International J.Math. Combin. Vol.4(2017), 91-102

The k-Distance Degree Index of Corona, Neighborhood Corona Products and Join of Graphs

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Abstract: The k-distance degree index (N_k-index) of a graph G have been introduced in [11], and is defined as $N_k(G) = \sum_{k=1}^{diam(G)} \left(\sum_{v\in V(G)} d_k(v)\right) \cdot k$, where $d_k(v) = |N_k(v)| = |\{u \in V(G) : d(v,u) = k\}|$ is the k-distance degree of a vertex v in G, $d(u,v)$ is the distance between vertices u and v in G and $diam(G)$ is the diameter of G. In this paper, we extend the study of N_k-index of a graph for other graph operations. Exact formulas of the N_k-index for corona $G \circ H$ and neighborhood corona $G \star H$ products of connected graphs G and H are presented. An explicit formula for the splitting graph $S(G)$ of a graph G is computed. Also, the N_k-index formula of the join $G + H$ of two graphs G and H is presented. Finally, we generalize the N_k-index formula of the join for more than two graphs.

Key Words: Vertex degrees, distance in graphs, k-distance degree, Smarandachely k-distance degree, k-distance degree index, corona, neighborhood corona.

AMS(2010): 05C07, 05C12, 05C76, 05C31.

§1. Introduction

In this paper, we consider only simple graph $G = (V,E)$, i.e., finite, having no loops no multiple and directed edges. A graph G is said to be connected if there is a path between every pair of its vertices. As usual, we denote by $n = |V|$ and $m = |E|$ to the number of vertices and edges in a graph G, respectively. The distance $d(u,v)$ between any two vertices u and v of G is the length of a minimum path connecting them. For a vertex $v \in V$ and a positive integer k, the open k-distance neighborhood of v in a graph G is $N_k(v/G) = \{u \in V(G) : d(u,v) = k\}$ and the closed k-neighborhood of v is $N_k[v/G] = N_k(v) \cup \{v\}$. The k-distance degree of a vertex v in G, denoted by $d_k(v/G)$ (or simply $d_k(v)$ if no misunderstanding) is defined as $d_k(v/G) = |N_k(v/G)|$, and generally, a Smarandachely k-distance degree $d_k(v/G : S)$ of v on vertex set $S \subset V(G)$ is $d_k(v/G) = |N_k(v/G : S)|$, where $N_k(v/G : S) = \{u \in V(G)\backslash S : d(u,v) = k\}$. Clearly, $d_k(v/G : \emptyset) = d_k(v/G)$ and $d_1(v/G) = d(v/G)$ for every $v \in V(G)$. A vertex of degree equals to zero in G is called an isolated vertex and a vertex of degree one is called a pendant vertex. The graph with just one vertex is referred to as trivial graph and denoted K_1. The complement

[1]Received January 10, 2017, Accepted, November 23, 2017.

$\overline{G}$ of a graph G is a graph with vertex set $V(G)$ and two vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in G. A totally disconnected graph $\overline{K_n}$ is one in which no two vertices are adjacent (that is, one whose edge set is empty). If a graph G consists of $s \geq 2$ disjoint copies of a graph H, then we write $G = sH$. For a vertex v of G, the eccentricity $e(v) = \max\{d(v,u) : u \in V(G)\}$. The radius of G is $rad(G) = \min\{e(v) : v \in V(G)\}$ and the diameter of G is $diam(G) = \max\{e(v) : v \in V(G)\}$. For any terminology or notation not mention here, we refer the reader to the books [3, 5].

A topological index of a graph G is a numerical parameter mathematically derived from the graph structure. It is a graph invariant thus it does not depend on the labeling or pictorial representation of the graph and it is the graph invariant number calculated from a graph representing a molecule. The topological indices of molecular graphs are widely used for establishing correlations between the structure of a molecular compound and its physic-chemical properties or biological activity. The topological indices which are definable by a distance function $d(.,.)$ are called a distance-based topological index. All distance-based topological indices can be derived from the distance matrix or some closely related distance-based matrix, for more information on this matter see [2] and a survey paper [20] and the references therein.

There are many examples of such indices, especially those based on distances, which are applicable in chemistry and computer science. The Wiener index (1947), defined as

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d(u,v)$$

is the first and most studied of the distance based topological indices [19]. The hyper-Wiener index,

$$WW(G) = \frac{1}{2} \sum_{\{u,v\} \subseteq V} (d(u,v) + d^2(u,v))$$

was introduced in (1993) by M. Randic [14]. The Harrary index

$$H(G) = \sum_{\{u,v\} \subseteq V} \frac{1}{d^2(u,v)}$$

was introduced in (1992) by Mihalic et al. [10]. In spite of this, the Harary index is nowadays defined [8, 12] as

$$H(G) = \sum_{\{u,v\} \subseteq V} \frac{1}{d(u,v)}.$$

The Schultz index

$$S(G) = \sum_{\{u,v\} \subseteq V} (d(u) + d(v)) d(u,v)$$

was introduced in (1989) by H. P. Schultz [16]. A. Dobrynin et al. in (1994) also proposed the Schultz index and called it the degree distance index and denoted $DD(G)$ [1]. S. Klavzar and

I Gutman, motivated by Schultz index, introduced in (1997) the second kind of Schultz index

$$S^*(G) = \sum_{\{u,v\}\subseteq V} d(u)d(v)d(u,v)$$

called modified Schultz (or Gutman) index of G [9]. The eccentric connectivity index

$$\xi^c = \sum_{v\in V} d(v)e(v)$$

was proposed by Sharma et al. [17]. For more details and examples of distance-based topological indices, we refer the reader to [2, 20, 13, 6] and the references therein.

Recently, The authors in [11], have been introduced a new type of graph topological index, based on distance and degree, called k-distance degree of a graph, for positive integer number $k \geq 1$. Which, for simplicity of notion, referred as N_k-index, denoted by $N_k(G)$ and defined by

$$N_k(G) = \sum_{k=1}^{diam(G)} \left(\sum_{v\in V(G)} d_k(v) \right) \cdot k$$

where $d_k(v) = d_k(v/G)$ and $diam(G)$ is the diameter of G. They have obtained some basic properties and bounds for N_k-index of graphs and they have presented the exact formulas for the N_k-index of some well-known graphs. They also established the N_k-index formula for a cartesian product of two graphs and generalize this formula for more than two graphs. The k-distance degree index, $N_k(G)$, of a graph G is the first derivative of the k-distance neighborhood polynomial, $N_k(G,x)$, of a graph evaluated at $x = 1$,see ([18]).

The following are some fundamental results which will be required for many of our arguments in this paper and which are finding in [11].

Lemma 1.1 *For $n \geq 1$, $N_k(\overline{K_n}) = N_k(K_1) = 0$.*

Theorem 1.2 *For any connected graph G of order n with size m and $diam(G) = 2$, $N_k(G) = 2n(n-1) - 2m$.*

Theorem 1.3 *For any connected nontrivial graph G, $N_k(G)$ is an even integer number.*

In this paper, we extend our study of N_k-index of a graph for other graph operations. Namely, exact formulas of the N_k-index for corona $G \circ H$ and neighborhood corona $G \star H$ products of connected graphs G and H are presented. An explicit formula for the splitting graph $S(G)$ of a graph G is computed. Also, the N_k-index formula of the join $G + H$ of two graphs G and H is presented. Finally, we generalize the N_k-index formula of the join for more than two graphs.

§2. The N_k-Index of Corona Product of Graphs

The corona of two graphs was first introduced by Frucht and Harary in [4].

Definition 2.1 *Let G and H be two graphs on disjoint sets of n_1 and n_2 vertices, respectively. The corona $G \circ H$ of G and H is defined as the graph obtained by taking one copy of G and n_1 copies of H, and then joining the i^{th} vertex of G to every vertex in the i^{th} copy of H.*

It is clear from the definition of $G \circ H$ that

$$\begin{aligned} n &= |V(G \circ H)| = n_1 + n_1 n_2, \\ m &= |E(G \circ H)| = m_1 + n_1(n_2 + m_2) \end{aligned}$$

and

$$diam(G \circ H) = diam(G) + 2,$$

where m_1 and m_2 are the sizes of G and H, respectively. In the following results, H^j, for $1 \leq j \leq n_1$, denotes the copy of a graph H which joining to a vertex v_j of a graph G,i.e., $H^j = \{v_j\} \circ H$, $D = diam(G)$ and $d_k(v/G)$ denotes the degree of a vertex v in a graph G. Note that in general this operation is not commutative.

Theorem 2.2 *Let G and H be connected graphs of orders n_1 and n_2 and sizes m_1 and m_2, respectively. Then*

$$N_k(G \circ H) = \left(1 + 2n_2 + n_2^2\right) N_k(G) + 2n_1 n_2 \left(n_1 + n_1 n_2 - 1\right) - 2n_1 m_2.$$

Proof Let G and H be connected graphs of orders n_1 and n_2 and sizes m_1 and m_2, respectively and let $D = diam(G)$, $n = |V(G \circ H)|$ and $m = |E(G \circ H)|$. Then by the definition of $G \circ H$ and for every $1 \leq k \leq diam(G \circ H)$, we have the following cases.

Case 1. For every $v \in V(G)$,

$$d_k(v/G \circ H) = d_k(v/G) + n_2 d_{k-1}(v/G).$$

Case 2. For every $u \in H^j$, $1 \leq j \leq n_1$,

- $d_1(u/G \circ H^j) = 1 + d_1(u/H)$;
- $d_2(u/G \circ H^j) = d_1(v_j/G) + (n_2 - 1) - d_1(u/H)$;
- $d_k(u/G \circ H^j) = d_{k-1}(v_j/G) + n_2 d_{k-2}(v_j/G)$, for every $3 \leq k \leq D + 2$.

Since for every $v \in V(G \circ H)$ either $v \in V(G)$ or $v \in V(H^j)$, for some $1 \leq j \leq n_1$, it follows that for $1 \leq k \leq diam(G \circ H)$,

$$\sum_{v \in V(G \circ H)} d_k(v/G \circ H) = \sum_{v \in V(G)} d_k(v/G \circ H) + \sum_{j=1}^{n_1} \sum_{u \in V(H^j)} d_k(u/G \circ H^j).$$

Hence, by using the hypothesis above

$$
\begin{aligned}
N_k(G\circ H) &= \sum_{k=1}^{diam(G\circ H)}\left[\sum_{v\in V(G\circ H)} d_k(v/G\circ H)\right]k\\
&= \sum_{k=1}^{D+2}\left[\sum_{v\in V(G)} d_k(v/G\circ H) + \sum_{j=1}^{n_1}\sum_{u\in V(H^j)} d_k(u/G\circ H^j)\right]k\\
&= \sum_{k=1}^{D+2}\left[\sum_{v\in V(G)} \Big(d_k(v/G) + n_2 d_{k-1}(v/G)\Big)\right].k + \sum_{k=1}^{D+2}\left[\sum_{j=1}^{n_1}\sum_{u\in V(H^j)} d_k(u/G\circ H^j)\right]k\\
&= \sum_{k=1}^{D+2}\Big(\sum_{v\in V(G)} d_k(v/G)\Big).k + n_2\sum_{k=1}^{D+2}\Big(\sum_{v\in V(G)} d_{k-1}(v/G)\Big)k\\
&\quad + \sum_{j=1}^{n_1}\sum_{u\in V(H^j)}\Big(1+d_1(u/H^j)\Big) + \sum_{j=1}^{n_1}\sum_{u\in V(H^j)}\Big(d_1(v_j/G)+(n_2-1)-d(u/H^j)\Big)2\\
&\quad + \sum_{k=3}^{D+2}\left[\sum_{j=1}^{n_1}\sum_{u\in V(H^j)}\Big(d_{k-1}(v_j/G)+n_2 d_{k-2}(v_j/G)\Big)\right]k
\end{aligned}
$$

Set $x = x_1 + x_2$, where

$$
\begin{aligned}
x_1 &= \sum_{k=1}^{D+2}\Big(\sum_{v\in V(G)} d_k(v/G)\Big)k\\
&= \sum_{k=1}^{D}\Big(\sum_{v\in V(G)} d_k(v/G)\Big)k + \Big(\sum_{v\in V(G)} d_{D+1}(v/G)\Big)(D+1) + \Big(\sum_{v\in V(G)} d_{D+2}(v/G)\Big)(D+2)\\
&= \sum_{k=1}^{D}\Big(\sum_{v\in V(G)} d_k(v/G)\Big)k + 0 + 0 = N_k(G).
\end{aligned}
$$

$$
\begin{aligned}
x_2 &= n_2\sum_{k=1}^{D+2}\Big(\sum_{v\in V(G)} d_{k-1}(v/G)\Big)k\\
&= n_2\Bigg[(\sum_{v\in V(G)} d_0(v/G))1 + (\sum_{v\in V(G)} d_1(v/G)).2 + \cdots + (\sum_{v\in V(G)} d_D(v/G))(D+1)\\
&\quad + (\sum_{v\in V(G)} d_{D+1}(v/G))(D+2)\Bigg] = n_2\left[n_1 + \sum_{k=1}^{D}(\sum_{v\in V(G)} d_k(v/G))(k+1) + 0\right]\\
&= n_2\left[n_1 + \sum_{k=1}^{D}(\sum_{v\in V(G)} d_k(v/G))k + \sum_{k=1}^{D}(\sum_{v\in V(G)} d_k(v/G))1\right]\\
&= n_2\Big[n_1 + N_k(G) + n_1(n_1-1)\Big].
\end{aligned}
$$

Thus, $x=(1+n_2)N_k(G)+n_1^2n_2$. Also, set $y=y_1+y_2+y_3$, where

$$\begin{aligned} y_1 &= \sum_{j=1}^{n_1}\sum_{u\in V(H^j)}(1+d_1(u/H))1 = n_1n_2+2n_1m_2, \\ y_2 &= \sum_{j=1}^{n_1}\sum_{u\in V(H^j)}(d_1(v_j/G)+(n_2-1)-d_1(u/H))2 = 2(2m_1n_2+n_1n_2(n_2-1)-2n_1m_2) \end{aligned}$$

and

$$\begin{aligned} y_3 &= \sum_{k=3}^{D+2}\left[\sum_{j=1}^{n_1}\sum_{u\in V(H^j)}\Big(d_{k-1}(v_j/G)+n_2d_{k-2}(v_j/G)\Big)\right]k \\ &= \sum_{k=3}^{D+2}\left[\sum_{j=1}^{n_1}\sum_{u\in V(H^j)}\Big(d_{k-1}(v_j/G)\Big)\right]k + n_2\sum_{k=3}^{D+2}\left[\sum_{j=1}^{n_1}\sum_{u\in V(H^j)}\Big(d_{k-2}(v_j/G)\Big)\right]k \\ &= n_2\left[\sum_{k=3}^{D+2}\Big(\sum_{j=1}^{n_1}(d_{k-1}(v_j/G)\Big)k\right] + n_2^2\left[\sum_{k=3}^{D+2}\Big(\sum_{j=1}^{n_1}(d_{k-2}(v_j/G)\Big)k\right]. \end{aligned}$$

Now set $y_3=y_3'+y_3''$, where

$$\begin{aligned} y_3' &= n_2\left[\sum_{k=3}^{D+2}(\sum_{j=1}^{n_1}(d_{k-1}(v_j/G))\right].k \\ &= n_2\left[(\sum_{v\in V(G)}d_2(v/G))3+(\sum_{v\in V(G)}d_2(v/G))4+\cdots+(\sum_{v\in V(G)}d_D(v/G))(D+1)+0\right] \\ &= n_2\left[\sum_{k=1}^{D}(\sum_{v\in V(G)}d_k(v/G))(k+1)-(\sum_{v\in V(G)}d_1(v/G))2\right] \end{aligned}$$

$$\begin{aligned} &= n_2\left[\sum_{k=1}^{D}(\sum_{v\in V(G)}d_k(v/G))k+\sum_{k=1}^{D}(\sum_{v\in V(G)}d_k(v/G))1-(\sum_{v\in V(G)}d_1(v/G))2\right] \\ &= n_2N_k(G)+n_1n_2(n_1-1)-4m_1n_2, \end{aligned}$$

and similarly

$$\begin{aligned} y_3'' &= n_2^2\left[\sum_{k=3}^{D+2}\Big(\sum_{j=1}^{n_1}(d_{k-2}(v_j/G)\Big)k\right] = n_2^2\left[\sum_{k=1}^{D}(\sum_{v\in V(G)}d_k(v/G))(k+2)\right] \\ &= n_2^2N_k(G)+2n_1n_2^2(n_1-1). \end{aligned}$$

Thus, $y_3=(n_2^2+n_2)N_k(G)+n_1n_2(n_1-1)-4m_1n_2+2n_1n_2^2(n_1-1)$.

Accordingly,

$$y=(n_2^2+n_2)N_k(G)+2n_1^2n_2^2+n_1^2n_2-2n_1n_2-2n_1m_2$$

and

$$N_k(G \circ H) = x + y.$$

Therefore,

$$N_k(G \circ H) = (1 + 2n_2 + n_2^2)N_k(G) + 2n_1n_2(n_1n_2 + n_1 - 1) - 2n_1m_2. \qquad \square$$

Corollary 2.3 *Let G be a connected graph of order $n \geq 2$ and size $m \geq 1$. Then*

(1) $N_k(K_1 \circ G) = 2(n^2 - m)$;
(2) $N_k(G \circ K_1) = 4N_k(G) + 2n(2n - 1)$;
(3) $N_k(G \circ \overline{K_p}) = (1 + 2p + p^2)N_k(G) + 2pn(pn + n - 1)$, *where $\overline{K_p}$ is a totally disconnected graph with $p \geq 2$ vertices.*

§3. The N_k-Index of Neighborhood Corona Product of Graphs

The neighborhood corona was introduced in [7].

Definition 3.1 *Let G and H be connected graphs of orders n_1 and n_2, respectively. Then the neighborhood corona of G and H, denoted by $G \star H$, is the graph obtained by taking one copy of G and n_1 copies of H, and joining every neighbor of the i^{th} vertex of G to every vertex in the i^{th} copy of H.*

It is clear from the definition of $G \circ H$ that

- In general $G \star H$ is not commutative.
- When $H = K_1$, $G \star H = S(G)$ is the splitting graph defined in [**?**].
- When $G = K_1$, $G \star H = G \cup H$.
- $n = |V(G \star H)| = n_1 + n_1n_2$
- $diam(G \star H) = \begin{cases} 3, & \text{if } diam(G) \leq 3; \\ diam(G), & \text{if } diam(G) \geq 3; \end{cases}$

In the following results, H^j, for $1 \leq j \leq n_1$, denotes the j^{th} copy of a graph H which corresponding to a vertex v_j of a graph G, i.e., $H^j = \{v_j\} \star H$, $D = diam(G)$ and $d_k(v/G)$ denotes the degree of a vertex v in a graph G.

Theorem 3.2 *Let G and H be connected graphs of orders and sizes n_1, n_2, m_1 and m_2 respectively such that $diam(G) \geq 3$. Then*

$$N_k(G \star H) = (1 + 2n_2 + n_2^2)N_k(G) + 2n_2^2(n_1 + m_1) + 2n_1(n_2 - m_2).$$

Proof Let G and H be connected graphs of orders and sizes n_1, m_1, n_2 and m_2 respectively and let $\{v_1, v_2, \cdots, v_{n_1}\}$ and $\{u_1, u_2, \cdots, u_{n_2}\}$ be the vertex sets of G and H respectively. Then for every $w|inv(G \star H)$ either $w = v \in V(G)$ or $w = u \in V(H)$. Since, for every $v \in V(G)$,

$$\begin{aligned} |N_1(v/G \star H)| &= |N_1(v/G)| + |V(H)||N_1(v/G)| \\ d_1(v/G \star H) &= d_1(v/G) + n_2 d_1(v/G) \\ &= (1 + n_2) d_1(v/G) \end{aligned}$$

and for every $u \in V(H^j)$, $1 \leq j \leq n_1$

$$\begin{aligned} |N_1(u/G \star H^j)| &= |N_1(u/H)| + |N_1(v_j/G)|, \\ d_1(u/G \star H^j) &= d_1(u/H) + d_1(v_j/G). \end{aligned}$$

Thus, for ever $w \in V(G \star H)$

$$\begin{aligned} \sum_{w \in V(G \star H)} d_1(w/G \star H) &= \sum_{v \in V(G)} d_1(v/G \star H) + \sum_{j=1}^{n_1} \sum_{u \in V(H^j)} d_1(u/G \star H^j) \\ &= \sum_{v \in V(G)} (1 + n_2) d_1(v/G) + \sum_{j=1}^{n_1} \sum_{u \in V(H^j)} (d_1(u/H^j) + d_1(v_j/G)) \\ &= (1 + n_2) \sum_{v \in V(G)} d_1(v/G) + \sum_{j=1}^{n_1} 2m_2 + n_2 \sum_{i=1}^{n_1} d_1(v_j/G) \\ &= (1 + 2n_2) \sum_{v \in V(G)} d_1(v/G) + 2n - 1m_2. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} |N_2(v_j/G \star H)| &= |N_2(v_j/G)| + |V(H^j)| + |V(H^j)||N_2(v_j/G)|, \\ d_2(v_j/G \star H) &= d_2(v_j/G) + n_2 + n_2 d_2(v/G) \\ &= (1 + n_2) d_2(v/G) + n_2 \end{aligned}$$

for every $v_j \in V(G)$, $1 \leq j \leq n_1$, and

$$\begin{aligned} |N_2(u/G \star H^j)| &= (|V(H^j)| - 1) - |N_1(u/H^j)| + |\{v_j\}| \\ &\quad + |V(H^j)||N_2(v_j/G)| + |N_2(v_j/G)| \\ d_2(u/G \star H^j) &= (n_2 - 1) - d_1(u/H) + 1 + n_2 d_2(v_j/G) + d_2(v/G) \\ &= n_2 + d_1(u/H) + (1 + n_2) d_2(v_j/G) \end{aligned}$$

for every $u \in H^j$, $1 \leq j \leq n_1$. Thus, for ever $w \in V(G \star H)$,

$$\begin{aligned}\sum_{w\in V(G\star H)} d_2(w/G\star H) &= \sum_{v\in V(G)} d_2(v/G\star H) + \sum_{j=1}^{n_1}\sum_{u\in V(H^j)} d_2(u/G\star H^j)\\ &= \sum_{v\in V(G)} \Big[(1+n_2)d_1(v/G) + n_2\Big]\\ &\quad + \sum_{j=1}^{n_1}\sum_{u\in V(H^j)} \Big[n_2 + d_1(u/H) + (1+n_2)d_1(v_j/G)\Big]\\ &= (1+n_2+n_2^2)\sum_{v\in V(G)} d_2(v/G) + n_1n_2^2 + n_1n_2 - 2n_1m_2.\end{aligned}$$

Also, for every $v \in V(G)$, $d_3(v/G\star H) = (1+n_2)d_3(v/G)$ and for every $u \in V(H^j)$,

$$d_3(u/G\star H^j) = n_2d_1(v_j/G) + (1+n_2)d_3(v_j/G).$$

Hence, For every $w \in V(G\star H)$,

$$\begin{aligned}d_3(w/G\star H) &= (1+n_2+n_2^2)\sum_{v\in V(G)} d_3(v/G)\\ &\quad + n_2^2\sum_{v\in V(G)} d_1(v/G).\end{aligned}$$

By continue in same process we get, for every $4 \leq k \leq diam(G\star H)$, that is, for every $v \in V(G)$,

$$d_k(v/G\star H) = (1+n_2)d_k(v/G)$$

and for every $u \in V(H^j)$,

$$d_k(u/G\star H^j) = (1+n+2)d_k(v_j/G),$$

and hence for every $w \in V(G\star H)$,

$$d_k(w/G\star H) = (1+2n_2+n_2^2)d_k(v/G).$$

Accordingly,

$$\begin{aligned}N_k(G\star H) &= \sum_{k=1}^{D}\Big(\sum_{w\in V(G\star H)} d_k(w/G\star H)\Big)\ k\\ &= \sum_{w\in V(G\star H)} d_1(w/G\star H))1 + \sum_{w\in V(G\star H)} d_2(w/G\star H))2 + \cdots\\ &\quad + \sum_{w\in V(G\star H)} d_D(w/G\star H))\ D\end{aligned}$$

$$= \Big[(1+2n_2) \sum_{v\in V(G)} d_1(v/G) + 2n_1m_2\Big]1$$
$$+ \Big[(1+2n_2+n_2^2) \sum_{v\in V(G)} d_2(v/G) + n_1n_2^2 + n_1n_2$$
$$- 2n_1m_2\Big]\ 2 + \Big[(1+2n_2+n_2^2) \sum_{v\in V(G)} d_3(v/G) + n_2^2 \sum_{v\in V(G)} d_1(v/G)\Big]\ 3$$
$$+ \Big[(1+2n_2+n_2^2) \sum_{v\in V(G)} d_4(v/G)\Big]\ 4 + \cdots + \Big[(1+2n_2+n_2^2) \sum_{v\in V(G)} d_D(v/G)\Big]\ D$$
$$= (1+2n_2+n_2^2)\Big[\sum_{v\in V(G)} d_1(v/G)\ 1 + \sum_{v\in V(G)} d_2(v/G)\ 2 + \cdots + \sum_{v\in V(G)} d_D(v/G)\ D\Big]$$
$$+ \Big[(-n_2^2 \sum_{v\in V(G)} d_1(v/G) + 2n_1m_2)\ 1 + (n_1n_2^2 + n_1n_2 - 2n_1m_2)\ 2$$
$$+ (n_2^2 \sum_{v\in V(G)} d_1(v/G))\ 3$$
$$= (1+2n_2+n_2^2)N_k(G) + 2n_2^2(n_1+m_1) + 2n_1(n_2-m_2). \qquad \square$$

Corollary 3.3 *Let G be a connected graph of order $n \geq 2$ and size m and let $S(G)$ be the splitting graph of G. Then*

$$N_k(S(G)) = 4N_k(G) + 2(2n+m).$$

§4. The N_k-Index of Join of Graphs

Definition 4.1([5]) *Let G_1 and G_2 be two graphs with disjoint vertex sets $V(G_1)$ and $V(G_2)$ and edge sets $E(G_1)$ and $E(G_2)$. Then the join $G_1 + G_2$ of G_1 and G_2 is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{uv|u \in V(G_1)\&v \in V(G_2)\}$.*

Definition 4.2 *It is clear that, $G_1 + G_2$ is a connected graph, $n = |V(G_1+G_2)| = |V(G_1)| + |V(G_2)|$, $m = |E(G_1+G_2)| = |V(G_1)||V(G_2)| + |E(G_1)| + |E(G_2)|$ and $diam(G_1+G_2) \leq 2$. Furthermore, $diam(G_1+G_2) = 1$ if and only if G_1 and G_2 are complete graphs. We denote by $d_k(v/G)$ to the k-distance degree of a vertex v in a graph G.*

Theorem 4.2 *Let G and H be connected graphs of order n_1 and n_2 and size m_1 and m_2, respectively. Then*

$$N_k(G+H) = 4\binom{n_1+n_2}{2} - 2(n_1n_2 + m_1 + m_2).$$

Proof The proof is an immediately consequences of Theorem 1.2. $\square$

Since, For any connected graph G, $G + K_1 = K_1 + G = K_1 \circ G$ then the next result follows

Corollary 2.3.

Corollary 4.3 *For any connected graph G with n vertices and m edges,*

$$N_k(G + K_1) = 2(n^2 - m).$$

The join of more than two graphs is defined inductively as following,

$$G_1 + G_2 + \cdots + G_t = (G_1 + G_2 + \cdots + G_{t-1}) + G_t$$

for some positive integer number $t \geq 2$. We denote by $\sum_{i=1}^{t} G_i$ to $G_1 + G_2 + \cdots + G_t$. It is clear for this definition that

- $n = |V(\sum_{i=1}^{t} G_i)| = \sum_{i=1}^{t} |V(G_i)|$.
- $m = |E(\sum_{i=1}^{t} G_i)| = \sum_{i=1}^{t} |E(G_i)| + \sum_{i=2}^{t} |V(G_i)| \Big(\sum_{j=1}^{i-1} |V(G_j)| \Big)$.
- $diam(\sum_{i=1}^{t} G_i) \leq 2$.

Accordingly, we can generalize Theorem 4.2 by using Theorem 1.2 as following.

Theorem 4.4 *For some positive integer number $t \geq 2$, let $G_1, G_2, \cdots, G_t$ be connected graphs of orders $n_1, n_2, \cdots, n_t$ and sizes $m_1, m_2, \cdots, m_t$, respectively. Then*

$$N_k(\sum_{i=1}^{t} G_i) = 4\binom{\sum_{i=1}^{t} n_i}{2} - 2\left[\sum_{i=1}^{t} m_i + \sum_{i=2}^{t} n_i \Big(\sum_{j=1}^{i-1} n_j \Big) \right].$$

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