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Slant Submanifolds of a Conformal (k,μ)-Contact Manifold

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Abstract: In this paper, we study the geometry of slant submanifolds of conformal (k,μ)-contact manifold when the tensor field Q is parallel. Further, we give a necessary and sufficient condition for a 3-dimensional slant submanifold of a 5-dimensional conformal (k,μ)-contact manifold to be a proper slant submanifold.

Key Words: (k,μ)-contact manifold; conformal (k,μ)-contact manifold; slant submanifold.

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§1. Introduction

Let (M^{2n},J,g) be a Hermitian manifold of even dimension $2n$, where J and g are the complex structure and Hermitian metric respectively. Then (M^{2n},J,g) is a locally conformal Kähler manifold if there is an open cover $\{U_i\}_{i\in I}$ of M^{2n} and a family $\{f_i\}_{i\in I}$ of C^∞ functions $f_i : U_i \to R$ such that each local metric $g_i = exp(-f_i)g|U_i$ is Kählerian. Here $g|U_i = \iota_i^* g$ where $\iota_i : U_i \to M^{2n}$ is the inclusion. Also (M^{2n},J,g) is globally conformal Kähler if there is a C^∞ function $f : M^{2n} \to R$ such that the metric $exp(f)g$ is Kählerian [11]. In 1955, Libermann [14] initiated the study of locally conformal Kähler manifolds. The geometrical conditions for locally conformal Kähler manifold have been obtained by Visman [22] and examples of these locally conformal Kähler manifolds were given by Triceri in 1982 [21]. In 2001, Banaru [2] succeeded to classify the sixteen classes of almost Hermitian Kirichenko's tensors. The locally conformal Kähler manifold is one of the sixteen classes of almost Hermitian manifolds. It is known that there is a close relationship between Kähler and contact metric manifolds because Kählerian structures can be made into contact structures by adding a characteristic vector field ξ. The contact structures consists of Sasakian and non-Sasakian cases. In 1972, Kenmotsu introduced a class of contact metric manifolds, called Kenmotsu manifolds, which are not Sasakian [13]. Later in 1995, Blair, Koufogiorgos and Papantoniou [4] introduced the notion of (k,μ)-contact manifold which consists of both Sasakian and non-Sasakian.

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On the other hand, Chen [7] introduced the notion of slant submanifold for an almost Hermitian manifold, as a generalization of both holomorphic and totally real submanifolds. Examples of slant submanifolds of C^2 and C^4 were given by Chen and Tazawa [8, 9, 10], while slant submanifolds of Kaehler manifold were given by Maeda, Ohnita and Udagawa [17]. The notion of slant immersion of a Riemannian manifold into an almost contact metric manifold was introduced by Lotta [15] and he has proved some properties of such immersions. Later, the study of slant submanifolds was enriched by the authors of [6, 12, 16, 18, 19] and many others. Recently, the authors of [1] introduced conformal Sasakian manifold and studied slant submanifolds of the conformal Sasakian manifold. As a generalization to the work of [1] in [20], we defined conformal (k,μ)-contact manifold and studied invariant and anti-invariant submanifolds of it. Our aim in the present paper is to extend the study of slant submanifold to the setting of conformal (k,μ)-contact manifold.

The paper is organized as follows: In section 2, we recall the notion and some results of (k,μ)-contact manifold and their submanifolds, which are used for further study. In section 3, we introduce a conformal (k,μ)-contact manifold and give some properties of submanifolds of it. Section 4 deals with the study of slant submaifolds of (k,μ)-contact manifold. Section 5 is devoted to the study of characterization of three-dimensional slant submanifolds of (k,μ)-contact manifold via covariant derivative of T and T^2, where T is the tangent projection of (k,μ)-contact manifold.

§2. Preliminaries

2.1 (k,μ)-Contact Manifold

Let $\tilde{M}$ be a $(2n+1)$-dimensional almost contact metric manifold with structure $(\tilde{\phi},\tilde{\xi},\tilde{\eta},\tilde{g})$, where $\tilde{\phi},\tilde{\xi},\tilde{\eta}$ are the tensor fields of type $(1,1)$, $(1,0)$, $(0,1)$ respectively, and $\tilde{g}$ is a Riemannian metric on $\tilde{M}$ satisfying

$$\begin{aligned} \tilde{\phi}^2 &= -I+\tilde{\eta}\otimes\tilde{\xi}, \quad \tilde{\eta}(\tilde{\xi})=1, \quad \tilde{\phi}\tilde{\xi}=0, \quad \tilde{\eta}\cdot\tilde{\phi}=0, \\ \tilde{g}(\tilde{\phi}X,\tilde{\phi}Y) &= \tilde{g}(X,Y)-\tilde{\eta}(X)\tilde{\eta}(Y), \quad \tilde{\eta}(X)=\tilde{g}(X,\xi), \end{aligned} \tag{2.1}$$

for all vector fields X,Y on $\tilde{M}$. An almost contact metric structure becomes a contact metric structure if

$$\tilde{g}(X,\tilde{\phi}Y)=d\tilde{\eta}(X,Y).$$

Then the 1-form $\tilde{\eta}$ is contact form and $\tilde{\xi}$ is a characteristic vector field.

We now define a $(1,1)$ tensor field $\tilde{h}$ by $\tilde{h}=\frac{1}{2}\mathfrak{L}_{\tilde{\xi}}\tilde{\phi}$, where $\mathcal{L}$ denotes the Lie differentiation, then $\tilde{h}$ is symmetric and satisfies $\tilde{h}\phi=-\phi\tilde{h}$. Further, a q-dimensional distribution on a manifold $\tilde{M}$ is defined as a mapping D on $\tilde{M}$ which assigns to each point $p\in\tilde{M}$, a q-dimensional subspace D_p of $T_p\tilde{M}$.

The (k,μ)-nullity distribution of a contact metric manifold $\tilde{M}(\phi,\xi,\eta,g)$ is a distribution

$$\begin{aligned} N(k,\mu): p \to N_p(k,\mu) = \{Z \in T_p\tilde{M} : \tilde{R}(X,Y)Z &= k[\tilde{g}(Y,Z)X - \tilde{g}(X,Z)Y] \\ &+\mu[\tilde{g}(Y,Z)\tilde{h}X - \tilde{g}(X,Z)\tilde{h}Y]\} \end{aligned}$$

for all $X, Y \in T\tilde{M}$. Hence if the characteristic vector field $\tilde{\xi}$ belongs to the (k,μ)-nullity distribution, then we have

$$\tilde{R}(X,Y)\tilde{\xi} = k[\tilde{\eta}(Y)X - \tilde{\eta}(X)Y] + \mu[\tilde{\eta}(Y)\tilde{h}X - \tilde{\eta}(X)\tilde{h}Y]. \tag{2.2}$$

The contact metric manifold satisfying the relation (2.2) is called (k,μ) contact metric manifold [4]. It consists of both k-nullity distribution for $\mu = 0$ and Sasakian for $k = 1$. A (k,μ)-contact metric manifold $\tilde{M}(\phi,\xi,\eta,g)$ satisfies

$$(\tilde{\nabla}_X\tilde{\phi})Y = \tilde{g}(X + \tilde{h}X, Y)\tilde{\xi} - \tilde{\eta}(Y)(X + \tilde{h}X) \tag{2.3}$$

for all $X, Y \in T\tilde{M}$, where $\tilde{\nabla}$ denotes the Riemannian connection with respect to $\tilde{g}$. From (2.3), we have

$$\tilde{\nabla}_X\tilde{\xi} = -\tilde{\phi}X - \tilde{\phi}\tilde{h}X \tag{2.4}$$

for all $X, Y \in T\tilde{M}$.

2.2 Submanifold

Assume M is a submanifold of a (k,μ)-contact manifold $\tilde{M}$. Let g and ∇ be the induced Riemannian metric and connections of M, respectively. Then the Gauss and Weingarten formulae are given respectively, by

$$\tilde{\nabla}_XY = \nabla_XY + \sigma(X,Y), \quad \tilde{\nabla}_XN = -A_NX + \nabla^{\perp}_XN \tag{2.5}$$

for all X, Y on TM and $N \in T^{\perp}M$, where $\nabla^{\perp}$ is the normal connection and A is the shape operator of M with respect to the unit normal vector N. The second fundamental form σ and the shape operator A are related by:

$$g(\sigma(X,Y),N) = g(A_NX,Y). \tag{2.6}$$

Let R and $\tilde{R}$ denote the curvature tensor of M and $\tilde{M}$, then, the Gauss and Ricci equations are given by

$$\begin{aligned} \tilde{g}(\tilde{R}(X,Y)Z,W) &= g(R(X,Y)Z,W) - g(\sigma(X,W),\sigma(Y,Z)) + g(\sigma(X,Z),\sigma(Y,W)), \\ \tilde{g}(\tilde{R}(X,Y)N_1,N_2) &= g(R^{\perp}(X,Y)N_1,N_2) - g([A_1,A_2]X,Y) \end{aligned}$$

for all $X, Y, Z, W \in TM$, $N_1, N_2 \in T^{\perp}M$ and A_1, A_2 are shape operators corresponding to N_1, N_2 respectively.

For each $x \in M$ and $X \in T_xM$, we decompose ϕX into tangential and normal components as:

$$\phi X = TX + FX, \tag{2.7}$$

where, T is an endomorphism and F is normal valued 1-form on T_xM. Similarly, for any $N \in T_x^{\perp}M$, we decompose ϕV into tangential and normal components as:

$$\phi N = tN + fN, \tag{2.8}$$

where, t is a tangent valued 1-form and f is an endomorphism on $T_x^{\perp}M$.

2.3 Slant Submanifolds of an Almost Contact Metric Manifold

For any $x \in M$ and $X \in T_xM$ such that X, ξ are linearly independent, the angle $\theta(x) \in [0, \frac{\pi}{2}]$ between ϕX and T_xM is a constant θ, that is θ does not depend on the choice of X and $x \in M$. θ is called the slant angle of M in $\tilde{M}$. Invariant and anti-invariant submanifolds are slant submanifolds with slant angle θ equal to 0 and $\frac{\pi}{2}$, respectively [**?**]. A slant submanifold which is neither invariant nor anti-invariant is called a proper slant submanifold.

We mention the following results for later use.

Theorem 2.1([6]) *Let M be a submanifold of an almost contact metric manifold $\tilde{M}$ such that $\xi \in TM$. Then, M is slant if and only if there exists a constant $\lambda \in [0, 1]$ such that*

$$T^2 = -\lambda(I - \eta \otimes \xi). \tag{2.9}$$

Further more, if θ is the slant angle of M, then $\lambda = cos^2\theta$.

Corollary 2.1([6]) *Let M be a slant submanifold of an almost contact metric manifold $\tilde{M}$ with slant angle θ. Then, for any $X, Y \in TM$, we have*

$$g(TX, TY) = cos^2\theta(g(X, Y) - \eta(X)\eta(Y)), \tag{2.10}$$

$$g(FX, FY) = sin^2\theta(g(X, Y) - \eta(X)\eta(Y)). \tag{2.11}$$

Lemma 2.1([15]) *Let M be a slant submanifold of an almost contact metric manifold $\tilde{M}$ with slant angle θ. Then, at each point x of M, $Q|D$ has only one eigenvalue $\lambda_1 = -cos^2\theta$.*

Lemma 2.2([15]) *Let M be a 3-dimensional slant submanifold of an almost contact metric manifold $\tilde{M}$. Suppose that M is not anti invariant. If $p \in M$, then in a neighborhood of p, there exist vector fields e_1, e_2 tangent to M, such that ξ, e_1, e_2 is a local orthonormal frame satisfying*

$$Te_1 = (cos\theta)e_2, \quad Te_2 = -(cos\theta)e_1. \tag{2.12}$$

§3. Conformal (k, μ)-Contact Manifold

A smooth manifold $(\bar{M}^{2n+1}, \bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ is called a conformal (k, μ)-contact manifold of a (k, μ)-

contact structure $(\tilde{M}^{2n+1}, \tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ if, there is a positive smooth function $f : \bar{M}^{2n+1} \to R$ such that

$$\tilde{g} = exp(f)\bar{g}, \quad \tilde{\phi} = \bar{\phi}, \quad \tilde{\eta} = (exp(f))^{\frac{1}{2}}\bar{\eta}, \quad \tilde{\xi} = (exp(-f))^{\frac{1}{2}}\bar{\xi}. \tag{3.1}$$

Example 3.1 Let R^{2n+1} be the $(2n+1)$-dimensional Euclidean space spanned by the orthogonal basis $\{\xi, X_1, X_2, \cdots, X_n, Y_1, Y_2, \cdots, Y_n\}$ and the Lie bracket defined as in [?]. Then, the almost contact metric structure $(\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ defined by

$$\begin{aligned}
\bar{\phi}\left(\sum_{i=1}^{n}\left(X_i\frac{\partial}{\partial x^i} + Y_i\frac{\partial}{\partial y^i} + z\frac{\partial}{\partial z}\right)\right) &= \sum_{i=1}^{n}\left(Y_i\frac{\partial}{\partial x^i} - X_i\frac{\partial}{\partial y^i}\right) + \sum_{i=1}^{n} Y_i y^i \frac{\partial}{\partial z}, \\
\bar{g} &= exp(-f)\{\bar{\eta}\otimes\bar{\eta} + \frac{1}{4}\sum_{i=1}^{n}\{(dx^i)^2 + (dy^i)^2\}\}, \\
\bar{\eta} &= (exp(-f))^{\frac{1}{2}}\left\{\frac{1}{2}(dz - \sum_{i=1}^{n} y^i dx^i)\right\}, \\
\bar{\xi} &= (exp(f))^{\frac{1}{2}}\left\{2\frac{\partial}{\partial z}\right\},
\end{aligned}$$

where $f = \sum_{i=1}^{n}(x^i)^2 + (y^i)^2 + z^2$.

It is easy to reveal that $(R^{2n+1}, \bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ is not a (k,μ)-contact manifold, but R^{2n+1} with the structure $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ defined by

$$\begin{aligned}
\tilde{\phi} &= \bar{\phi}, \\
\tilde{g} &= \tilde{\eta}\otimes\tilde{\eta} + \frac{1}{4}\sum_{i=1}^{n}\{(dx^i)^2 + (dy^i)^2\}, \\
\tilde{\eta} &= \frac{1}{2}(dz - \sum_{i=1}^{n} y^i dx^i), \\
\tilde{\xi} &= 2\frac{\partial}{\partial z},
\end{aligned}$$

is a (k,μ)-space form.

Let $\bar{M}$ be a conformal (k,μ)-contact manifold, let $\tilde{\nabla}$ and $\bar{\nabla}$ denote the Riemannian connections of $\bar{M}$ with respect to metrics $\tilde{g}$ and $\bar{g}$, respectively. Using the Koszul formula, we obtain the following relation between the connections $\tilde{\nabla}$ and $\bar{\nabla}$

$$\tilde{\nabla}_X Y = \bar{\nabla}_X Y + \frac{1}{2}\{\omega(X)Y + \omega(Y)X - \bar{g}(X,Y)\omega^\sharp\} \tag{3.2}$$

such that $\omega(X) = X(f)$ and $\omega^\sharp = gradf$ is a vector field metrically equivalent to 1-form ω, that is $\bar{g}(\omega^\sharp, X) = \omega(X)$.

Then with a straight forward computation we will have

$$\begin{aligned} exp(-f)(\tilde{R}(X,Y,Z,W)) &= \bar{R}(X,Y,Z,W) + \frac{1}{2}\{B(X,Z)\bar{g}(Y,W) - B(Y,Z) \\ &\quad \bar{g}(X,W) + B(Y,W)\bar{g}(X,Z) - B(X,W)\bar{g}(Y,Z)\} \\ &\quad + \frac{1}{4}\|\omega^\sharp\|^2\{\bar{g}(X,Z)\bar{g}(Y,W) - \bar{g}(Y,Z)\bar{g}(X,W)\} \end{aligned} \tag{3.3}$$

for all vector fields X, Y, Z, W on $\bar{M}$, where $B = \bar{\nabla}\omega - \frac{1}{2}\omega \otimes \omega$ and $\bar{R}$, $\tilde{R}$ are the curvature tensors of M related to connections of $\bar{\nabla}$ and $\tilde{\nabla}$, respectively. Furthermore, by the relations, (2.1), (2.3) and (3.2) we get

$$\begin{aligned} (\bar{\nabla}_X\bar{\phi})Y &= (exp(f))^{\frac{1}{2}}\{\bar{g}(X + \bar{h}X, Y)\xi - \bar{\eta}(Y)(X + \bar{h}X)\} \\ &\quad -\frac{1}{2}\{\omega(\bar{\phi}Y)X - \omega(Y)\bar{\phi}X + g(X,Y)\bar{\phi}\omega^\sharp - g(X,\bar{\phi}Y)\omega^\sharp\} \end{aligned} \tag{3.4}$$

$$\bar{\nabla}_X\bar{\xi} = -(exp(f))^{\frac{1}{2}}\{\bar{\phi}X + \bar{\phi}hX\} + \frac{1}{2}\{\bar{\eta}(X)\omega^\sharp - \omega(\bar{\xi})X\} \tag{3.5}$$

for all vector fields X, Y on $\bar{M}$. Now assume M is a submanifold of a conformal (k,μ)-contact manifold $\bar{M}$ and ∇, R are the connection, curvature tensor on M, respectively, and g is an induced metric on M.

For all $X, Y \in TM$ and $N \in T^\perp M$, from the Gauss, Weingarten formulas and (3.4), we obtain the following relations:

$$\begin{aligned} (\nabla_X T)Y &= A_{FY}X + t\sigma(X,Y) + (exp(f))^{\frac{1}{2}}\{g(X + hX, Y)\xi - \eta(Y)(X + hX)\} \\ &\quad -\frac{1}{2}\{\omega(\phi Y)X - \omega(Y)TX + g(X,Y)(\phi\omega^\sharp)^\top - g(X,TY)(\omega^\sharp)^\top\}, \end{aligned} \tag{3.6}$$

$$(\nabla_X F)Y = f\sigma(X,Y) - \sigma(X,TY) + \frac{1}{2}\{\omega(Y)FX - g(X,Y)F\omega^\sharp + g(X,TY)\omega^{\sharp\perp}\}, \tag{3.7}$$

$$(\nabla_X t)N = A_{fN}X - PA_NX - \frac{1}{2}\{\omega(\phi N)X - \omega(N)PX + g(X,tN)(\omega^\sharp)^\top\}, \tag{3.8}$$

$$(\nabla_X f)N = -\sigma(X,tN) - FA_NX + \frac{1}{2}\{\omega(N)FX + g(X,tN)(\omega^\sharp)^\perp\}, \tag{3.9}$$

where, $g = \bar{g}|M$, $\eta = \bar{\eta}|M$, $\xi = \bar{\xi}|M$ and $\phi = \bar{\phi}|M$.

§4 Slant Submanifolds of Conformal (k,μ)-Contact Manifolds

In this section, we prove a characterization theorem for slant submanifolds of a conformal (k,μ)-contact manifold.

Theorem 4.1 *Let M be a slant submanifold of conformal (k,μ)-contact manifold $\bar{M}$ such that $\omega^\sharp \in T^\perp M$ and $\xi \in TM$. Then Q is parallel if and only if one of the following is true:*

(i) M is anti-invariant;
(ii) $dim(M) \geq 3$;
(iii) M is trivial.

Proof Let θ be the slant angle of M in $\bar{M}$, then for any $X, Y \in TM$ and by equation (2.9), we infer

$$T^2Y = QY = cos^2\theta(-Y + \eta(Y)\xi). \tag{4.1}$$

$$\Rightarrow Q(\nabla_X Y) = cos^2\theta(-\nabla_X Y + \eta(\nabla_X Y)\xi). \tag{4.2}$$

Differentiating (4.1) covariantly with respect to X, we get

$$\nabla_X QY = cos^2\theta(-\nabla_X Y + \eta(\nabla_X Y)\xi - g(Y, \nabla_X \xi)\xi + \eta(Y)\nabla_X \xi). \tag{4.3}$$

Subtracting (4.2) from (4.3), we obtain

$$(\nabla_X Q)Y = cos^2\theta[g(\nabla_X Y, \xi)\xi + \eta(Y)\nabla_X \xi]. \tag{4.4}$$

If Q is parallel, then from (**??**) it follows that either $cos(\theta) = 0$ i.e. M is anti-invariant or

$$g(\nabla_X Y, \xi)\xi + \eta(Y)\nabla_X \xi = 0. \tag{4.5}$$

We know $g(\nabla_X \xi, \xi) = 0$, since $g(\nabla_X \xi, \xi) = -g(\xi, \nabla_X \xi)$, which implies $\nabla_X \xi \in D$.

Suppose $\nabla_X \xi \neq 0$, then (4.5) yields $\eta(Y) = 0$ i.e. $Y \in D$. But then (4.5) implies $\nabla_X \xi \in D^\perp \oplus < \xi >$, which is absurd.

Hence $\nabla_X \xi = 0$ and therefore either $D = 0$ or we can take at least two linearly independent vectors X and TX to span D. In this case the eigenvalue must be non-zero as $\theta = \frac{\pi}{2}$ has already been taken. Hence $dim(M) \geq 3$. □

Now, we state the the main result of this section.

Theorem 4.2 *Let M be a slant submanifold of conformal (k, μ)-contact manifold $\bar{M}$ such that $\xi \in TM$. Then M is slant if and only if*

(1) *The endomorphism $Q|D$ has only one eigen value at each point of M;*

(2) *There exists a function $\lambda : M \to [0, 1]$ such that*

$$\begin{aligned}(\nabla_X Q)Y \quad = \quad & \lambda\{(exp(f))^{\frac{1}{2}}[g(Y, TX + ThX)\xi - \eta(Y)(TX + ThX)] \\ & -\frac{1}{2}\{\omega(\xi)g(X, Y)\xi - \eta(X)\omega(Y)\xi + \omega(\xi)\eta(Y)X - \eta(X)\eta(Y)\omega^{\sharp^T}\}\},\end{aligned} \tag{4.6}$$

for any $X, Y \in TM$. Moreover, if θ is the slant angle of M, then $\lambda = cos^2\theta$.

Proof Statement 1 gets from Lemma (2.1). So, it remains to prove statement 2. Let M be a slant submanifold, then by (4.4) we have

$$(\nabla_X Q)Y = cos^2\theta(-g(Y, \nabla_X \xi) + \eta(Y)\nabla_X \xi). \tag{4.7}$$

By putting (3.5) in (4.7), we find (4.6). Conversely, let $\lambda_1(x)$ is the only eigenvalue of $Q|D$ at each point $x \in M$ and $Y \in D$ be a unit eigenvector associated with λ_1, i.e., $QY = \lambda_1 Y$.

Then from statement (2), we have

$$X(\lambda_1)Y + \lambda_1 \nabla_X Y = \nabla_X(QY) = Q(\nabla_X Y) + \lambda\{(exp(f))^{\frac{1}{2}} g(X, TY + ThY)\xi - \frac{1}{2}\{\omega(\xi)g(X,Y)\xi - \eta(X)\omega(Y)\xi\}\}, \tag{4.8}$$

for any $X \in TM$. Since both $\nabla_X Y$ and $Q(\nabla_X Y)$ are perpendicular to Y, we conclude that $X(\lambda_1) = 0$. Hence λ_1 is constant. So it remains to prove M is slant. For proof one can refer to Theorem (4.3) in [6].

§5. Slant Submanifolds of Dimension Three

Theorem 5.1 *Let M be a 3-dimensional proper slant submanifold of a conformal (k,μ)-contact manifold $\bar{M}$, such that $\xi \in TM$, then*

$$\begin{aligned}(\nabla_X T)Y &= cos^2\theta(exp(f))^{\frac{1}{2}}\{g(X + hX, Y)\xi - \eta(Y)(X + hX)\} + \frac{1}{2}\{\omega(\xi)g(TX,Y)\xi \\ &\quad -\eta(X)\omega(TY)\xi + \omega(\xi)\eta(Y)TX - \eta(X)\eta(Y)T\omega^{\sharp^T}\}\end{aligned} \tag{5.1}$$

for any $X, Y \in TM$ and θ is the slant angle of M.

Proof Let $X, Y \in TM$ and $p \in M$. Let ξ, e_1, e_2 be the orthonormal frame in a neighborhood U of p given by Lemma (2.2). Put $\xi|U = e_0$ and let α_i^j be the structural 1-forms defined by

$$\nabla_X e_i = \sum_{j=0}^{2} \alpha_i^j e_j. \tag{5.2}$$

In view of orthonormal frame ξ, e_1, e_2, we have

$$Y = \eta(Y)e_0 + g(Y, e_1)e_1 + g(Y, e_2)e_2. \tag{5.3}$$

Thus, we get

$$(\nabla_X T)Y = \eta(Y)(\nabla_X T)e_0 + g(Y, e_1)(\nabla_X T)e_1 + g(Y, e_2)(\nabla_X T)e_2. \tag{5.4}$$

Therefore, for obtaining $(\nabla_X T)Y$, we have to get $(\nabla_X T)e_0$, $(\nabla_X T)e_1$ and $(\nabla_X T)e_2$. By applying (3.5), we get

$$\begin{aligned}(\nabla_X T)e_0 &= \nabla_X(Te_0) - T(\nabla_X e_0) \\ &= (exp(f))^{\frac{1}{2}}(T^2X + T^2hX) + \frac{1}{2}\{\omega(\xi)TX - \eta(X)T\omega^{\sharp^T}\}.\end{aligned} \tag{5.5}$$

Moreover, by using (2.12) we obtain

$$\begin{aligned}(\nabla_X T)e_1 &= \nabla_X(Te_1) - T(\nabla_X e_1)\\ &= \nabla_X((cos\theta)e_2) - T(\alpha_1^0(X)e_0 + \alpha_1^1(X)e_1 + \alpha_1^2(X)e_2)\\ &= (cos\theta)\alpha_2^0(X)e_0. \end{aligned} \tag{5.6}$$

Similarly, we get

$$(\nabla_X T)e_2 = -(cos\theta)\alpha_1^0(X)e_0. \tag{5.7}$$

By substituting (5.5)-(5.7) in (5.4), we have

$$\begin{aligned}(\nabla_X T)Y &= (exp(f))^{\frac{1}{2}}\eta(Y)(T^2X + T^2hX) + \frac{1}{2}\{\eta(Y)\omega(\xi)TX - \eta(X)\eta(Y)T\omega^{\sharp^T}\}\\ &\quad +cos(\theta)\{g(Y,e_1)\alpha_2^0(X)e_0 - g(Y,e_2)\alpha_1^0(X)e_0\}. \end{aligned} \tag{5.8}$$

Now, we obtain $\alpha_1^0(X)$ and $\alpha_2^0(X)$ as follows:

$$\begin{aligned}\alpha_1^0(X) &= g(\nabla_X e_1, e_0)\\ &= Xg(e_1,e_0) - g(e_1,\nabla_X e_0)\\ &= -(exp(f))^{\frac{1}{2}}g(e_2, X+hX) + \frac{1}{2}\{\omega(\xi)g(e_1,X) - \eta(X)\omega(e_1)\} \end{aligned} \tag{5.9}$$

and similarly we get

$$\alpha_2^0(X) = cos\theta g(e_1,X) + cos\theta g(e_1,hX). \tag{5.10}$$

By using (5.9) and (5.10) in (5.8) and in view of (5.3) and (2.9) we obtain (5.1). □

From, Theorems 4.3 and 5.4, we can state the following:

Corollary 5.1 *Let M be a three dimensional submanifold of a (k,μ)-contact manifold tangent to ξ. Then the following statements are equivalent:*

(1) *M is slant;*

(2) $(\nabla_X T)Y = cos^2\theta(exp(f))^{\frac{1}{2}}\{g(X+hX,Y)\xi - \eta(Y)(X+hX)\} + \frac{1}{2}\{\omega(\xi)g(TX,Y)\xi$

$$-\eta(X)\omega(TY)\xi + \omega(\xi)\eta(Y)TX - \eta(X)\eta(Y)T\omega^{\sharp^T}\};$$

(3) $(\nabla_X Q)Y = \lambda\{(exp(f))^{\frac{1}{2}}[g(X,TX+ThX)\xi - \eta(Y)(TX+ThX)] - \frac{1}{2}\{\omega(\xi)g(X,Y)\xi$

$$-\eta(X)\omega(Y)\xi + \omega(\xi)\eta(Y)X - \eta(X)\eta(Y)\omega^{\sharp^T}\}\}.$$

The next result characterizes 3-dimensional slant submanifold in terms of the Weingarten map.

Theorem 5.2 *Let M be a 3-dimensional proper slant submanifold of a conformal (k,μ)-contact* *manifold $\bar{M}$, such that $\xi \in TM$. Then, there exists a function $C : M \to [0,1]$ such that*

$$\begin{aligned} A_{FX}Y &= A_{FY}X + C(exp(f))^{\frac{1}{2}}(\eta(X)(Y+hY) - \eta(Y)(X+hX)) + \omega(\xi)g(TX,Y)\xi \\ &+g(X,TY)\omega^{\sharp} + \frac{1}{2}\{\eta(X)\omega(TY)\xi - \eta(Y)\omega(TX)\xi + \eta(X)\omega(\xi)TY \\ &-\eta(Y)\omega(\xi)TX - \omega(X)TY + \omega(Y)TX + \omega(TX)Y - \omega(TY)X\}, \end{aligned} \tag{5.11}$$

for any $X, Y \in TM$. Moreover in this case, if θ is the slant angle of M then we have $C = sin^2\theta$.

Proof Let $X, Y \in TM$ and M is a slant submanifold. From (3.6) and Theorem 5.1, we have

$$\begin{aligned} t\sigma(X,Y) &= (\lambda-1)(exp(f))^{\frac{1}{2}}\{g(Y,X+hX)\xi - \eta(Y)(X+hX)\} + \frac{1}{2}\{\omega(\xi)g(X,TY)\xi \\ &-\eta(X)\omega(TY)\xi + \omega(\xi)\eta(Y)TX - \eta(X)\eta(Y)T\omega^{\sharp^T} + \omega(TY)X - \omega(Y)TX \\ &+g(X,Y)T\omega^{\sharp} - g(X,TY)\omega^{\sharp}\} - A_{FY}X. \end{aligned} \tag{5.12}$$

Now by using the fact that $\sigma(X,Y) = \sigma(Y,X)$, we obtain (5.11). □

Next, we assume that M is a three dimensional proper slant submanifold M of a five-dimensional conformal (k,μ)-contact manifold $\bar{M}$ with slant angle θ. Then for a unit tangent vector field e_1 of M perpendicular to ξ, we put

$$e_2 = (sec\theta)Te_1, \quad e_3 = \xi, \quad e_4 = (csc\theta)Fe_1, \quad e_5 = (csc\theta)Fe_2. \tag{5.13}$$

It is easy to show that $e_1 = -(sec\theta)Te_2$ and by using Corollary 2.1, $\{e_1, e_2, e_3, e_4, e_5\}$ form an orthonormal frame such that e_1, e_2, e_3 are tangent to M and e_4, e_5 are normal to M. Also we have

$$te_4 = -sin\theta e_1, \quad te_5 = -sin\theta e_2, \quad fe_4 = -cos\theta e_5, \quad fe_5 = -cos\theta e_4. \tag{5.14}$$

If we put $\sigma^r_{ij} = g(\sigma(e_i,e_j),e_r)$, $i,j = 1,2,3$, $r = 4,5$, then we have the following result:

Lemma 5.1 *In the above conditions, we have*

$$\begin{aligned} \sigma^4_{12} &= \sigma^5_{11}, \ \sigma^4_{22} = \sigma^5_{12}, \\ \sigma^4_{13} &= \sigma^5_{23} = -(exp(f))^{\frac{1}{2}}sin\theta \\ \sigma^4_{32} &= \sigma^4_{33} = \sigma^5_{33} = \sigma^5_{13} = 0. \end{aligned} \tag{5.15}$$

Proof Apply (5.11) by setting $X = e_1$ and $Y = e_2$, we obtain

$$A_{e_4}e_2 = A_{e_5}e_1 + (cot\theta)\{\omega(\xi)\xi - \omega^{\sharp} + \omega(e_1)e_1 + \omega(e_2)e_2\}.$$

Using (2.6) in the above relation, we get

$$\sigma^4_{12} = \sigma^5_{11}, \ \ \sigma^4_{22} = \sigma^5_{12}, \sigma^4_{23} = \sigma^5_{13}.$$

Further, by taking $X = e_1$ and $Y = e_3$ in (5.11), we have

$$A_{e_4} e_3 = -(exp(f))^{\frac{1}{2}}(sin\theta)(e_1 + he_1). \tag{5.16}$$

After applying (2.6) in (5.16), we obtain

$$\sigma^4_{13} = -(exp(f))^{\frac{1}{2}}(sin\theta), \ \sigma^4_{23} = \sigma^4_{33} = 0.$$

In the similar manner by putting $X = e_2$ and $Y = e_3$, we get

$$\sigma^5_{23} = -(exp(f))^{\frac{1}{2}}(sin\theta), \ \sigma^5_{33} = 0. \qquad \square$$

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