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Homology of a Type of Octahedron

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Abstract: In this paper we find out the homology of a type of octahedron with six vertices, twelve edges and eight faces and have shown that it is analogous with the homology of a chain complex $0 \longrightarrow \mathbf{R}^6 \longrightarrow \mathbf{R}^{12} \to \mathbf{R}^8 \to 0$ and also find out the singular homology and the Euler characteristic of this type of octahedron which is equal to $\sum\limits_{n=0}^{\infty} dim_R(H_n(S))$, where S is a octahedron.

Key Words: Homology module, singular homology, homotopy.

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§1. Introduction

Homology classes were first defined rigorously by Henri Poincaré in his seminal paper "Analysis situs" in 1895 referring to the work of Riemann,Betti and von Dyck. The homology group was further developed by Emmy Noether [1] and, independently, by Leopold Vietoris and Walther Mayer [2] in the period 1925-28.

In mathematics (especially algebraic topology and abstract algebra), homology is a certain general procedure to associate a sequence of abelian groups or modules with a given mathematical object such as a topological space or a group. So, in algebraic topology, singular homology refers to the study of a certain set of algebraic invariants of a topological space X, the so-called homology groups $H_n(X)$. Intuitively spoken, singular homology counts, for each dimension n, the n-dimensional holes of a space.

The abstract algebra invariants such as ring,field were used to make concept of homology more rigorous and these developments give rise to mathematical branches such as homological algebra and K-Theory.

Homological algebra is a tool used to prove nonconstructive existence theorems in algebra (and in algebraic topology). It also provides obstructions to carrying out various kinds of constructions; when the obstructions are zero, the construction is possible. Finally, it is detailed enough so that actual calculations may be performed in important cases.

Let f and g be matrices whose product is zero. If $g.v = 0$ for some column vector v, say, of length n, we can not always write $v = f.u$ for some row vector u. This failure is measured by the defect

$$d = n - rank(f) - rank(g).$$

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Let R be a ring and let U, V and W be the modules over R. In modern language, f and g represent linear maps

$$U \xrightarrow{\quad f \quad} V \xrightarrow{\quad g \quad} W$$

with $gf = 0$, and d is the dimension of the homology module

$$H = \ker(g)/f(U)$$

Given an R-module homomorphism $f : A \longrightarrow B$, one is immediately led to study the kernel $\ker f$ and image $\mathrm{im} f$. Given another map $g : B \longrightarrow C$, we can form the sequence

$$A \xrightarrow{\quad f \quad} B \xrightarrow{\quad g \quad} C, \tag{1}$$

where A, B and C are the modules over R. We say that such a sequence is exact (at B) if $\ker(g) = \mathrm{im}(f)$. This implies in particular that the composite $gf : A \longrightarrow C$ is zero and finally brings our attention to sequence (1) such that $gf = 0$.

The word polyhedron has slightly different meanings in geometry and algebraic geometry. In elementary geometry, a polyhedron is simply a three-dimensional solid which consists of a collection of polygons, usually joined at their edges. In [4], S. Dey et al. studied homology of a type of heptahedron. Here we consider polyhedron octahedron with eight faces, six vertices and twelve edges.

In this paper, first we find out that homology of a type of octahedron is analogous to the homology of a chain complex, $0 \longrightarrow \mathbf{R}^7 \longrightarrow \mathbf{R}^{12} \longrightarrow \mathbf{R}^7 \longrightarrow 0$ and we also find out the matrices of this complex. Next we show computationally, $H_2(S) \cong H_0(S) \cong \mathbf{R}$ and $H_1(S) = 0$ and the Euler characteristic of this type of octahedron which is equal to $\sum\limits_{n=0}^{\infty} dim_R(H_n(S))$.

§2. Homology of a Octahedron

We can obtain a chain complex from a geometric object. We refer to the Weibel's book [3] for some details of the construction. We illustrate it with a octahedron S in Fig.1 following.

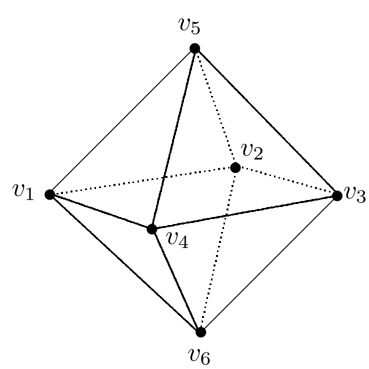

Fig.1

Level the vertex set of S as $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and then the twelve edges $e_{12}, e_{23}, e_{34}, e_{41}, e_{15}, e_{25}, e_{35}, e_{45}, e_{16}, e_{26}, e_{36}, e_{46}$, where e_{ij}=e_{ji} for $i, j = 1, 2, 3, 4, 5, 6$ can be ordered as

$$\begin{aligned} E \quad = \quad & \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_1\}, \{v_1, v_5\}, \{v_2, v_5\}, \\ & \{v_3, v_5\}, \{v_4, v_5\}, \{v_1, v_6\}, \{v_2, v_6\}, \{v_3, v_6\}, \{v_4, v_6\}\} \end{aligned}$$

and seven faces $f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8$ can be ordered as

$$\begin{aligned} F \quad = \quad & \{\{v_1, v_2, v_5\}, \{v_2, v_3, v_5\}, \{v_3, v_4, v_5\}, \\ & \{v_4, v_1, v_5\}, \{v_1, v_2, v_6\}, \{v_2, v_3, v_6\}, \{v_4, v_1, v_6\}\}. \end{aligned}$$

Let R be a ring and let $C_i(S)$ be the free R-module on the set V, E, F, respectively. Define maps ∂_0, ∂_1, ∂_2: $F \to E$ by removing the first, second, and third vertices, respectively except first face for each map. For the first face we define each map in such a way so that we can construct the homology. So, ∂_0, ∂_1, ∂_2 are given by

$$\begin{aligned} \partial_0 : f_1 &\longrightarrow e_{25} & \qquad \partial_1 : f_1 &\longrightarrow e_{15} & \qquad \partial_2 : f_1 &\longrightarrow e_{12} \\ f_2 &\longrightarrow e_{35} & f_2 &\longrightarrow e_{25} & f_2 &\longrightarrow e_{23} \\ f_3 &\longrightarrow e_{45} & f_3 &\longrightarrow e_{35} & f_3 &\longrightarrow e_{34} \\ f_4 &\longrightarrow e_{15} & f_4 &\longrightarrow e_{45} & f_4 &\longrightarrow e_{41} \\ f_5 &\longrightarrow e_{26} & f_5 &\longrightarrow e_{16} & f_5 &\longrightarrow e_{12} \\ f_6 &\longrightarrow e_{36} & f_6 &\longrightarrow e_{26} & f_6 &\longrightarrow e_{23} \\ f_7 &\longrightarrow e_{46} & f_7 &\longrightarrow e_{36} & f_7 &\longrightarrow e_{34} \\ f_7 &\longrightarrow e_{16} & f_7 &\longrightarrow e_{46} & f_7 &\longrightarrow e_{41} \end{aligned}$$

The set maps ∂_i yield $k + 1$ module maps $C_k \longrightarrow C_{k-1}$, which we also call ∂_i, their alternating sum $d_i = \Sigma(-1)^i \partial_i$ is the map $C_k \longrightarrow C_{k-1}$, where $(0 \leq i \leq k \leq n)$ in the chain complex C. We can then define the map

$$d_2 = \partial_0 - \partial_1 + \partial_2 : C_2 \longrightarrow C_1,$$

which is given by

$$\begin{aligned} f_1 &\longrightarrow e_{25} - e_{15} + e_{12} \\ f_2 &\longrightarrow e_{35} - e_{25} + e_{23} \\ f_3 &\longrightarrow e_{45} - e_{35} + e_{34} \\ f_4 &\longrightarrow e_{15} - e_{45} + e_{41} \\ f_5 &\longrightarrow e_{26} - e_{16} + e_{12} \\ f_6 &\longrightarrow e_{36} - e_{26} + e_{23} \end{aligned}$$

$$f_7 \longrightarrow e_{46} - e_{36} + e_{34}$$
$$f_8 \longrightarrow e_{16} - e_{46} + e_{41}$$

We can define maps $\partial_0, \partial_1 : E \longrightarrow V$ by removing the first, second vertices, respectively. Therefore we have

$$\begin{aligned}
\partial_0 : e_{12} &\longrightarrow v_2 & \qquad \partial_1 : e_{12} &\longrightarrow v_1 \\
e_{23} &\longrightarrow v_3 & e_{23} &\longrightarrow v_2 \\
e_{34} &\longrightarrow v_4 & e_{34} &\longrightarrow v_3 \\
e_{41} &\longrightarrow v_1 & e_{41} &\longrightarrow v_4 \\
e_{15} &\longrightarrow v_5 & e_{15} &\longrightarrow v_1 \\
e_{25} &\longrightarrow v_5 & e_{25} &\longrightarrow v_2 \\
e_{35} &\longrightarrow v_5 & e_{35} &\longrightarrow v_3 \\
e_{45} &\longrightarrow v_5 & e_{45} &\longrightarrow v_4 \\
e_{16} &\longrightarrow v_6 & e_{16} &\longrightarrow v_1 \\
e_{26} &\longrightarrow v_6 & e_{26} &\longrightarrow v_2 \\
e_{36} &\longrightarrow v_6 & e_{36} &\longrightarrow v_3 \\
e_{46} &\longrightarrow v_6 & e_{46} &\longrightarrow v_4
\end{aligned}$$

We can define map $d_1 = \partial_0 - \partial_1$ from C_1 *to* C_0, and it is given by

$$\begin{aligned}
e_{12} &\longrightarrow v_2 - v_1 \\
e_{23} &\longrightarrow v_3 - v_2 \\
e_{34} &\longrightarrow v_4 - v_3 \\
e_{41} &\longrightarrow v_1 - v_4 \\
e_{15} &\longrightarrow v_5 - v_1 \\
e_{25} &\longrightarrow v_5 - v_2 \\
e_{35} &\longrightarrow v_5 - v_3 \\
e_{45} &\longrightarrow v_5 - v_4 \\
e_{16} &\longrightarrow v_6 - v_1 \\
e_{26} &\longrightarrow v_6 - v_2 \\
e_{36} &\longrightarrow v_6 - v_3 \\
e_{46} &\longrightarrow v_6 - v_4
\end{aligned}$$

By viewing $C_0 = \mathbf{R}^6$, $C_1 = \mathbf{R}^{12}$, and $C_2 = \mathbf{R}^8$, the maps d_1 and d_2 are given by the following matrices

$$d_1 = \begin{bmatrix} -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

and

$$d_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 \end{bmatrix}$$

Because $d_1 d_2$ is easily computed to be zero matrix, the sequence

$$\cdots C_2 \xrightarrow{\quad d_2 \quad} C_1 \xrightarrow{\quad d_1 \quad} C_0 \longrightarrow 0$$

is a complex. We compute the homology $V_n(S)$ of this complex with the help of Matlab. By Finding the column space of d_1, we find $\mathrm{im}(d_1)$. This space has a basis consisting of the vectors

$$\{(-1,1,0,0,0,0),(0,-1,1,0,0,0),(0,0,-1,1,0,0),(-1,0,0,0,1,0),(-1,0,0,0,0,1)\}.$$

We note that by adding $(0,0,0,0,0,1)$ that we get a basis for $\mathbf{R}^6$. Therefore

$$C_0/\mathrm{im}(d_1) \cong \mathbf{R}.$$

Thus

$$V_0(S) = \mathbf{R}.$$

Now, $\ker(d_1)$ has a basis

$$\left\{\begin{array}{cc} (1,1,1,1,1,1,1), & (1,0,1,1,0,1,1), \\ (1,0,0,1,0,0,1), & (1,0,0,0,0,0,0), \\ (0,-1,-1,-1,0,0,0), & (0,1,0,0,0,0,0,0), \\ (0,0,1,0,0,0,0), & (0,0,0,1,0,0,0), \\ (0,0,0,0,-1,-1,-1), & (0,0,0,0,1,0,0), \\ (0,0,0,0,0,1,0), & (0,0,0,0,0,0,1) \end{array}\right\}$$

Again, $\mathrm{im}(d_2)$ has a basis

$$\left\{\begin{array}{cc} \{(1,0,0,0,0,0,0,0), & (0,1,0,0,0,0,0), \\ (0,0,1,0,0,0,0), & (0,0,0,1,0,0,0), \\ (0,0,0,0,1,0,0), & (0,0,0,0,0,1,0), \\ (0,0,0,0,0,0,1), & (0,0,0,0,-1,-1,-1) \\ (-1,0,0,1,-1,0,0), & (1,-1,0,0,0,-1,0), \\ (0,1,-1,0,0,0,-1), & (0,0,1,-1,1,1,1)\} \end{array}\right\}$$

If u_i are the basis vectors of $\ker(d_1)$,then the following vectors of $\mathrm{im}(d_2)$ can be constructed in the following way:

$$\begin{array}{lll} (-1,0,0,1,-1,0,0) & \text{is} & u_9-u_4+u_{12}+u_{10}+u_{11}, \\ (1,-1,0,0,0,-1,0) & \text{is} & u_4-u_6-u_{11}, \\ (0,1,-1,0,0,0,-1) & \text{is} & u_6-u_7-u_{12} \qquad\qquad \text{and} \\ (0,0,1,-1,1,1,1) & \text{is} & u_7-u_8+u_{10}+u_{12}+u_{11}. \end{array}$$

The rest of the elements of $\mathrm{im}(d_2)$ can be found in $\ker(d_1)$.

Thus we see that $\mathrm{im}(d_2)=\ker(d_1)$. Therefore, $V_1(S)=0$. Finally, $\ker(d_2)$ has a basis of one element $\{(-1,-1,-1,-1,1,1,1,1)\}$. So, $V_2(S)=\ker d_2=\mathbf{R}$. To summarize, the singular homology $V_n(S)$ of the Octahedron is

$$\begin{array}{rcl} V_0(S) & = & V_2(S)=\mathbf{R}, \\ V_1(S) & = & 0, \\ V_n(S) & = & 0 \ \text{ if } \ n\geq 3. \end{array}$$

The *Euler characteristic* is a fundamental invariant for the classification of surfaces, so it is particulary useful that it can be calculated with homological algebra. The Euler characteristic of such surface H is $v-e+f$, where v is the number of vertices, e is the number of edges and f is the number of faces. Now, the Euler characteristic of octahedron is 2, which is equal to $\sum\limits_{n=0}^{\infty} dim_R(V_n(S))$. This is the same as the Euler Characteristic of a sphere as a octahedron is homeomorphic to a sphere, so it is homotopic to a sphere.

§3. Conclusion

One can find the homology of other polyhedron like prism, decahedron etc.

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