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Semifull Line (Block) Signed Graphs

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Abstract: In this paper we introduced the new notions semifull signed graph and semifull line (block) signed graph of a signed graph and its properties are obtained. Also, we obtained the structural characterizations of these notions. Further, we presented some switching equivalent characterizations.

Key Words: Signed graphs, balance, switching, semifull signed graph, semifull line (block) signed graph, negation of a signed graph, semifull Smarandachely graph.

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§1. Introduction

For all terminology and notation in graph theory we refer the reader to consult any one of the standard text-books by Chartrand and Zhang [2], Harary [3] and West [12].

If $B = \{u_1, u_2, \cdots, u_r, r \geq 2\}$ is a block of a graph Γ, then we say that vertex u_1 and block B are incident with each other, as are u_2 and B and so on. If two blocks B_1 and B_2 of G are incident with a common cut vertex, then they are adjacent blocks. If $B = \{e_1, e_2, \cdots, e_s, s \geq 1\}$ is a block of a graph Γ, then we say that an edge e_1 and block B are incident with each other, as are e_2 and B and so on. This concept was introduced by Kulli [7]. The vertices, edges and blocks of a graph are called its members.

The line graph $L(\Gamma)$ of a graph Γ is the graph whose vertex set is the set of edges of Γ in which two vertices are adjacent if the corresponding edges are adjacent (see [3]).

The semifull graph $\mathcal{SF}(\Gamma)$ of a graph Γ is the graph whose vertex set is the union of vertices, edges and blocks of Γ in which two vertices are adjacent if the corresponding members of Γ are adjacent or one corresponds to a vertex and the other to an edge incident with it or one corresponds to a block B of Γ and the other to a vertex v of Γ and v is in B. In fact, this notion was introduced by Kulli [8]. Generally, for a subset $B' \subset B$, a semifull Smarandachely graph $\mathcal{SSF}(\Gamma)$ of a graph Γ on B' is the graph with $V(\mathcal{SSF}(\Gamma)) = V(\Gamma) \bigcup E(\Gamma) \bigcup B'$, and two vertices are adjacent in $\mathcal{SSF}(\Gamma)$ if the corresponding members of Γ are adjacent or one corresponds to a vertex and the other to an edge incident with it or one corresponds to a block

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B' of Γ and the other to a vertex v of Γ with $v \in B'$. Clearly, $\mathcal{SSF}(\Gamma) = \mathcal{SF}(\Gamma)$ if $B' = B$.

In [9], the author introduced the new notions called "*semifull line graphs and semifull block graphs*" as follows: The semifull line graph $\mathcal{SFL}(\Gamma)$ of a graph Γ is the graph whose vertex set is the union of the set of vertices, edges and blocks of Γ in which two vertices are adjacent $\mathcal{SFL}(\Gamma)$ if the corresponding vertices and edges of Γ are adjacent or one corresponds to a vertex of Γ and other to an edge incident with it or one corresponds to a block B of Γ and other to a vertex v of Γ and v is in B.

The semifull block graph $\mathcal{SFB}(\Gamma)$ of a graph Γ is the graph whose vertex set is the union of the set of vertices, edges and blocks of Γ in which two vertices are adjacent in $\mathcal{SFB}(\Gamma)$ if the corresponding vertices and blocks of Γ are adjacent or one corresponds to a vertex of Γ and other to an edge incident with it or one corresponds to a block B of Γ and other to a vertex v of Γ and v is in B.

A *signed graph* is an ordered pair $\Sigma = (\Gamma, \sigma)$, where $\Gamma = (V, E)$ is a graph called *underlying graph of* Σ and $\sigma : E \to \{+, -\}$ is a function. We say that a signed graph is *connected* if its underlying graph is connected. A signed graph $\Sigma = (\Gamma, \sigma)$ is *balanced*, if every cycle in Σ has an even number of negative edges (See [4]). Equivalently, a signed graph is balanced if product of signs of the edges on every cycle of Σ is positive.

Signed graphs Σ_1 and Σ_2 are isomorphic, written $\Sigma_1 \cong \Sigma_2$, if there is an isomorphism between their underlying graphs that preserves the signs of edges.

The theory of balance goes back to Heider [6] who asserted that a social system is balanced if there is no tension and that unbalanced social structures exhibit a tension resulting in a tendency to change in the direction of balance. Since this first work of Heider, the notion of balance has been extensively studied by many mathematicians and psychologists. In 1956, Cartwright and Harary [4] provided a mathematical model for balance through graphs.

A *marking* of Σ is a function $\zeta : V(\Gamma) \to \{+, -\}$. Given a signed graph Σ one can easily define a marking ζ of Σ as follows: For any vertex $v \in V(\Sigma)$,

$$\zeta(v) = \prod_{uv \in E(\Sigma)} \sigma(uv),$$

the marking ζ of Σ is called *canonical marking* of Σ.

The following are the fundamental results about balance, the second being a more advanced form of the first. Note that in a bipartition of a set, $V = V_1 \cup V_2$, the disjoint subsets may be empty.

Theorem 1.1 *A signed graph Σ is balanced if and only if either of the following equivalent conditions is satisfied:*

(1)(Harary [4]) *Its vertex set has a bipartition $V = V_1 \cup V_2$ such that every positive edge joins vertices in V_1 or in V_2, and every negative edge joins a vertex in V_1 and a vertex in V_2.*

(2)(Sampathkumar [10]) *There exists a marking μ of its vertices such that each edge uv in Γ satisfies $\sigma(uv) = \zeta(u)\zeta(v)$.*

Let $\Sigma = (\Gamma, \sigma)$ be a signed graph. *Complement* of Σ is a signed graph $\overline{\Sigma} = (\overline{\Gamma}, \sigma')$, where for any edge $e = uv \in \overline{\Gamma}$, $\sigma'(uv) = \zeta(u)\zeta(v)$. Clearly, $\overline{\Sigma}$ as defined here is a balanced signed graph due to Theorem 1.1.

A switching function for Σ is a function $\zeta : V \rightarrow \{+, -\}$. The switched signature is $\sigma^{\zeta}(e) := \zeta(v)\sigma(e)\zeta(w)$, where e has end points v, w. The switched signed graph is $\Sigma^{\zeta} := (\Sigma|\sigma^{\zeta})$. We say that Σ switched by ζ. Note that $\Sigma^{\zeta} = \Sigma^{-\zeta}$ (see [1]).

If $X \subseteq V$, switching Σ by X (or simply switching X) means reversing the sign of every edge in the cutset $E(X, X^c)$. The switched signed graph is Σ^X. This is the same as Σ^{ζ} where $\zeta(v) := -$ if and only if $v \in X$. Switching by ζ or X is the same operation with different notation. Note that $\Sigma^X = \Sigma^{X^c}$.

Signed graphs Σ_1 and Σ_2 are switching equivalent, written $\Sigma_1 \sim \Sigma_2$ if they have the same underlying graph and there exists a switching function ζ such that $\Sigma_1^{\zeta} \cong \Sigma_2$. The equivalence class of Σ,

$$[\Sigma] := \{\Sigma' : \Sigma' \sim \Sigma\}$$

is called the its switching class.

Similarly, Σ_1 and Σ_2 are switching isomorphic, written $\Sigma_1 \cong \Sigma_2$, if Σ_1 is isomorphic to a switching of Σ_2. The equivalence class of Σ is called its switching isomorphism class.

Two signed graphs $\Sigma_1 = (\Gamma_1, \sigma_1)$ and $\Sigma_2 = (\Gamma_2, \sigma_2)$ are said to be *weakly isomorphic* (see [11]) or *cycle isomorphic* (see [13]) if there exists an isomorphism $\phi : \Gamma_1 \rightarrow \Gamma_2$ such that the sign of every cycle Z in Σ_1 equals to the sign of $\phi(Z)$ in Σ_2. The following result is well known (see [13]):

Theorem 1.2(T. Zaslavsky, [13]) *Two signed graphs Σ_1 and Σ_2 with the same underlying graph are switching equivalent if and only if they are cycle isomorphic.*

§2. Semifull Line Signed Graphs

Motivated by the existing definition of complement of a signed graph, we now extend the notion called semifull line graphs to realm of signed graphs: the *semifull line signed graph* $\mathcal{SFL}(\Sigma)$ of a signed graph $\Sigma = (\Gamma, \sigma)$ as a signed graph $\mathcal{SFL}(\Sigma) = (\mathcal{SFL}(\Gamma), \sigma')$, where for any edge e_1e_2 in $\mathcal{SFL}(\Gamma)$, $\sigma'(e_1e_2) = \sigma(e_1)\sigma(e_2)$. Further, a signed graph $\Sigma = (\Gamma, \sigma)$ is called semifull line signed graph, if $\Sigma \cong \mathcal{SFL}(\Sigma')$ for some signed graph Σ'. The following result indicates the limitations of the notion of semifull line signed graphs as introduced above, since the entire class of unbalanced signed graphs is forbidden to be semifull line signed graphs.

Theorem 2.1 *For any signed graph $\Sigma = (\Gamma, \sigma)$, its semifull line signed graph $\mathcal{SFL}(\Sigma)$ is balanced.*

Proof Let $\sigma^{'}$ denote the signing of $\mathcal{SFL}(\Sigma)$ and let the signing σ of Σ be treated as a marking of the vertices of $\mathcal{SFL}(\Sigma)$. Then by definition of $\mathcal{SFL}(\Sigma)$, we see that $\sigma'(e_1e_2) = \sigma(e_1)\sigma(e_2)$, for every edge e_1e_2 of $\mathcal{SFL}(\Sigma)$ and hence, by Theorem 1, the result follows. □

For any positive integer k, the k^{th} iterated semifull line signed graph, $\mathcal{SFL}^k(\Sigma)$ of Σ is defined as follows:

$$\mathcal{SFL}^0(\Sigma) = \Sigma, \mathcal{SFL}^k(\Sigma) = \mathcal{SFL}(\mathcal{SFL}^{k-1}(\Sigma))$$

Corollary 2.2 *For any signed graph $\Sigma = (\Gamma, \sigma)$ and for any positive integer k, $\mathcal{SFL}^k(\Sigma)$ is balanced.*

Proposition 2.3 *For any two signed graphs Σ_1 and Σ_2 with the same underlying graph, their semifull line signed graphs are switching equivalent.*

Proof Suppose $\Sigma_1 = (\Gamma, \sigma)$ and $\Sigma_2 = (\Gamma', \sigma')$ be two signed graphs with $\Gamma \cong \Gamma'$. By Theorem 2.1, $\mathcal{SFL}(\Sigma_1)$ and $\mathcal{SFL}(\Sigma_2)$ are balanced and hence, the result follows from Theorem 1.2. □

The *semifull signed graph* $\mathcal{SF}(\Sigma)$ of a signed graph $\Sigma = (\Gamma, \sigma)$ as a signed graph $\mathcal{SF}(\Sigma) = (\mathcal{SF}(\Gamma), \sigma')$, where for any edge e_1e_2 in $\mathcal{SF}(\Gamma)$, $\sigma'(e_1e_2) = \sigma(e_1)\sigma(e_2)$. Further, a signed graph $\Sigma = (\Gamma, \sigma)$ is called semifull signed graph, if $\Sigma \cong \mathcal{SF}(\Sigma')$ for some signed graph Σ'. The following result indicates the limitations of the notion of semifull signed graphs as introduced above, since the entire class of unbalanced signed graphs is forbidden to be semifull signed graphs.

Theorem 2.4 *For any signed graph $\Sigma = (\Gamma, \sigma)$, its semifull signed graph $\mathcal{SF}(\Sigma)$ is balanced.*

Proof Let $\sigma^{'}$ denote the signing of $\mathcal{SF}(\Sigma)$ and let the signing σ of Σ be treated as a marking of the vertices of $\mathcal{SF}(\Sigma)$. Then by definition of $\mathcal{SF}(\Sigma)$, we see that $\sigma'(e_1e_2) = \sigma(e_1)\sigma(e_2)$, for every edge e_1e_2 of $\mathcal{SF}(\Sigma)$ and hence, by Theorem 1, the result follows. □

For any positive integer k, the k^{th} iterated semifull line signed graph, $\mathcal{SF}^k(\Sigma)$ of Σ is defined as follows:

$$\mathcal{SF}^0(\Sigma) = \Sigma, \mathcal{SF}^k(\Sigma) = \mathcal{SF}(\mathcal{SF}^{k-1}(\Sigma))$$

Corollary 2.5 *For any signed graph $\Sigma = (\Gamma, \sigma)$ and for any positive integer k, $\mathcal{SF}^k(\Sigma)$ is balanced.*

Proposition 2.6 *For any two signed graphs Σ_1 and Σ_2 with the same underlying graph, their semifull signed graphs are switching equivalent.*

Proof Suppose $\Sigma_1 = (\Gamma, \sigma)$ and $\Sigma_2 = (\Gamma', \sigma')$ be two signed graphs with $\Gamma \cong \Gamma'$. By Theorem 2.4, $\mathcal{SF}(\Sigma_1)$ and $\mathcal{SF}(\Sigma_2)$ are balanced and hence, the result follows from Theorem 1.2. □

In [9], the author characterizes graphs such that semifull line graphs and semifull graphs are isomorphic.

Theorem 2.7 *Let Γ be a nontrivial connected graph. The graphs $\mathcal{SFL}(\Gamma)$ and $\mathcal{SF}(\Gamma)$ are isomorphic if and only if Γ is a block.*

In view of the above result, we have the following result that characterizes the family of signed graphs satisfies $\mathcal{SFL}(\Sigma) \sim \mathcal{SF}(\Sigma)$.

Theorem 2.8 *For any signed graph $\Sigma = (\Gamma, \sigma)$, $\mathcal{SFL}(\Sigma) \sim \mathcal{SF}(\Sigma)$ if and only if Γ is a block.*

Proof Suppose that $\mathcal{SFL}(\Sigma) \sim \mathcal{SF}(\Sigma)$. Then clearly, $\mathcal{SFL}(\Gamma) \cong \mathcal{SF}(\Gamma)$. Hence by Theorem 2.7, Γ is a block.

Conversely, suppose that Σ is a signed graph whose underlying graph is a block. Then by Theorem 2.7, $\mathcal{SFL}(\Gamma)$ and $\mathcal{SF}(\Gamma)$ are isomorphic. Since for any signed graph Σ, both $\mathcal{SFL}(\Sigma)$ and $\mathcal{SF}(\Sigma)$ are balanced, the result follows by Theorem 1.2. □

The following result characterize signed graphs which are semifull line signed graphs.

Theorem 2.9 *A signed graph $\Sigma = (\Gamma, \sigma)$ is a semifull line signed graph if and only if Σ is balanced signed graph and its underlying graph Γ is a semifull line graph.*

Proof Suppose that Σ is balanced and Γ is a semifull line graph. Then there exists a graph Γ' such that $\mathcal{SFL}(\Gamma') \cong \Gamma$. Since Σ is balanced, by Theorem 1.1, there exists a marking ζ of Γ such that each edge uv in Σ satisfies $\sigma(uv) = \zeta(u)\zeta(v)$. Now consider the signed graph $\Sigma' = (\Gamma', \sigma')$, where for any edge e in Γ', $\sigma'(e)$ is the marking of the corresponding vertex in Γ. Then clearly, $\mathcal{SFL}(\Sigma') \cong \Sigma$. Hence Σ is a semifull line signed graph.

Conversely, suppose that $\Sigma = (\Gamma, \sigma)$ is a semifull line signed graph. Then there exists a signed graph $\Sigma' = (\Gamma', \sigma')$ such that

$$\mathcal{SFL}(\Sigma') \cong \Sigma.$$

Hence, Γ is the semiful line graph of Γ' and by Theorem 2.1, Σ is balanced. □

In view of the above result, we can easily characterize signed graphs which are semifull signed graphs.

The notion of *negation* $\eta(\Sigma)$ of a given signed graph Σ defined in [5] as follows:

$\eta(\Sigma)$ has the same underlying graph as that of Σ with the sign of each edge opposite to that given to it in Σ. However, this definition does not say anything about what to do with nonadjacent pairs of vertices in Σ while applying the unary operator $\eta(.)$ of taking the negation of Σ.

For a signed graph $\Sigma = (\Gamma, \sigma)$, the $\mathcal{SFL}(\Sigma)$ $(\mathcal{SF}(\Sigma))$ is balanced. We now examine, the conditions under which negation $\eta(\Sigma)$ of $\mathcal{SFL}(\Sigma)$ $(\mathcal{SF}(\Sigma))$ is balanced.

Theorem 2.10 *Let $\Sigma = (\Gamma, \sigma)$ be a signed graph. If $\mathcal{SFL}(\Gamma)$ $(\mathcal{SF}(\Gamma))$ is bipartite then $\eta(\mathcal{SFL}(\Sigma))$ $(\eta(\mathcal{SF}(\Sigma)))$ is balanced.*

Proof Since $\mathcal{SFL}(\Sigma)$ $(\mathcal{SF}(\Sigma))$ is balanced, if each cycle C in $\mathcal{SFL}(\Sigma)$ $(\mathcal{SF}(\Sigma))$ contains even number of negative edges. Also, since $\mathcal{SFL}(\Gamma)$ $(\mathcal{SF}(\Gamma))$ is bipartite, all cycles have even length; thus, the number of positive edges on any cycle C in $\mathcal{SFL}(\Sigma)$ $(\mathcal{SF}(\Sigma))$ is also even. Hence $\eta(\mathcal{SFL}(\Sigma))$ $(\eta(\mathcal{SF}(\Sigma)))$ is balanced. □

§3. Semifull Block Signed Graphs

Motivated by the existing definition of complement of a signed graph, we now extend the notion called semifull block graphs to realm of signed graphs: the *semifull block signed graph* $\mathcal{SFB}(\Sigma)$ of a signed graph $\Sigma = (\Gamma, \sigma)$ as a signed graph $\mathcal{SFB}(\Sigma) = (\mathcal{SFB}(\Gamma), \sigma')$, where for any edge e_1e_2 in $\mathcal{SFB}(\Gamma)$, $\sigma'(e_1e_2) = \sigma(e_1)\sigma(e_2)$. Further, a signed graph $\Sigma = (\Gamma, \sigma)$ is called semifull block signed graph, if $\Sigma \cong \mathcal{SFL}(\Sigma')$ for some signed graph Σ'. The following result indicates the limitations of the notion of semifull block signed graphs as introduced above, since the entire class of unbalanced signed graphs is forbidden to be semifull block signed graphs.

Theorem 3.1 *For any signed graph $\Sigma = (\Gamma, \sigma)$, its semifull block signed graph $\mathcal{SFB}(\Sigma)$ is balanced.*

Proof Let $\sigma^{'}$ denote the signing of $\mathcal{SFB}(\Sigma)$ and let the signing σ of Σ be treated as a marking of the vertices of $\mathcal{SFB}(\Sigma)$. Then by definition of $\mathcal{SFB}(\Sigma)$, we see that $\sigma'(e_1e_2) = \sigma(e_1)\sigma(e_2)$, for every edge e_1e_2 of $\mathcal{SFB}(\Sigma)$ and hence, by Theorem 1, the result follows. □

For any positive integer k, the k^{th} iterated semifull block signed graph, $\mathcal{SFB}^k(\Sigma)$ of Σ is defined as follows:

$$\mathcal{SFB}^0(\Sigma) = \Sigma,\ \mathcal{SFB}^k(\Sigma) = \mathcal{SFB}(\mathcal{SFB}^{k-1}(\Sigma))$$

Corollary 3.2 *For any signed graph $\Sigma = (\Gamma, \sigma)$ and for any positive integer k, $\mathcal{SFB}^k(\Sigma)$ is balanced.*

Proposition 3.3 —it For any two signed graphs Σ_1 and Σ_2 with the same underlying graph, their semifull block signed graphs are switching equivalent.

Proof Suppose $\Sigma_1 = (\Gamma, \sigma)$ and $\Sigma_2 = (\Gamma', \sigma')$ be two signed graphs with $\Gamma \cong \Gamma'$. By Theorem 3.1, $\mathcal{SFB}(\Sigma_1)$ and $\mathcal{SFB}(\Sigma_2)$ are balanced and hence, the result follows from Theorem 1.2. □

In [9], the author characterizes graphs such that semifull block graphs and semifull graphs are isomorphic.

Theorem 3.4 *Let Γ be a nontrivial connected graph. The graphs $\mathcal{SFB}(\Gamma)$ and $\mathcal{SF}(\Gamma)$ are isomorphic if and only if Γ is P_2.*

In view of the above result, we have the following result that characterizes the family of signed graphs satisfies $\mathcal{SFB}(\Sigma) \sim \mathcal{SF}(\Sigma)$.

Theorem 3.5 *For any signed graph $\Sigma = (\Gamma, \sigma)$, $\mathcal{SFB}(\Sigma) \sim \mathcal{SF}(\Sigma)$ if and only if Γ is P_2.*

Proof Suppose that $\mathcal{SFB}(\Sigma) \sim \mathcal{SF}(\Sigma)$. Then clearly, $\mathcal{SFB}(\Gamma) \cong \mathcal{SF}(\Gamma)$. Hence by Theorem 16, Γ is P_2.

Conversely, suppose that Σ is a signed graph whose underlying graph is P_2. Then by Theorem 16, $\mathcal{SFB}(\Gamma)$ and $\mathcal{SF}(\Gamma)$ are isomorphic. Since for any signed graph Σ, both $\mathcal{SFB}(\Sigma)$ and $\mathcal{SF}(\Sigma)$ are balanced, the result follows by Theorem 2. □

In view of the Theorem 2.9, we can easily characterize signed graphs which are semifull block signed graphs.

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