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Minimum Equitable Dominating Randić Energy of a Graph

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Abstract: Let G be a graph with vertex set $V(G)$, edge set $E(G)$ and d_i is the degree of its i-th vertex v_i, then the Randić matrix $R(G)$ of G is the square matrix of order n, whose (i,j)-entry is equal to $\frac{1}{\sqrt{d_i d_j}}$ if the i-th vertex v_i and j-th vertex v_j of G are adjacent, and zero otherwise. The Randic energy [3] $RE(G)$ of the graph G is defined as the sum of the absolute values of the eigenvalues of the Randić matrix $R(G)$. A subset ED of $V(G)$ is called an equitable dominating set [11], if for every $v_i \in V(G) - ED$ there exists a vertex $v_j \in ED$ such that $v_i v_j \in E(G)$ and $|d_i(v_i) - d_j(v_j)| \leq 1$. In the contrast, such a dominating set ED is Smarandachely if $|d_i(v_i) - d_j(v_j)| \geq 1$. Recently, Adiga, et.al. introduced, the minimum covering energy $Ec(G)$ of a graph [1] and S. Burcu Bozkurt, et.al. introduced, Randić Matrix and Randić Energy of a graph [3]. Motivated by these papers, Minimum equitable dominating Randić energy of a graph $RE_{ED}(G)$ of some graphs are worked out and bounds on $RE_{ED}(G)$ are obtained.

Key Words: Randić matrix and its energy, minimum equitable dominating set and minimum equitable dominating Randić energy of a graph.

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§1. Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The adjacency matrix $A(G)$ of the graph G is a square matrix, whose (i,j)-entry is equal to 1 if the vertices v_i and v_j are adjacent, otherwise zero [12]. Since $A(G)$ is symmetric, its eigenvalues are all real. Denote them by $\lambda_1, \lambda_2, \ldots, \lambda_n$, and as a whole, they are called the spectrum of G and denoted by Spec(G). The energy of graph [12] G is

$$\varepsilon(G) = \sum_{i=1}^{n} |\lambda_i|.$$

The literature on energy of a graph and its bounds can refer [4,8,9,10,12]. The Randić matrix $R(G) = (r_{ij})$ of G is the square matrix of order n, where

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$$(r_{ij}) = \begin{cases} \frac{1}{\sqrt{d_i d_j}}, & \text{if } v_i \text{ and } v_j \text{ are adjacent vertices in } G; \\ 0, & \text{otherwise.} \end{cases}$$

The Randic energy [3] $RE(G)$ of the graph G is defined as the sum of the absolute values of the eigenvalues of the Randić matrix $R(G)$. Let $\rho_1, \rho_2, \cdots, \rho_n$ be the eigenvalues of the Randić matrix $R(G)$. Since $R(G)$ is symmetric matrix, these eigenvalues are real numbers and their sum is zero. Randić energy [3] can be defined as

$$RE(G) = \textstyle\sum_{i=1}^{n} |\rho_i|$$

For details of Randić energy and its bounds, can refer [2, 3, 5, 6, 7].

Let G be a graph with vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$ and ED is minimum equitable dominating set of G. Minimum equitable dominating Randić matrix of G is $n \times n$ matrix $R_{ED}(G) = (r_{ij})$, where

$$(r_{ij}) = \begin{cases} \frac{1}{\sqrt{d_i d_j}}, & \text{if } v_i \text{ and } v_j \text{ are adjacent vertices in } G; \\ 1, & \text{if } i = j \text{ and } v_i \in ED; \\ 0, & \text{otherwise.} \end{cases}$$

The characteristic polynomial of $R_{ED}(G)$ is denoted by $det(\rho I - R_{ED}(G)) = |\rho I - R_{ED}(G)|$. Since $R_{ED}(G)$ is symmetric, its eigenvalues are real numbers. If the distinct eigenvalues of $R_{ED}(G)$ are $\rho_1 > \rho_2 > \cdots > \rho_r$ with their multiplicities are $m_1, m_2, \cdots, m_r$ then spectrum of $R_{ED}(G)$ is denoted by

$$SpecR_{ED}(G) = \begin{pmatrix} \rho_1 & \rho_2 & \cdots & \rho_r \\ m_1 & m_2 & \cdots & m_r \end{pmatrix}.$$

The minimum equitable dominating Randić energy of G is defined as

$$RE_{ED}(G) = \sum_{i=1}^{n} |\rho_i|.$$

Example 1.1 Let W_5 be a wheel graph, with vertex set $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$, and let its minimum equitable dominating set be $ED = \{v_1\}$. Then minimum equitable dominating Randić matrix $R_{ED}(W_5)$ is

$$R_{ED}(W_5) = \begin{bmatrix} 1 & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} \\ \frac{1}{\sqrt{12}} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{\sqrt{12}} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ \frac{1}{\sqrt{12}} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{\sqrt{12}} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \end{bmatrix}$$

$$SpecR_{ED}(W_5) = \begin{pmatrix} -0.6666 & 0 & 0.2324 & 1.4342 \\ 1 & 2 & 1 & 1 \end{pmatrix}.$$

The minimum equitable dominating Randić energy of W_5 is $RE_{ED}(W_5) = 2.3332$.

§2. Bounds for the Minimum Equitable Dominating Randić Energy of a Graph

Lemma 2.1 *If $\rho_1, \rho_2, \cdots, \rho_n$ are the eigenvalues of $R_{ED}(G)$. Then*

$$\sum_{i=1}^{n} \rho_i = |ED|$$

and

$$\sum_{i=1}^{n} \rho_i^2 = |ED| + 2\sum_{i<j} \frac{1}{d_i d_j},$$

where ED is minimum equitable dominating set.

Proof (i) The sum of eigenvalues of $R_{ED}(G)$ is

$$\sum_{i=1}^{n} \rho_i = \sum_{i=1}^{n} r_{ii} = |ED|.$$

(ii) Consider, the sum of squares of $\rho_1, \rho_2, \ldots, \rho_n$ is,

$$\begin{aligned}
\sum_{i=1}^{n} \rho_i^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n} r_{ij}\ r_{ji} = \sum_{i=1}^{n} (r_{ii})^2 + \sum_{i\neq j} r_{ij}\ r_{ji} \\
&= \sum_{i=1}^{n} (r_{ii})^2 + 2\sum_{i<j} (r_{ij})^2 \\
\sum_{i=1}^{n} \rho_i^2 &= |ED| + 2\sum_{i<j} \frac{1}{d_i d_j}.
\end{aligned}$$

□

Upper and lower bounds for $RE_{ED}(G)$ is similar proof to McClelland's inequalities [10], are given below

Theorem 2.2(Upper Bound) *Let G be a graph with ED is minimum equitable dominating set. Then*

$$RE_{ED}(G) \ \leq \ \sqrt{n\left[|ED| + 2\sum_{i<j} \frac{1}{d_i d_j}\right]}.$$

Proof Let $\rho_1, \rho_2, \cdots, \rho_n$ be the eigenvalues of $R_{ED}(G)$. By Cauchy-Schwartz inequality, we have

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right), \tag{1}$$

where a and b are any real numbers.

If $a_i = 1$, $b_i = |\rho_i|$ in (1), we get

$$\left(\sum_{i=1}^{n} |\rho_i|\right)^2 \leq \left(\sum_{i=1}^{n} 1^2\right)\left(\sum_{i=1}^{n} |\rho_i|^2\right), \quad [RE_{ED}(G)]^2 \leq n\left[|ED| + 2\sum_{i<j}\frac{1}{d_i d_j}\right]$$

by Lemma 2.1,

$$RE_{ED}(G) \leq \sqrt{n\left[|ED| + 2\sum_{i<j}\frac{1}{d_i d_j}\right]}.$$

Theorem 2.3(Lower Bound) *Let G be a graph with $|ED|$ is minimum equitable dominating set and d_i is degree of v_i. Then*

$$RE_{ED}(G) \geq \sqrt{|ED| + 2\sum_{i<j}\frac{1}{d_i d_j} + n(n-1)D^{\frac{2}{n}}},$$

where $D = \prod_{i=1}^{n} |\rho_i|$.

Proof Consider

$$[RE_{ED}(G)]^2 = \left[\sum_{i=1}^{n} |\rho_i|\right]^2 = \sum_{i=1}^{n} |\rho_i|^2 + \sum_{i \neq j} |\rho_i|\ |\rho_j|. \tag{2}$$

By using arithmetic and geometric mean inequality, we have

$$\begin{aligned}
\frac{1}{n(n-1)}\sum_{i\neq j} |\rho_i|\ |\rho_j| &\geq \left(\prod_{i\neq j} |\rho_i|\ |\rho_j|\right)^{\frac{1}{n(n-1)}} \\
\sum_{i\neq j} |\rho_i|\ |\rho_j| &\geq n(n-1)\left(\prod_{i=1}^{n} |\rho_i|^{2(n-1)}\right)^{\frac{1}{n(n-1)}} \\
\sum_{i\neq j} |\rho_i|\ |\rho_j| &\geq n(n-1)\left(\prod_{i=1}^{n} |\rho_i|\right)^{2/n}.
\end{aligned} \tag{3}$$

Now using (3) in (2), we get

$$[RE_{ED}(G)]^2 \geq \sum_{i=1}^{n} |\rho_i|^2 + n(n-1)\left(\prod_{i=1}^{n} |\rho_i|\right)^{2/n},$$

$$[RE_{ED}(G)]^2 \geq |ED| + 2\sum_{i<j}\frac{1}{d_i d_j} + n(n-1)D^{\frac{2}{n}}, \quad \text{where} \quad D = \prod_{i=1}^{n} |\rho_i|,$$

$$RE_{ED}(G) \geq \sqrt{|ED| + 2\sum_{i<j}\frac{1}{d_i d_j} + n(n-1)D^{\frac{2}{n}}}.$$

§3. Bounds for Largest Eigenvalue of $R_{ED}(G)$ and its Energy

Proposition 3.1 *Let G be a graph and $\rho_1(G) = max_{1\leq i\leq n}\{|\rho_i|\}$ be the largest eigenvalue of $R_{ED}(G)$. Then*

$$\sqrt{\frac{1}{n}\left[|ED| + 2\sum_{i<j}\frac{1}{d_id_j}\right]} \leq \rho_1(G) \leq \sqrt{|ED| + 2\sum_{i<j}\frac{1}{d_id_j}}.$$

Proof Consider,

$$\begin{aligned}\rho_1^2(G) &= max_{1\leq i\leq n}\ \{|\rho_i|^2\} \leq \sum_{i=1}^{n}|\rho_i|^2 = |ED| + 2\sum_{i<j}\frac{1}{d_id_j}\\ \rho_1(G) &\leq \sqrt{|ED| + 2\sum_{i<j}\frac{1}{d_id_j}}\end{aligned}$$

Next,

$$\begin{aligned} n\ \rho_1^2(G) &\geq \sum_{i=1}^{n}\rho_i^2 = |ED| + 2\sum_{i<j}\frac{1}{d_id_j}\\ \rho_1^2(G) &\geq \frac{1}{n}\left[|ED| + 2\sum_{i<j}\frac{1}{d_id_j}\right]\\ \rho_1(G) &\geq \sqrt{\frac{1}{n}\left[|ED| + 2\sum_{i<j}\frac{1}{d_id_j}\right]}\end{aligned}$$

Therefore,

$$\sqrt{\frac{1}{n}\left[|ED| + 2\sum_{i<j}\frac{1}{d_id_j}\right]} \leq \rho_1(G) \leq \sqrt{|ED| + 2\sum_{i<j}\frac{1}{d_id_j}}.$$

Proposition 3.2 *If G is a graph and $n \leq |ED| + 2\sum_{i<j}\frac{1}{d_id_j}$, then*

$$RE_{ED}(G) \leq \frac{|ED| + 2\sum_{i<j}\frac{1}{d_id_j}}{n} + \sqrt{(n-1)\left[|ED| + 2\sum_{i<j}\frac{1}{d_id_j} - \left(\frac{|ED| + 2\sum_{i<j}\frac{1}{d_id_j}}{n}\right)^2\right]}.$$

Proof We know that,

$$\sum_{i=1}^{n}\rho_i^2 = |ED| + 2\sum_{i<j}\frac{1}{d_id_j}, \quad \sum_{i=2}^{n}\rho_i^2 = |ED| + 2\sum_{i<j}\frac{1}{d_id_j} - \rho_1^2 \tag{4}$$

By Cauchy-Schwarz inequality, we have

$$\left(\sum_{i=2}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=2}^{n} a_i^2\right)\left(\sum_{i=2}^{n} b_i^2\right).$$

If $a_i = 1$ and $b_i = |\rho_i|$, we have

$$\begin{aligned}
\left(\sum_{i=2}^{n} |\rho_i|\right)^2 &\leq (n-1)\sum_{i=2}^{n} |\rho_i|^2 \\
[RE_{ED}(G) - |\rho_1|]^2 &\leq (n-1)\left[|ED| + 2\sum_{i<j}\frac{1}{d_i d_j} - \rho_1^2\right] \\
RE_{ED}(G) &\leq \rho_1 + \sqrt{(n-1)\left[|ED| + 2\sum_{i<j}\frac{1}{d_i d_j} - \rho_1^2\right]} \qquad (5)
\end{aligned}$$

Consider the function,

$$F(x) = x + \sqrt{(n-1)\left[|ED| + 2\sum_{i<j}\frac{1}{d_i d_j} - x^2\right]}.$$

Then,

$$F'(x) = 1 - \frac{x\sqrt{(n-1)}}{\sqrt{|ED| + 2\sum_{i<j}\frac{1}{d_i d_j} - x^2}}.$$

Here $F(x)$ is decreasing in

$$\left(\frac{1}{n}\left[|ED| + 2\sum_{i<j}\frac{1}{d_i d_j}\right], |ED| + 2\sum_{i<j}\frac{1}{d_i d_j}\right).$$

We know that $F'(x) \leq 0$,

$$1 - \frac{x\sqrt{(n-1)}}{\sqrt{|ED| + 2\sum_{i<j}\frac{1}{d_i d_j} - x^2}} \leq 0.$$

We have

$$x \geq \sqrt{\frac{|ED| + 2\sum_{i<j}\frac{1}{d_i d_j}}{n}}.$$

Since,

$$n \leq |ED| + 2\sum_{i<j}\frac{1}{d_i d_j} \quad \text{and} \quad \frac{|ED| + 2\sum_{i<j}\frac{1}{d_i d_j}}{n} \leq \rho_1,$$

we have

$$\sqrt{\frac{|ED|+2\sum_{i<j}\frac{1}{d_id_j}}{n}} \leq \frac{|ED|+2\sum_{i<j}\frac{1}{d_id_j}}{n} \leq \rho_1 \leq |ED|+2\sum_{i<j}\frac{1}{d_id_j}.$$

Then, equation (5) become

$$RE_{ED}(G) \leq \frac{|ED|+2\sum_{i<j}\frac{1}{d_id_j}}{n} + \sqrt{(n-1)\left[|ED|+2\sum_{i<j}\frac{1}{d_id_j} - \left(\frac{|ED|+2\sum_{i<j}\frac{1}{d_id_j}}{n}\right)^2\right]}.$$

$\square$

§4. Minimum Equitable Dominating Randić Energy of Some Graphs

Theorem 4.1 *If K_n is complete graph with n vertices, then minimum equitable dominating Randić energy of K_n is*

$$RE_{ED}(K_n) = \frac{3n-5}{n-1}.$$

Proof Let K_n be the complete graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and minimum equitable dominating set is $ED = \{v_1\}$, we have characteristic polynomial of $R_{ED}(K_n)$ is

$$|\rho I - R_{ED}(K_n)| = \begin{vmatrix} \rho-1 & \frac{-1}{n-1} & \frac{-1}{n-1} & \cdots & \frac{-1}{n-1} & \frac{-1}{n-1} & \frac{-1}{n-1} \\ \frac{-1}{n-1} & \rho & \frac{-1}{n-1} & \cdots & \frac{-1}{n-1} & \frac{-1}{n-1} & \frac{-1}{n-1} \\ \frac{-1}{n-1} & \frac{-1}{n-1} & \rho & \cdots & \frac{-1}{n-1} & \frac{-1}{n-1} & \frac{-1}{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{-1}{n-1} & \frac{-1}{n-1} & \frac{-1}{n-1} & \cdots & \rho & \frac{-1}{n-1} & \frac{-1}{n-1} \\ \frac{-1}{n-1} & \frac{-1}{n-1} & \frac{-1}{n-1} & \cdots & \frac{-1}{n-1} & \rho & \frac{-1}{n-1} \\ \frac{-1}{n-1} & \frac{-1}{n-1} & \frac{-1}{n-1} & \cdots & \frac{-1}{n-1} & \frac{-1}{n-1} & \rho \end{vmatrix}_{n\times n}.$$

$R'_k = R_k - R_2,\ k = 3, 4, \cdots, n-1, n$. Then, we get $\left(\rho + \frac{1}{n-1}\right)$ common from R_3 to R_n and we have

$$|\rho I - R_{ED}(K_n)| = \left(\rho + \frac{1}{n-1}\right)^{n-2} \begin{vmatrix} \rho-1 & \frac{-1}{n-1} & \frac{-1}{n-1} & \cdots & \frac{-1}{n-1} & \frac{-1}{n-1} & \frac{-1}{n-1} \\ \frac{-1}{n-1} & \rho & \frac{-1}{n-1} & \cdots & \frac{-1}{n-1} & \frac{-1}{n-1} & \frac{-1}{n-1} \\ 0 & -1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & -1 & 0 & \cdots & 1 & 0 & 0 \\ 0 & -1 & 0 & \cdots & 0 & 1 & 0 \\ 0 & -1 & 0 & \cdots & 0 & 0 & 1 \end{vmatrix}_{n\times n}.$$

$C_2' = C_2 + C_3 + \cdots + C_n$, we get the characteristic polynomial

$$\begin{aligned} |\rho I - R_{ED}(K_n)| &= \left(\rho + \frac{1}{n-1}\right)^{n-2}\left[\rho^2 - \left(\frac{2n-3}{n-1}\right)\rho + \left(\frac{n-3}{n-1}\right)\right], \\ SpecR_{ED}(K_n) &= \begin{pmatrix} \frac{-1}{n-1} & \frac{(2n-3)-\sqrt{4n-3}}{2(n-1)} & \frac{(2n-3)+\sqrt{4n-3}}{2(n-1)} \\ n-2 & 1 & 1 \end{pmatrix}. \end{aligned}$$

The minimum equitable dominating Randić energy of K_n is

$$RE_{ED}(K_n) = \frac{3n-5}{n-1}. \qquad \square$$

Theorem 4.2 *If $S_n, (n \geq 4)$ is star graph with n vertices, then minimum equitable dominating Randić energy of S_n is*

$$RE_{ED}(S_n) = n.$$

Proof Let $S_n, (n \geq 4)$ be the star graph with vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$ and minimum equitable dominating set is $ED = \{v_1, v_2, \cdots, v_n\}$, we have characteristic polynomial of $R_{ED}(S_n)$ is

$$|\rho I - R_{ED}(S_n)| = \begin{vmatrix} \rho-1 & \frac{-1}{\sqrt{n-1}} & \frac{-1}{\sqrt{n-1}} & \cdots & \frac{-1}{\sqrt{n-1}} & \frac{-1}{\sqrt{n-1}} & \frac{-1}{\sqrt{n-1}} \\ \frac{-1}{\sqrt{n-1}} & \rho-1 & 0 & \cdots & 0 & 0 & 0 \\ \frac{-1}{\sqrt{n-1}} & 0 & \rho-1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{-1}{\sqrt{n-1}} & 0 & 0 & \cdots & \rho-1 & 0 & 0 \\ \frac{-1}{\sqrt{n-1}} & 0 & 0 & \cdots & 0 & \rho-1 & 0 \\ \frac{-1}{\sqrt{n-1}} & 0 & 0 & \cdots & 0 & 0 & \rho-1 \end{vmatrix}_{n\times n}.$$

$R_k' = R_k - R_2,\ k = 3, 4, \cdots, n$. Then taking $(\rho - 1)$ common from R_3 to R_n, we get

$$|\rho I - R_{ED}(S_n)| = (\rho-1)^{n-2}\begin{vmatrix} \rho-1 & \frac{-1}{\sqrt{n-1}} & \frac{-1}{\sqrt{n-1}} & \cdots & \frac{-1}{\sqrt{n-1}} & \frac{-1}{\sqrt{n-1}} & \frac{-1}{\sqrt{n-1}} \\ \frac{-1}{\sqrt{n-1}} & \rho-1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & -1 & 0 & \cdots & 1 & 0 & 0 \\ 0 & -1 & 0 & \cdots & 0 & 1 & 0 \\ 0 & -1 & 0 & \cdots & 0 & 0 & 1 \end{vmatrix}_{n\times n}.$$

$C_2' = C_2 + C_3 + \cdots + C_n$. Then, the characteristic polynomial

$$|\rho I - R_{ED}(S_n)| = \rho(\rho-1)^{n-2}(\rho-2),$$

$$SpecR_{ED}(S_n) = \begin{pmatrix} 0 & 1 & 2 \\ \\ 1 & n-2 & 1 \end{pmatrix}.$$

The minimum equitable dominating Randić energy of S_n is $RE_{ED}(S_n) = n$. □

Theorem 4.3 *If $K_{m,n}$, where $m < n$ and $|m-n| \geq 2$ is complete bipartite graph with $m+n$ vertices, then minimum equitable dominating Randić energy of $K_{m,n}$ is $RE_{ED}(K_{m,n}) = m+n$.*

Proof Let $K_{m,n}$, where $m < n$ and $|m-n| \geq 2$ be the complete bipartite graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_m, u_1, u_2, \cdots, u_n\}$ and minimum equitable dominating set is $ED = V(G)$, we have characteristic polynomial of $R_{ED}(K_{m,n})$ is

$$|\rho I - R_{ED}(K_{m,n})| = \begin{vmatrix} \rho-1 & 0 & \ldots & 0 & 0 & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & \cdots & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} \\ 0 & \rho-1 & \ldots & 0 & 0 & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & \cdots & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \rho-1 & 0 & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & \cdots & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} \\ 0 & 0 & \cdots & 0 & \rho-1 & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & \cdots & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} \\ \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & \cdots & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & \rho-1 & 0 & \cdots & 0 & 0 \\ \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & \cdots & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & 0 & \rho-1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & \cdots & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & 0 & 0 & \cdots & \rho-1 & 0 \\ \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & \cdots & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & 0 & 0 & \cdots & 0 & \rho-1 \end{vmatrix}.$$

$R'_k = R_k - R_m,\ k = 1, 2, 3, \cdots, m-1$ and $R'_d = R_d - R_{m+1},\ d = m+2, m+3, \cdots, m+n$. Then, taking $(\rho - 1)$ common from R_1 to R_{m-1} and R_{m+2} to R_{m+n}, we get

$$(\rho-1)^{m+n-2} \begin{vmatrix} 1 & 0 & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & \rho-1 & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & \cdots & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} \\ \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & \cdots & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & \rho-1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & -1 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & -1 & 0 & \cdots & 0 & 1 \end{vmatrix}.$$

$C'_{m+1} = C_{m+1} + C_{m+2} + \cdots + C_{m+n}$,

$$(\rho-1)^{m+n-2}\begin{vmatrix} 1 & 0 & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & \rho-1 & \frac{-n}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & \cdots & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} \\ \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & \cdots & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & \rho-1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{vmatrix}.$$

The characteristic polynomial $|\rho I - R_{ED}(K_{m,n})| = \rho\,(\rho-1)^{m+n-2}\,(\rho-2)$,

$$SpecR_{ED}(K_{m,n}) = \begin{pmatrix} 0 & 1 & 2 \\ 1 & m+n-2 & 1 \end{pmatrix}.$$

The minimum equitable dominating Randić energy of $K_{m,n}$ is $RE_{ED}(K_{m,n}) = m+n$. $\square$

Theorem 4.4 *If $K_{n\times 2}$, $(n \geq 3)$ is cocktail party graph with $2n$ vertices, then minimum equitable dominating Randić energy of $K_{n\times 2}$ is $\frac{4n-6}{n-1}$.*

Proof Let $K_{n\times 2}$, $(n \geq 3)$ be the cocktail party graph with vertex set $V(G) = \{v_1, v_2, \cdots, v_n, u_1, u_2, \cdots, u_n\}$ and minimum equitable dominating set is $ED = \{v_1, u_1\}$, we have characteristic polynomial of $R_{ED}(K_{n\times 2})$ is

$$|\rho I - R_{ED}(K_{n\times 2})| = \begin{vmatrix} \lambda-1 & 0 & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \cdots & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \cdots & \frac{-1}{2n-2} & \frac{-1}{2n-2} \\ 0 & \lambda-1 & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \cdots & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \cdots & \frac{-1}{2n-2} & \frac{-1}{2n-2} \\ \frac{-1}{2n-2} & \frac{-1}{2n-2} & \lambda & \frac{-1}{2n-2} & \cdots & \frac{-1}{2n-2} & 0 & \cdots & \frac{-1}{2n-2} & \frac{-1}{2n-2} \\ \frac{-1}{2n-2} & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \lambda & \cdots & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \cdots & \frac{-1}{2n-2} & \frac{-1}{2n-2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{-1}{2n-2} & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \cdots & \lambda & \frac{-1}{2n-2} & \cdots & \frac{-1}{2n-2} & 0 \\ \frac{-1}{2n-2} & \frac{-1}{2n-2} & 0 & \frac{-1}{2n-2} & \cdots & \frac{-1}{2n-2} & \lambda & \cdots & \frac{-1}{2n-2} & \frac{-1}{2n-2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{-1}{2n-2} & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \cdots & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \cdots & \lambda & \frac{-1}{2n-2} \\ \frac{-1}{2n-2} & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \cdots & 0 & \frac{-1}{2n-2} & \cdots & \frac{-1}{2n-2} & \lambda \end{vmatrix}.$$

$R'_k = R_k - R_3,\ k = 4, 5, \cdots, n$ and $R'_{n+k} = R_{n+k} - R_{k+1},\ k = 2, 3, \cdots, n$, we get

$$|\rho I - R_{ED}(K_{n\times 2})| = \begin{vmatrix} \lambda-1 & 0 & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \cdots & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \cdots & \frac{-1}{2n-2} & \frac{-1}{2n-2} \\ 0 & \lambda-1 & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \cdots & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \cdots & \frac{-1}{2n-2} & \frac{-1}{2n-2} \\ \frac{-1}{2n-2} & \frac{-1}{2n-2} & \lambda & \frac{-1}{2n-2} & \cdots & \frac{-1}{2n-2} & 0 & \cdots & \frac{-1}{2n-2} & \frac{-1}{2n-2} \\ 0 & 0 & \frac{-1}{2n-2}-\lambda & \lambda+\frac{1}{2n-2} & \cdots & 0 & \frac{-1}{2n-2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \frac{-1}{2n-2}-\lambda & 0 & \cdots & \lambda+\frac{1}{2n-2} & \frac{-1}{2n-2} & \cdots & 0 & \frac{1}{2n-2} \\ 0 & 0 & -\lambda & 0 & \cdots & 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & \lambda & 0 \\ 0 & 0 & 0 & 0 & \cdots & -\lambda & 0 & \cdots & 0 & \lambda \end{vmatrix}.$$

$C'_3 = C_3 + C_4 + \cdots + C_n + C_{n+1} + \cdots + C_{2n}$ and $C'_k = C_k + C_{n+(k-1)},\ k = 4, 5, \cdots, n+1$. We get

$$|\rho I - R_{ED}(K_{n\times 2})| = \begin{vmatrix} \lambda-1 & 0 & -1 & \frac{-1}{n-1} & \cdots & \frac{-1}{n-1} & \frac{-1}{2n-2} & \cdots & \frac{-1}{2n-2} & \frac{-1}{2n-2} \\ 0 & \lambda-1 & -1 & \frac{-1}{n-1} & \cdots & \frac{-1}{n-1} & \frac{-1}{2n-2} & \cdots & \frac{-1}{2n-2} & \frac{-1}{2n-2} \\ \frac{-1}{2n-2} & \frac{-1}{2n-2} & \lambda-\frac{(2n-4)}{(2n-2)} & \frac{-1}{n-1} & \cdots & \frac{-1}{n-1} & 0 & \cdots & \frac{-1}{2n-2} & \frac{-1}{2n-2} \\ 0 & 0 & 0 & \lambda+\frac{1}{n-1} & \cdots & 0 & \frac{-1}{2n-2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda+\frac{1}{n-1} & \frac{-1}{2n-2} & \cdots & 0 & \frac{1}{2n-2} \\ 0 & 0 & 0 & 0 & \cdots & 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & \lambda & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \lambda \end{vmatrix}.$$

The characteristic polynomial

$$|\rho I - R_{ED}(K_{n\times 2})| = \rho^{n-1}(\rho-1)\left(\rho+\frac{1}{n-1}\right)^{n-2}\left[\rho^2-\left(\frac{2n-3}{n-1}\right)\rho+\left(\frac{n-3}{n-1}\right)\right],$$

$$SpecR_{ED}(K_{n\times 2}) = \begin{pmatrix} \frac{-1}{n-1} & 0 & 1 & \frac{(2n-3)-\sqrt{4n-3}}{2(n-1)} & \frac{(2n-3)+\sqrt{4n-3}}{2(n-1)} \\ n-2 & n-1 & 1 & 1 & 1 \end{pmatrix}.$$

The minimum equitable dominating Randić energy of $K_{n\times 2}$ is

$$RE_{ED}(K_{n\times 2}) = \frac{4n-6}{n-1},$$

where $n \geq 3$. □

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