-\eta(X)R(\xi,V)\xi]\}+b\{\eta(R(\xi,V)Y)[QX-2nk\eta(X)\xi]\\ &- [R(QX,V)Y-2nk\eta(X)R(\xi,V)Y]-\eta(V)[R(\xi,QX)Y\\ &- 2nk\eta(X)R(\xi,\xi)Y]-\eta(Y)[R(\xi,V)QX-2nk\eta(X)R(\xi,V)\xi]\}.\end{aligned} \tag{6.3}$$

Taking inner product of (6.3) with respect to ξ and then by virtue of (2.5) and (2.6), we obtain

$$0=k\eta(Y)\left[S(V,X)-A''g(V,X)-B''\eta(V)\eta(X)\right], \tag{6.4}$$

where $A''=\left[\frac{r}{b(2n+1)}\left(\frac{a}{2n}+2b\right)-2nk-\frac{ka}{b}\right]$ and $B''=\left[4nk+\frac{ka}{b}-\frac{r}{b(2n+1)}\left(\frac{a}{2n}+2b\right)\right]$.

Since for an $N(k)$-contact metric manifolds $\eta(Y)\neq 0$, then (6.4) yields either $k=0$ or

$$S(V,X)=A''g(V,X)+B''\eta(V)\eta(X).$$

This leads us to the following:

Theorem 6.1 *Let M be a $(2n+1)$-dimensional $N(k)$-contact metric manifold satisfying the condition $\widetilde{C}_e(\xi,X)\cdot R=0$. Then M reduces to η-Einstein manifold or it is locally isometric to $E^{n+1}\times S^n(4)$ for $n>1$ and flat for $n=1$.*

Next we prove the following result:

Theorem 6.2 *Let M be an $(2n+1)$-dimensional $N(k)$-contact metric manifold satisfying $\widetilde{C}_e(\xi,X)\cdot S=0$. Then the Ricci tensor S is given by the equation (6.7).—*

Proof Let us consider an $N(k)$-contact metric manifold satisfying the condition $\widetilde{C}_e(\xi, X)\cdot S = 0$. Then it can be easily seen that

$$S(\widetilde{C}_e(\xi, X)Y, W) + S(Y, \widetilde{C}_e(\xi, X)W) = 0. \tag{6.5}$$

By virtue of (2.10), it follows from above equation that

$$\begin{aligned} 0 = &\left[a\left(k - \frac{r}{2n(2n+1)}\right) + 2b\left(nk - \frac{r}{2n+1}\right)\right]\{\eta(Y)[S(X, W) - \eta(X)S(\xi, W)] \\ &+ \eta(W)[S(Y, X) - \eta(X)S(Y, \xi)]\} + b\{\eta(Y)[S(QX, W) - 2nk\eta(X)S(\xi, W)] \\ &+ \eta(W)[S(Y, QX) - 2nk\eta(X)S(Y, \xi)]\}. \end{aligned} \tag{6.6}$$

On plugging $Y = \xi$ in (6.6) and making necessary calculation, we have

$$S(QX, W) = MS(X, Y) + N\eta(X)\eta(W), \tag{6.7}$$

where,

$$\begin{aligned} M &= \left[\frac{r}{2n+1}\left(\frac{a}{2n} + 2b\right) - \frac{ak}{b} - 2nk\right], \\ N &= \left[\frac{2nk^2a}{b} + 8n^2k^2 - \frac{2nkr}{b(2n+1)}\left(\frac{a}{2n} + 2b\right)\right]. \end{aligned}$$

Hence the proof. □

References

[1] C. Baikoussis and T. Koufogiorgos, On a type of contact manifold, *Journal of Geometry*, 1-2 (1993), 19.

[2] D.E. Blair, Two remarks on Contact manifold, *Tohoku Math. J.*, 29 (1977), 319-324.

[3] D.E. Blair, J.S. Kim and M.M. Tripathi, On the concircular curvature tensor of a contact metric manifold, *J. Korean Math. Soc.*, 42 (2005), 5, 883-892.

[4] D.E. Blair, T. Koufogiorgos and Papantoniou, Contact metric manifolds satisfying a nullity condition, *Israel J. Math.*, 91, 189 (1995).

[5] W.M. Boothby and H.C. Wang, On contact manifolds, *Ann. of Math.*, 68 (1958), 721-734.

[6] Cihan Ozgur and Sibel Sular, On $N(k)$-contact metric manifolds satisfying certain conditions, *SUT Journal of Mathematics*, 44, 1 (2008), 89-99.

[7] Cihan Ozgur and U.C. De, On the quasi-conformal curvature tensor of a kenmotsu manifold, *Mathematica Pannonica*, 17 (2) (2006), 221-228.

[8] U.C. De and Avijit Sarkar, On the quasi-conformal curvature tensor of a $(\kappa.\mu)$ contact metric manifold, *Math. Reports*, 14(64),2(2012), 115-129

[9] U.C. De and Y. Matsuyama, Quasi-conformally flat manifolds satisfying certain condition on the Ricci tensor, *SUT J. Math.*, 42 (2006), 295-303.

[10] U.C De, Ahmet Yildiz and Sujith Ghosh, On a class of $N(k)$-contact metric manifolds,

Lecture Notes in Mathematics, Math. Reports, 16 (66), 2 (2014), 207-217.

[11] U.C. De, J.B. Jun and A.K. Gazi, Sasakian manifolds with quasi-conformal curvature tensor, *Bull. Korean Math. Soc.*, 45 (2008), 313-319.

[12] J-S. Kim, J. Choi, C. Ozgur and M.M. Tripathi, On the contact conformal curvature tensor of a contact metric manifold, *Indian J. Pure Appl. Math.*, 37 (2006), 4, 199-207.

[13] M. Kon, Invariant submanifolds in Sasakian manifolds, *Mathematische Annalen*, 219, 3 (1976), 277-290.

[14] B.J. Papantoniou, Contact Riemannian manifolds satisfying $R(\xi, X)\cdot R = 0$ and $k \in (k, \mu)-$ nullity distribution, *Yokohama Math. J.*, 40 (1993), 149-161.

[15] J.S. Pak and Y.J. Shin, A note on contact conformal curvature tensor, *Korean Math. Soc.*, 13 (1998), 2, 337-343.

[16] D. Perrone, Contact Riemannian manifolds satisfying $R(\xi, X) \cdot R = 0$, *Yokohama Math. J.*, 39 (1992), 141-149.

[17] D.G. Prakasha, C.S. Bagewadi and Venkatesha, Conformally and quasi-conformally conservative curvature tensors on a trans-Sasakian manifold with respect to semi-symmetric metric connections, *Differential Geometry - Dynamical Systems*, Vol. 10, (2008), 263-274.

[18] S. Sasaki, *Lecture Notes on Almost Contact Manifold,* Tohoku University I, 1965.

[19] S. Tanno, Ricci curvature of contact riemannian manifolds, *Tohoku Math. J.*, 40 (1988), 441-448.

[20] K. Yano and S. Sawaki, Riemannian manifolds admitting a conformal transformation group, *J. Differential Geom.*, 2 (1968), 161-184.