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The Examination of Eigenvalues and Eigenfunctions of the Sturm-Liouville Fuzzy Problem According to Boundary Conditions

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Abstract: In this paper, the eigenvalues and the eigenfunctions of the homogeneous fuzzy Sturm-Liouville problem are examined under the three different situations according to the boundary conditions. This examination is studied under the approach of Hukuhara differentiability. The examples are given for this situations.

Key Words: Fuzzy boundary value problems, second-order fuzzy differential equations, Hukuhara differentiability, eigenvalue, eigenfunction.

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§1. Introduction

Approaches to fuzzy boundary value problems can be of two types. The first approach assumes that the derivative in the boundary value problem can be considered as a derivative of fuzzy function. This derivative can be Hukuhara derivative or a derivative in generalized sense [1,2,11,12]. The second approach is based on generating the fuzzy solution from crisp solution. In particular case, this approach can be of three ways. The first one uses the extension principle [3]. The second way uses the concept of differential inclusion [10]. In the third way the fuzzy problem is considered to be a set of crisp problem [6,7].

In this paper, the eigenvalues and the eigenfunctions of the homogeneous fuzzy Sturm-Liouville problem are examined under the approach of Hukuhara differentiability. It is seen that applied method for this examination is different according to given boundary conditions. Therefore, the eigenvalues and the eigenfunctions of the problem are examined under the three different situation and the examples are given for this situations.

§2. Preliminaries

In this section, we give some definitions and introduce the necessary notation which will be used throughout the paper.

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Definition 2.1([11]) *A fuzzy number is a function $u : \mathbb{R} \to [0,1]$ satisfying the following properties:*

(i) *u is normal;*

(ii) *u is convex fuzzy set;*

(iii) *u is upper semi-continuous on $\mathbb{R}$;*

(iv) *$cl\{x \in \mathbb{R} \mid u(x) > 0\}$ is compact where cl denotes the closure of a subset.*

Let $\mathbb{R}_F$ denote the space of fuzzy numbers.

Definition 2.2([12]) *Let $u \in \mathbb{R}_F$. The α-level set of u, denoted , $[u]^\alpha$, $0 < \alpha \leq 1$, is*

$$[u]^\alpha = \{x \in \mathbb{R} \mid u(x) \geq 0\}.$$

If $\alpha = 0$, the support of u is defined

$$[u]^0 = cl\{x \in \mathbb{R} \mid u(x) > 0\}.$$

The notation, denotes explicitly the α-level set of u. The notation, $[u]^\alpha = [\underline{u}_\alpha, \overline{u}_\alpha]$ denotes explicitly the α-level set of u.We refer to $\underline{u}$ and $\overline{u}$ as the lower and upper branches of u, respectively.

The following remark shows when $[\underline{u}_\alpha, \overline{u}_\alpha]$ is a valid α-level set.

Remark 2.3([11]) The sufficient and necessary conditions for $[\underline{u}_\alpha, \overline{u}_\alpha]$ to define the parametric form of a fuzzy number as follows:

(i) $\underline{u}_\alpha$ is bounded monotonic increasing (nondecreasing) left-continuous function on $(0,1]$ and right-continuous for $\alpha = 0$,

(ii) $\overline{u}_\alpha$ is bounded monotonic decreasing (nonincreasing) left-continuous function on $(0,1]$ and right-continuous for $\alpha = 0$,

(iii) $\underline{u}_\alpha \leq \overline{u}_\alpha$, $0 \leq \alpha \leq 1$.

Definition 2.4([13]) *If A is a symmetric triangular numbers with supports $[\underline{a}, \overline{a}]$, the $\alpha-$level sets of $[A]^\alpha$ is $[A]^\alpha = \left[\underline{a} + \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha, \overline{a} - \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right]$.*

Definition 2.5([12]) *For $u, v \in \mathbb{R}_F$ and $\lambda \in \mathbb{R}$, the sum $u + v$ and the product λu are defined by $[u+v]^\alpha = [u]^\alpha + [v]^\alpha$, $[\lambda u]^\alpha = \lambda [u]^\alpha$, $\forall \alpha \in [0,1]$, where means the usual addition of two intervals (subsets) of $\mathbb{R}$ and $\lambda [u]^\alpha$ means the usual product between a scalar and a subset of $\mathbb{R}$.*

The metric structure is given by the Hausdorff distance

$$D : \mathbb{R}_F \times \mathbb{R}_F \to \mathbb{R}_+ \cup \{0\}, \quad ([10])$$

by

$$D(u,v) = \sup_{\alpha \in [0,1]} \max\{|\underline{u}_\alpha - \underline{v}_\alpha|, |\overline{u}_\alpha - \overline{v}_\alpha|\}.$$

Definition 2.6([13]) *Let $u, v \in \mathbb{R}_F$. If there exists $w \in \mathbb{R}_F$ such that $u = v + w$, then w is called the Hukuhara difference of fuzzy numbers u and v, and it is denoted by $w = u \ominus v$.*

Definition 2.7([11]) *Let $f : [a, b] \to \mathbb{R}_F$ and $t_0 \in [a, b]$. We say that f is Hukuhara differential at t_0, if there exists an element $f^{'}(t_0) \in \mathbb{R}_F$ such that for all $h > 0$ sufficiently small, $\exists f(t_0 + h) \ominus f(t_0)$, $f(t_0) \ominus f(t_0 - h)$ and the limits*

$$\lim_{h\to 0^+} \frac{f(t_0+h) \ominus f(t_0)}{h} = \lim_{h\to 0^+} \frac{f(t_0) \ominus f(t_0-h)}{h} = f^{'}(t_0).$$

Here the limits are taken in the metric space $(\mathbb{R}_F, D)$.

Theorem 2.8([5]) *Let $f : I \to \mathbb{R}_F$ be a function and denote $[f(t)]^\alpha = \left[\underline{f}_\alpha(t), \overline{f}_\alpha(t)\right]$, for each $\alpha \in [0, 1]$. If f is Hukuhara differentiable, then $\underline{f}_\alpha$ and $\overline{f}_\alpha$ are differentiable functions and $\left[f^{'}(t)\right]^\alpha = \left[\underline{f}^{'}_\alpha(t), \overline{f}^{'}_\alpha(t)\right]$.*

Definition 2.9([9]) *Let $f : A \subset \mathbb{R} \to \mathbb{R}$ be a function. f^+ and f^- are not the negative function defined as*

$$f^+(x) = \begin{cases} f(x), & f(x) \geq 0 \\ 0, & f(x) < 0 \end{cases}, \quad f^-(x) = \begin{cases} -f(x), & f(x) \leq 0 \\ 0, & f(x) > 0 \end{cases}$$

The function f^+ and f^- are called the positive piece and negative piece of f, respectively.

§3. The Eigenvalues and the Eigenfunctions of the Sturm-Liouville Fuzzy Problem According to the Boundary Conditions

Let

$$\begin{aligned} Ly &= p(x) y^{''} + q(x) y, \ p^{'}(x) = 0, \\ A, B, C, D &\geq 0, \ A^2 + B^2 \neq 0 \text{ and } C^2 + D^2 \neq 0. \end{aligned}$$

(I) Consider the eigenvalues of the fuzzy boundary value problem

$$Ly + \lambda y = 0, \ x \in [a, b] \tag{3.1}$$

$$Ay(a) + By^{'}(a) = 0, \tag{3.2}$$

$$Cy(b) + Dy^{'}(b) = 0. \tag{3.3}$$

Let be functions $\underline{\phi}_\alpha$, $\underline{\psi}_\alpha$, $\overline{\phi}_\alpha$, $\overline{\psi}_\alpha$ the solution of the fuzzy boundary value problem (3.1)-(3.3). The eigenvalues of the fuzzy boundary value problem (3.1)-(3.3) if and only if are consist of the zeros of functions $\underline{W}_\alpha(\lambda)$ and $\overline{W}_\alpha(\lambda)$, where ([8])

$$\underline{W}_\alpha(\lambda) = W\left(\underline{\phi}_\alpha, \underline{\psi}_\alpha\right)(x, \lambda) = \underline{\phi}_\alpha(x, \lambda)\, \underline{\psi}^{'}_\alpha(x, \lambda) - \underline{\psi}_\alpha(x, \lambda)\, \underline{\phi}^{'}_\alpha(x, \lambda),$$

$$\overline{W}_{\alpha}(\lambda) = W\left(\overline{\phi}_{\alpha}, \overline{\psi}_{\alpha}\right)(x, \lambda) = \overline{\phi}_{\alpha}(x, \lambda)\,\overline{\psi}'_{\alpha}(x, \lambda) - \overline{\psi}_{\alpha}(x, \lambda)\,\overline{\phi}'_{\alpha}(x, \lambda).$$

Example 3.1 Consider the fuzzy Sturm-Liouville problem

$$y'' + \lambda y = 0,\ y(0) = 0,\ y(1) + y'(1) = 0. \tag{3.4}$$

Let be $\lambda = k^2$, $k > 0$,

$$\phi(x, \lambda) = \sin(kx)$$

be the solution of the classical differential equation $y'' + \lambda y = 0$ satisfying the condition $y(0) = 0$ and

$$\psi(x, \lambda) = \left(\cos(k) + \frac{\sin(k)}{k}\right)\cos(kx) + \left(\sin(k) - \frac{\cos(k)}{k}\right)\sin(kx)$$

be the solution satisfying the condition $y(1) = 1$, $y'(1) = -1$. Then,

$$[\phi(x, \lambda)]^{\alpha} = \left[\underline{\phi}_{\alpha}(x, \lambda), \overline{\phi}_{\alpha}(x, \lambda)\right] = [\alpha, 2 - \alpha]\sin(kx) \tag{3.5}$$

is the solution of the fuzzy differential equation $y'' + \lambda y = 0$ satisfying the condition $y(0) = 0$ and

$$[\psi(x, \lambda)]^{\alpha} = \left[\underline{\psi}_{\alpha}(x, \lambda), \overline{\psi}_{\alpha}(x, \lambda)\right] = [\alpha, 2 - \alpha]\,\psi(x, \lambda) \tag{3.6}$$

is the solution satisfying the condition $y(1) = 1$, $y'(1) = -1$. Since the eigenvalues of the fuzzy Sturm-Liouville problem (3.4) are zeros the functions $\underline{W}_{\alpha}(\lambda)$ and $\overline{W}_{\alpha}(\lambda)$, $\underline{W}_{\alpha}(\lambda)$ is obtained as

$$\underline{W}_{\alpha}(\lambda) = \alpha^2\{(-k\cos(k) - \sin(k))\sin^2(kx) + (k\sin(k) - \cos(k))\cos(kx)\sin(kx) +$$

$$+ (-k\cos(k) - \sin(k))\cos^2(kx) - (k\sin(k) - \cos(k))\cos(kx)\sin(kx)\}$$

$$\underline{W}_{\alpha}(\lambda) = -\alpha^2(k\cos(k) + \sin(k))$$

and similarly $\overline{W}_{\alpha}(\lambda)$ is obtained as

$$\overline{W}_{\alpha}(\lambda) = -(2 - \alpha)^2(k\cos(k) + \sin(k)).$$

From here, yields

$$\underline{W}_{\alpha}(\lambda) = 0 \Rightarrow k\cos(k) + \sin(k) = 0,$$

$$\overline{W}_{\alpha}(\lambda) = 0 \Rightarrow k\cos(k) + \sin(k) = 0.$$

Computing the values k satisfying the equation $k\cos(k) + \sin(k) = 0$, we have

$$k_1 = 2.028757838,\ k_2 = 4.913180439,\ k_3 = 7.978665712,\ k_4 = 11.08553841, \cdots$$

We show that this values are k_n, $n = 1, 2, \cdots$ Substituing this values in (3.5),(3.6), we obtain

$$[\phi_n(x)]^{\alpha} = \left[\underline{\phi_n}_{\alpha}(x), \overline{\phi_n}_{\alpha}(x)\right] = [\alpha, 2 - \alpha]\sin(k_n x),$$

$$[\psi_n(x)]^\alpha = \left[\underline{\psi_n}_\alpha(x), \overline{\psi_n}_\alpha(x)\right] = [\alpha, 2-\alpha]\left(\sin(k_n) - \frac{\cos(k_n)}{k_n}\right)\sin(k_n x).$$

As

$$[\alpha, 2-\alpha](\sin(k_n x))^+$$

and

$$[\alpha, 2-\alpha]\left(\left(\sin(k_n) - \frac{\cos(k_n)}{k_n}\right)\sin(k_n x)\right)^+, \ [\phi_n(x)]^\alpha$$

and $[\psi_n(x)]^\alpha$ are a valid $\alpha-$level set. Let be $k_n x \in [(n-1)\pi, n\pi]$, $n = 1, 2, \cdots$

(i) If n is single, $\sin(k_n x) \geq 0$. Then $[\phi_n(x)]^\alpha$ is a valid $\alpha-$level set.

(ii) If n is double, $\sin(k_n x) \leq 0$. Also, since $x \in [0,1]$, $k_n \in [(n-1)\pi, n\pi]$, and according to Fig.1, $\sin(k_n) - \frac{\cos(k_n)}{k_n} < 0$ for n is double. Then $[\psi_n(x)]^\alpha$ is a valid $\alpha-$level set.

Consequently, $k_n x \in [(n-1)\pi, n\pi]$, $n = 1, 2, \cdots$

(i) If n is single, the eigenvalues are $\lambda_n = k_n^2$, with associated eigenfunctions

$$[y_{1n}(x)]^\alpha = [\alpha, 2-\alpha]\sin(k_n x),$$

(ii) If n is double, the eigenvalues are $\lambda_n = k_n^2$, with associated eigenfunctions

$$[y_{2n}(x)]^\alpha = [\alpha, 2-\alpha]\left(\sin(k_n) - \frac{\cos(k_n)}{k_n}\right)\sin(k_n x),$$

(iii) If $\alpha = 1$, the eigenvalues are $\lambda_n = k_n^2$, with associated eigenfunctions

$$[y_n(x)]^\alpha = \sin(k_n x).$$

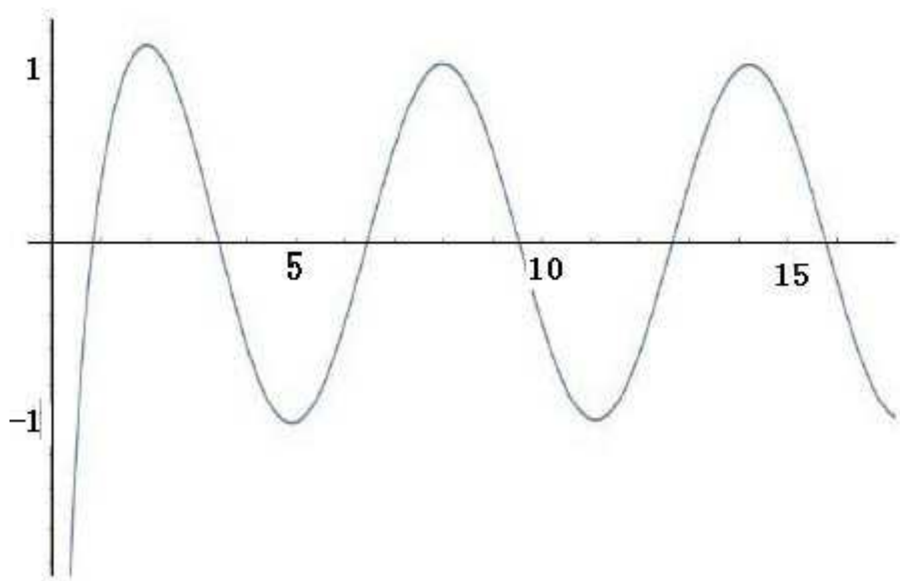

Fig.1 The graphic of the function $f(k) = \sin(k) - \frac{\cos(k)}{k}$

(II) Consider the eigenvalues of the fuzzy boundary value problem,

$$Ly + \lambda y = 0,\ x \in (a,b) \tag{3.7}$$

$$Ay(a) = By^{'}(a), \tag{3.8}$$

$$Cy(b) = Dy^{'}(b). \tag{3.9}$$

$$[\phi(x,\lambda)]^{\alpha} = \left[\underline{\phi}_{\alpha}(x,\lambda), \overline{\phi}_{\alpha}(x,\lambda)\right]$$

is the solution of the fuzzy differential equation (3.7) satisfying the conditions $y(a) = B$, $y^{'}(a) = A$ and

$$[\psi(x,\lambda)]^{\alpha} = \left[\underline{\psi}_{\alpha}(x,\lambda), \overline{\psi}_{\alpha}(x,\lambda)\right]$$

is the solution satisfying the conditions

$$y(b) = D,\ y^{'}(b) = C.$$

Hence, the method which is applied for the fuzzy boundary value problem (3.1)-(3.3) is valid for the problem (3.7)-(3.9).

Example 3.2 Consider the fuzzy Sturm-Liouville problem

$$y^{''} + \lambda y = 0,\ y(0) = 0,\ y^{'}(1) = y(1). \tag{3.10}$$

Let $\lambda = k^2$, $k > 0$. Then,

$$[\phi(x,\lambda)]^{\alpha} = \left[\underline{\phi}_{\alpha}(x,\lambda), \overline{\phi}_{\alpha}(x,\lambda)\right] = [\alpha, 2-\alpha]\sin(kx) \tag{3.11}$$

is the solution of the fuzzy differential equation $y^{''} + \lambda y = 0$ satisfying the condition $y(0) = 0$ and

$$[\psi(x,\lambda)]^{\alpha} = \left[\underline{\psi}_{\alpha}(x,\lambda), \overline{\psi}_{\alpha}(x,\lambda)\right] = [\alpha, 2-\alpha]\,\psi(x,\lambda) \tag{3.12}$$

is the solution satisfying the condition $y^{'}(1) = y(1)$, where

$$\psi(x,\lambda) = \left(\cos(k) - \frac{\sin(k)}{k}\right)\cos(kx) + \left(\sin(k) + \frac{\cos(k)}{k}\right)\sin(kx).$$

Since the eigenvalues of the fuzzy Sturm-Liouville problem (3.10) are zeros the functions $\underline{W}_{\alpha}(\lambda)$ and $\overline{W}_{\alpha}(\lambda)$, we obtained

$$\underline{W}_{\alpha}(\lambda) = -\alpha^2(k\cos(k) - \sin(k)),\ \overline{W}_{\alpha}(\lambda) = -(2-\alpha)^2(k\cos(k) - \sin(k)).$$

Computing the values k satisfying the equation $k\cos(k) - \sin(k) = 0$, we have

$$k_1 = 4.493409458,\ k_1 = 7.725251837,\ k_3 = 10.90412166,\ k_4 = 14.06619391, ...$$

We show that this values are k_n, $n = 1, 2, \cdots$. Substituting this values in (3.11), (3.12), we obtain

$$[\phi_n(x)]^\alpha = \left[\underline{\phi_n}_\alpha(x), \overline{\phi_n}_\alpha(x)\right] = [\alpha, 2-\alpha]\sin(k_n x),$$

$$[\psi_n(x)]^\alpha = \left[\underline{\psi_n}_\alpha(x), \overline{\psi_n}_\alpha(x)\right] = [\alpha, 2-\alpha]\left(\sin(k_n) + \frac{\cos(k_n)}{k_n}\right)\sin(k_n x).$$

Let $k_n x \in [(n-1)\pi, n\pi]$, $n = 1, 2, \cdots$

(i) If n is single, $\sin(k_n x) \geq 0$. Then $[\phi_n(x)]^\alpha$ is a valid $\alpha-$level set.

(ii) If n is double, $\sin(k_n x) \leq 0$. Also, since $x \in [0,1]$, $k_n \in [(n-1)\pi, n\pi]$ and according to Fig.2, $\sin(k_n) + \frac{\cos(k_n)}{k_n} < 0$ for n is double. Then $[\psi_n(x)]^\alpha$ is a valid $\alpha-$level set.

Consequently, $k_n x \in [(n-1)\pi, n\pi]$, $n = 1, 2, \cdots$

(i) If n is single, the eigenvalues are $\lambda_n = k_n^2$, with associated eigenfunctions

$$[y_{1n}(x)]^\alpha = [\alpha, 2-\alpha]\sin(k_n x),$$

(ii) If n is double, the eigenvalues are $\lambda_n = k_n^2$, with associated eigenfunctions

$$[y_{2n}(x)]^\alpha = [\alpha, 2-\alpha]\left(\sin(k_n) + \frac{\cos(k_n)}{k_n}\right)\sin(k_n x),$$

(iii) If $\alpha = 1$, the eigenvalues are $\lambda_n = k_n^2$, with associated eigenfunctions

$$[y_n(x)]^\alpha = \sin(k_n x).$$

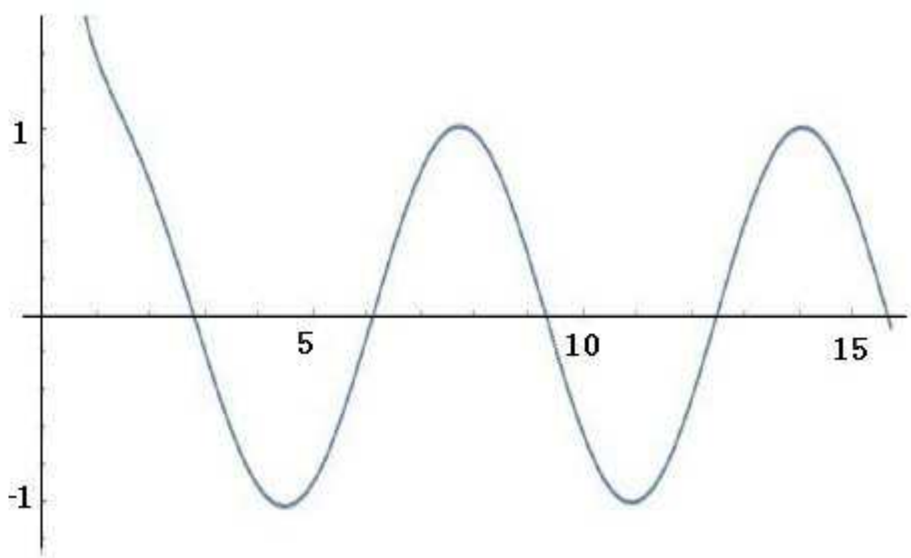

Fig.2 The graphic of the function $f(k) = \sin(k) + \frac{\cos(k)}{k}$

(III) Consider the eigenvalues of the fuzzy boundary value problem

$$Ly + \lambda y = 0, \ x \in (a,b), \tag{3.13}$$

$$-Ay(a) + By'(a) = 0, \tag{3.14}$$

$$-Cy(b) + Dy'(b) = 0. \tag{3.15}$$

Let be $[y]^{\alpha} = \left[\underline{y}_{\alpha}, \overline{y}_{\alpha}\right]$ the general solution of the fuzzy differential equation (3.13). From the boundary condition (3.14)

$$-A\left[\underline{y}_{\alpha}(a,\lambda), \overline{y}_{\alpha}(a,\lambda)\right] + B\left[\underline{y}_{\alpha}(a,\lambda), \overline{y}_{\alpha}(a,\lambda)\right]' = 0.$$

Using the Hukuhara differentiability, the fuzzy arithmetic and -$[y]^{\alpha} = \left[-\overline{y}_{\alpha}, -\underline{y}_{\alpha}\right]$, we obtained

$$\left[-A\overline{y}_{\alpha}(a,\lambda) + B\underline{y}_{\alpha}'(a,\lambda), -A\underline{y}_{\alpha}(a,\lambda) + B\overline{y}_{\alpha}'(a,\lambda)\right] = 0.$$

From here, the equations

$$-A\overline{y}_{\alpha}(a,\lambda) + B\underline{y}_{\alpha}'(a,\lambda) = 0 \tag{3.16}$$

$$-A\underline{y}_{\alpha}(a,\lambda) + B\overline{y}_{\alpha}'(a,\lambda) = 0 \tag{3.17}$$

are obtained. So we can not decompose the lower solution and upper solution. Therefore we can not find the function $\underline{\phi}_{\alpha}(x,\lambda)$ satisfying the condition (3.16) and the function $\overline{\phi}_{\alpha}(x,\lambda)$ satisfying the condition (3.17) of the fuzzy differential equation (3.13). Consequently, there is not the function

$$[\phi(x,\lambda)]^{\alpha} = \left[\underline{\phi}_{\alpha}(x,\lambda), \overline{\phi}_{\alpha}(x,\lambda)\right]$$

satisfying the condition (3.14) of the fuzzy differential equation (3.13). Similarly, there is not the function

$$[\psi(x,\lambda)]^{\alpha} = \left[\underline{\psi}_{\alpha}(x,\lambda), \overline{\psi}_{\alpha}(x,\lambda)\right]$$

satisfying the condition (3.15) of the fuzzy differential equation (3.13).

Therefore, the method which is applied for the fuzzy boundary value problem (3.1)-(3.3) is not valid for the problem (3.13)-(3.15).

Example 3.3 Consider the fuzzy Sturm-Liouville problem

$$y'' + \lambda y = 0,\ y(0) = 0,\ y'(1) - y(1) = 0. \tag{3.18}$$

Let be $[y]^{\alpha} = \left[\underline{y}_{\alpha}, \overline{y}_{\alpha}\right]$ and $\lambda = k^2,\ k > 0$. Then, the lower and upper solutions of the fuzzy diferential equation in (3.18) are

$$\underline{y}_{\alpha}(x,\lambda) = c_1(\alpha)\cos(kx) + c_2(\alpha)\sin(kx), \tag{3.19}$$

$$\overline{y}_{\alpha}(x,\lambda) = c_3(\alpha)\cos(kx) + c_4(\alpha)\sin(kx). \tag{3.20}$$

From the boundary conditon $y(0) = 0$

$$\underline{y}_{\alpha}(0,\lambda) = c_1(\alpha) = 0,\ \overline{y}_{\alpha}(0,\lambda) = c_3(\alpha) = 0.$$

From the boundary conditon $y'(1) - y(1) = 0$,

$$\underline{y}'_{\alpha}(1,\lambda) - \overline{y}_{\alpha}(1,\lambda) = 0,\ \overline{y}'_{\alpha}(1,\lambda) - \underline{y}_{\alpha}(1,\lambda) = 0$$

and from here, we obtain the system of equations

$$\underline{y}'_{\alpha}(1,\lambda) - \overline{y}_{\alpha}(1,\lambda) = kc_2(\alpha)\cos(k) - c_4(\alpha)\sin(k) = 0,$$

$$\overline{y}'_{\alpha}(1,\lambda) - \underline{y}_{\alpha}(1,\lambda) = kc_4(\alpha)\cos(k) - c_2(\alpha)\sin(k) = 0.$$

If

$$\begin{vmatrix} k\cos(k) & -\sin(k) \\ -\sin(k) & k\cos(k) \end{vmatrix} = k^2\cos^2(k) - \sin^2(k) = 0,$$

there is the non-zero solution of the system of equation. Computing the values k satisfying this equation, we have

$$k_1 = 2.028757838,\ k_2 = 4.493409458,\ k_3 = 4.913180439,\ k_4 = 7.725251837, \cdots$$

We show that this values are k_n, $n = 1, 2, \cdots$ Substituing this values in (3.19), (3.20), we obtain

$$\underline{y}_{n\alpha}(x) = c_2(\alpha)\sin(k_n x),\ \overline{y}_{n\alpha}(x) = c_4(\alpha)\sin(k_n x),$$

$$[y_n(x)]^{\alpha} = \left[\underline{y}_{n\alpha}(x), \overline{y}_{n\alpha}(x)\right].$$

As

$$\frac{\partial \underline{y}_{n\alpha}(x)}{\partial\alpha} \geq 0,\ \frac{\partial \overline{y}_{n\alpha}(x)}{\partial\alpha} \leq 0 \text{ and } \underline{y}_{n\alpha}(x) \leq \overline{y}_{n\alpha}(x),$$

$[y_n(x)]^{\alpha}$ is valid $\alpha-$level set. Then $k_n x \in [m\pi, (m+1)\pi]$, $m = 0, 1, \cdots$

(i) If m is double, since $\sin(k_n x) \geq 0$, it must be $\frac{\partial c_2(\alpha)}{\partial\alpha} \geq 0$, $\frac{\partial c_4(\alpha)}{\partial\alpha} \leq 0$ and $c_2(\alpha) \leq c_4(\alpha)$,

(ii) If m is single, since $\sin(k_n x) \leq 0$, it must be $\frac{\partial c_2(\alpha)}{\partial\alpha} \leq 0$, $\frac{\partial c_4(\alpha)}{\partial\alpha} \geq 0$ and $c_2(\alpha) \geq c_4(\alpha)$.

Consequently, $k_n x \in [m\pi, (m+1)\pi]$, $m = 0, 1, \cdots$

(i) If m is double, the eigenvalues are $\lambda_n = k_n^2$, with associated eigenfunctions

$$[y_{1n}(x)]^{\alpha} = [c_2(\alpha)\sin(k_n x),\ c_4(\alpha)\sin(k_n x)],$$

for $c_2(\alpha)$ and $c_4(\alpha)$ satisfying the inequalities $\frac{\partial c_2(\alpha)}{\partial\alpha} > 0$, $\frac{\partial c_4(\alpha)}{\partial\alpha} < 0$ and $c_2(\alpha) < c_4(\alpha)$.

(ii) If m is single, the eigenvalues are $\lambda_n = k_n^2$, with associated eigenfunctions

$$[y_{2n}(x)]^{\alpha} = [c_2(\alpha)\sin(k_n x),\ c_4(\alpha)\sin(k_n x)],$$

for $c_2(\alpha)$ and $c_4(\alpha)$ satisfying the inequalities $\frac{\partial c_2(\alpha)}{\partial\alpha} < 0$, $\frac{\partial c_4(\alpha)}{\partial\alpha} > 0$ and $c_2(\alpha) > c_4(\alpha)$.

(iii) If $\frac{\partial c_2(\alpha)}{\partial\alpha} = 0$, $\frac{\partial c_4(\alpha)}{\partial\alpha} = 0$ and $c_2(\alpha) = c_2 = c_4 = c_4(\alpha)$, the eigenvalues are $\lambda_n = k_n^2$, with associated eigenfunctions

$$[y_n(x)]^{\alpha} = \sin(k_n x).$$

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