The world lines of a dust collapsar

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Abstract In a previous article it was shown that the end state for the dust metric of Oppenheimer and Snyder has most of its mass concentrated just inside the gravitational radius; it is proposed that the resulting object be considered as an idealized *shell collapsar*. Here the treatment is extended to include the family of interior metrics described by Choquet-Bruhat. The end state is again a shell collapsar, and its structure depends on the density profile at the beginning of the collapse. What is lacking in most previous commentaries on the Oppenheimer-Snyder article is the recognition that Oppenheimer and Snyder matched the time coordinate at the surface, and that implies a finite upper limit for the comoving time coordinate inside the collapsar. A collapse process having all the matter going inside the gravitational radius would require comoving times which go outside that upper limit.

1 Introduction

Since the inception of General Relativity (GR), solutions have been sought for the evolution of a mass distribution under its own gravity. The first attempt at an *equilibrium* GR solution was the uniform density of Schwarzschild[1] in 1916, but progress on "the problem of motion"[2] came very slowly. The first time-dependent solutions, with the idealized equation of state $p = 0$, were those of Tolman[3]; for simplicity, especially because of the absence of gravitational waves, the Tolman solutions were all spherosymmetric. Models based on this Tolman solution are known as *dust models*.

The 1939 article of Oppenheimer and Snyder[4] (OS) was a particular Tolman solution and played a central role in the birth of the black hole, especially because it was used as a basis for Penrose's[5] Theorem, which states that if certain conditions hold, the end state of a gravitational collapse must be a point singularity having infinite density. Penrose claimed that the OS dust metric, which at that time was the only known solution of the time-dependent field equations, satisfied those conditions. However, it has now been demonstrated[6][7] that the OS metric, describing collapse from an initially uniform density, has an end state quite different from that described by Penrose; the end state of OS is a shell with most of the dust material concentrated just inside the gravitational radius.

A set of metrics was described more recently by Choquet-Bruhat[8]. This latter author came to the same conclusion as Penrose regarding the end state of OS, and went on to generalize that conclusion to the extended family[8]. Here I seek to preserve the analysis of the OS article, and shall show below that the YCB family then has a similar end state to OS. The part of OS which was not fully implemented, by either Penrose or Choquet-Bruhat, is the mapping by OS, from the comoving coordinates used to describe the interior of the collapsar, onto the exterior Schwarzschild coordinates. OS glued together these two metrics by imposing continuity conditions at the surface. It would seem that the only treatment, since the black-hole era, which maintains the continuity conditions of OS is that of Weinberg[9]. On the whole the OS analysis has been forgotten.

The greater part of the mass of the OS collapsar is concentrated just inside the gravitational radius; in the limit $t \to +\infty$ the density at the surface, like that at the centre of a black hole, becomes infinite, but with this more extended shell version of collapse it is possible that a real collapsar with a nontrivial equation of state will be a smeared out version of an OS shell, a possibility which we discussed in two previous articles[6][10]. Such collapsars may be considered as a revival of the *frozen stars*[11][12][13] discussed as early alternatives to black holes, a more recent version of which is the *gravastar*[14].

2 The YCB metric

The metric of Choquet-Bruhat[8] (YCB) describes a dust star for which the initial velocity profile, as in OS, remains zero, but in which the density is nonuniform. The end configuration in YCB has the same crowding together of world lines as in OS; I propose to call it a *shell collapsar*. It should be especially remarked that the property in question may belong to an initial state having either increasing or decreasing initial density as the radius goes towards $R = 0$; it is in contradiction with Choquet-Bruhat.

The generalized OS metric of YCB[1] in the region $R < R_b = 2m$ is

$$ds^2 = d\tau^2 - r'^2 dR^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \quad , \tag{1}$$

where

$$r = Rz = R\left(1 - \frac{3\tau}{2}\sqrt{\frac{2M(R)}{R^3}}\right)^{2/3} \quad , \quad r' = \frac{\partial r}{\partial R} = z + R\frac{\partial z}{\partial R} \quad , \tag{2}$$

and $M(R_b) = M(2m) = m$. The coordinate R is comoving, so it is constant along all the world lines during collapse, including at the surface $R = R_b = 2m$.

In this coordinate frame the stress tensor $T^{\mu\nu}$ has the single nonzero component

$$T^{\tau\tau} = -\frac{1}{16\pi}\left[\frac{2}{g_{RR}}\frac{\partial^2 g_{RR}}{\partial\tau^2} + \frac{4}{g_{\theta\theta}}\frac{\partial^2 g_{\theta\theta}}{\partial\tau^2} - \left(\frac{1}{g_{RR}}\frac{\partial g_{RR}}{\partial\tau}\right)^2 - 2\left(\frac{1}{g_{\theta\theta}}\frac{\partial g_{\theta\theta}}{\partial\tau}\right)^2\right] \quad , \tag{3}$$

which reduces to

$$T^{\tau\tau} = -\frac{1}{4\pi}\left[\frac{1}{\sqrt{-g_{RR}}}\frac{\partial^2\sqrt{-g_{RR}}}{\partial\tau^2} + \frac{2}{\sqrt{-g_{\theta\theta}}}\frac{\partial^2\sqrt{-g_{\theta\theta}}}{\partial\tau^2}\right] \quad , \tag{4}$$

[1] We are using here the notation of the OS article.

that is

$$T^{\tau\tau} = -\frac{1}{4\pi}\left[\frac{1}{r}\frac{\partial^2 r}{\partial\tau^2} + \frac{2}{r'}\frac{\partial^2 r'}{\partial\tau^2}\right] \quad . \tag{5}$$

This is the quantity denoted by $\mu/8\pi$ in YCB eq (12.3), and its value is

$$T^{\tau\tau} = \frac{M'(R)}{4\pi r^2 r'} = \frac{R^2\mu_0(R)}{4\pi r^2 r'} \quad , \tag{6}$$

in agreement with YCB eqs (12.9) and (12.12). For the OS case, $r' = z = r/R$ and $M(R) = mR^3/R_b^3$, so we have the constant density

$$T^{\tau\tau} = \frac{3mR^3}{4\pi r^3 R_b^3}, \quad \mu_0(R) = \frac{3m}{R_b^3} \quad \text{(OS)} \quad . \tag{7}$$

The volume element for this metric is

$$\sqrt{-g}\sin\theta dR d\theta d\phi = r^2 r' \sin\theta dR d\theta d\phi \quad ,$$

so that

$$\int_{R_1<R} \mu(\tau, R_1)\sqrt{-g}\sin\theta dR_1 d\theta d\phi = M(R) \quad , \tag{8}$$

which gives a hint for the meaning of $M(R)$ that we shall take up later.

To change from the comoving radius R to one matching the exterior radius r, we put

$$r' dR = dr - \frac{\partial r}{\partial\tau}d\tau = dr + \sqrt{\frac{2M}{r}}d\tau \quad , \tag{9}$$

so the metric becomes

$$ds^2 = \left(1 - \frac{2M}{r}\right)d\tau^2 - 2\sqrt{\frac{2M}{r}}dr d\tau - dr^2 - r^2 d\theta^2 - r^2\sin^2\theta d\phi^2 \quad . \tag{10}$$

This is YCB eq (12.19), apart from an error in its first term given as $(1 - \sqrt{2M/r})d\tau^2$. Then, completing the square, we obtain

$$\begin{aligned} ds^2 &= \frac{r}{r-2M}\left[\left(1 - \frac{2M}{r}\right)d\tau - \sqrt{\frac{2M}{r}}dr\right]^2 - \frac{r}{r-2M}dr^2 \\ &\quad -r^2 d\theta^2 - r^2\sin^2\theta d\phi^2 \quad . \end{aligned} \tag{11}$$

Following a procedure of Weinberg[9], the above square-bracketed quantity will now be put in the form of a multiple of a perfect differential dy. This may be achieved by changing the variable τ to z so that

$$\tau = -\frac{2}{3\sqrt{F}}\left(z^{3/2} - 1\right), \quad \frac{\partial\tau}{\partial z} = -R\sqrt{\frac{r}{2M}} \quad , \tag{12}$$

where $F(R) = 2M(R)/R^3$, giving

$$d\tau = -R\sqrt{\frac{r}{2M}}\left(dz - \frac{F'}{3F\sqrt{z}}\left(z^{3/2} - 1\right)dR\right) \quad . \tag{13}$$

and hence the square-bracketed quantity is

$$-R\sqrt{\frac{r}{2M}}\left(dz+FRdR-\frac{F'}{3F\sqrt{z}}\left(z^{3/2}-1\right)\left(1-\frac{2M}{r}\right)dR\right) \quad . \tag{14}$$

The metric then becomes

$$ds^2=\frac{R^2r^2}{2M(r-2M)}[dz+GdR]^2-\frac{r}{r-2M}dr^2-r^2d\theta^2-r^2\sin^2\theta d\phi^2 \quad , \tag{15}$$

where

$$G(R,z)=\frac{F'}{3F}\left(-z+R^2F+\frac{3RF^2}{F}+z^{-1/2}-R^2Fz^{-3/2}\right) \quad . \tag{16}$$

We have identified the term in dr^2 with the corresponding term in the exterior Schwarzschild metric by having put the gravitational mass m equal to $M(R_b)$. Then, since the metric is spherosymmetric with zero density in $R>R_b$, by Birkhoff's Theorem the first term in the metric of the exterior region $R>R_b$ must be

$$g_{tt}dt^2=\frac{r-2m}{r}dt^2 \quad (R>R_b) \quad , \tag{17}$$

and then, by a suitable mapping $t=t(z,R)$ in the interior region $R<R_b$, the metric of (15) must make g_{tt} continuous at $R=R_b$.

In the constant-density OS case, $F(R)=2m/R_b^3$, $F'=0$, and the content of the square bracket is $dz+2mRdR/R_b^3$ which is a perfect differential (see [4] equation (32)), that is

$$dz+GdR=dy, \quad y=\frac{rR_b}{2mR}+\frac{R^2}{2R_b^2}-\frac{1}{2} \quad \text{(OS)} \quad , \tag{18}$$

and the variable y is then the *cotime*, related by OS equation (36) to the exterior time t. In the general case we have to generalize the OS metric in such a way that the content of the square bracket, times an integrating factor $f(R,z)$ with $f(R_b,z)=1$, is a perfect differential $dy(R,z)$ for $0\le R\le R_b, z\ge 1$. The perfect differential has to satisfy

$$\frac{\partial f}{\partial R}=\frac{\partial}{\partial z}(fG) \quad , \tag{19}$$

and the cotime is then

$$y=z-\int_R^{R_b}G(R_1,z)f(R_1,z)dR_1 \quad , \tag{20}$$

with the metric given by

$$ds^2=\frac{R^2r^2}{2f^2M(r-2M)}dy^2-\frac{r}{r-2M}dr^2-r^2d\theta^2-r^2\sin^2\theta d\phi^2 \quad . \tag{21}$$

We shall see how to calculate the integrating factor $f(R,z)$ in the next section.

The correspondence between the cotime and the time coordinate of the exterior Schwarzschild metric is the same as in OS equation (36), namely

$$\frac{t}{2m} = -\frac{2}{3}y^{3/2} - 2\sqrt{y} + \ln\frac{\sqrt{y}+1}{\sqrt{y}-1} \quad . \tag{22}$$

The motivation, in the OS case, came from (20), putting, at $R = R_b$,

$$y(R_b, z) = \frac{zR_b}{2m} = \frac{r}{2m} \quad . \tag{23}$$

The freefall world lines for $R > R_b$, including the surface $R = R_b$, are

$$\frac{t}{2m} = -\frac{2}{3}\left(\frac{r}{2m}\right)^{3/2} - 2\sqrt{\frac{r}{2m}} + \ln\frac{\sqrt{r}+\sqrt{2m}}{\sqrt{r}-\sqrt{2m}} + \text{const.} \quad , \tag{24}$$

so the world lines either side of $R = R_b$ fit continuously, if const.=0 there. Under the mapping (22), the metric becomes

$$ds^2 = \frac{R^2 r^2 (y-1)^2}{2R_b^2 f^2 M(r-2M)y^3}dt^2 - \frac{r}{r-2M}dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \quad , \tag{25}$$

which differs from the OS case solely by the integrating factor f^2, and we find that, for all $y > 1$ at the surface $R = R_b$, where $f = 1$, the matching condition (17) is satisfied, so that the entire metric satisfies the continuity requirement. Furthermore the entire history going from $t = -\infty$ to $t = +\infty$ is covered; this is because of the logarithmic singularity in (22) at $y = 1$. It implies that gravitational contraction stops at $r = 2m$, that is the gravitational radius. The "end of time" at $y = 1$, corresponding to $t \to +\infty$, is precisely what OS, in their equation (37), gave as their proper time limit $\tau = \tau_0(R)$

$$\tau = \tau_0(R) = \frac{2}{3}\sqrt{\frac{R_b^3}{r_0}} - \frac{r_0}{3\sqrt{2}}\left(3 - \frac{R^2}{R_b^2}\right)^{3/2} \quad , \tag{26}$$

where $r_0 = 2m$ is the gravitational radius; there is no time beyond $\tau = \tau_0(R)$.

3 Asymptotics of the integrating factor

In (16) it is convenient to use dimensionless variables, that is

$$F(R) = \frac{2m}{R_b^3}E(\rho) = \frac{1}{4m^2}E(\rho), \quad \rho = \frac{R}{2m} \quad , \tag{27}$$

where $E(1) = 1$. Then (19) takes the form

$$\frac{\partial f}{\partial \rho} = \frac{E'}{3E}\frac{\partial}{\partial z}(G_1 f) \quad (f(1,z) = 1) \quad , \tag{28}$$

where

$$G_1 = -z + \rho^2 E + 3\rho E^2/E' + z^{-1/2} - \rho^2 E z^{-3/2} \quad . \tag{29}$$

We put $f = \phi E^{-1/3}$ and change the variable ρ to E, so that

$$E\frac{\partial\phi}{\partial E} - \frac{1}{3}\phi = \frac{1}{3}\frac{\partial}{\partial z}(G_1\phi) \quad . \tag{30}$$

Then, putting $\phi = \phi_0 + \phi_1 z^{-1/2} + \phi_2 z^{-1} + \ldots$, the leading term is $\phi_0 = 1$. The next nonzero term is ϕ_3, and it satisfies

$$\frac{d\phi_3}{dE} - \frac{1}{2E}\phi_3 = -\frac{1}{6E} \quad , \tag{31}$$

that is, since $\phi_3(1) = 0$,

$$\phi_3 = \frac{1}{3}\left(1 - E^{1/2}\right) \quad , \tag{32}$$

and ϕ_6 is obtained similarly as

$$\phi_6 = \frac{2}{9}\left(1 - 2E^{1/2} + E\right) \quad . \tag{33}$$

However, ϕ_5 requires an integration

$$\phi_5(R) = E^{5/6}\int_\rho^1 E^{-5/6}\left(\frac{3\rho E - 2\rho^2 E'}{6E'} - \frac{3\rho E + \rho^2 E'}{6E'}E^{1/2}\right) d\rho \quad . \tag{34}$$

This may be carried out algebraically for the case $E(\rho) = 1 - \lambda(1-\rho)$ with $\lambda < 1$, by changing the variable of integration to E; for example $\lambda = -1$ in that case gives

$$\phi_5 = 8E - \frac{2}{7}E^2 - \frac{1}{13}E^3 + \frac{1}{4}(2-E)^2 E^{3/2} - \frac{2871}{364}E^{5/6} \quad . \tag{35}$$

Further terms are obtained, as polynomials in $E^{1/6}$, from the recurrence relation

$$\frac{d\phi_n}{dE} - \frac{n}{6E}\phi_n = -\frac{n-2}{6E}\left(\rho E(3E/\lambda + \rho)\phi_{n-2} + \phi_{n-3} - E\rho^2\phi_{n-5}\right) \quad , \tag{36}$$

for example $\lambda = -1$ in this case gives

$$\begin{aligned}\phi_7 = -32E^2 + \frac{2840}{77}E^3 - \frac{15720}{1547}E^4 + \frac{170}{2093}E^5 + \frac{20}{377}E^6 \\ -E^{1/2}\left(\frac{5}{2}E^2 - 5E^3 + \frac{15}{4}E^4 - \frac{5}{4}E^5 + \frac{5}{32}E^6\right) \\ +\alpha E^{5/6}\left(5E - 5E^2 + \frac{5}{4}E^3\right) + E^{7/6}\left(\beta + \frac{5}{32} - \frac{5}{4}\alpha\right) \quad ,\end{aligned} \tag{37}$$

where

$$\alpha = \frac{2871}{364}, \quad \beta = \frac{58388558}{11350339} \quad . \tag{38}$$

These may be readily adapted to the more general case $\lambda < 1$, and then computed up to any value of n. The resulting power series in $\zeta = z^{-1/2}$ has been shown to have radius of convergence beyond $|\zeta| = 1$.

The cotime variable y now has to satisfy

$$\frac{\partial y}{\partial z} = f, \quad \frac{\partial y}{\partial \rho} = \frac{E'}{3E} G_1 f \quad (y(z,1) = z) \quad . \tag{39}$$

We have just obtained

$$f(z,\rho) = E^{-1/3} \left(1 + \sum_{n=3}^{\infty} \phi_n(\rho) z^{-n/2} \right) \quad , \tag{40}$$

so, integrating the first of these with respect to z,

$$y(z,\rho) = y_0(\rho) + E^{-1/3} \left[z - \sum_{n=3}^{\infty} \frac{2}{n-2} \phi_n z^{-(n-2)/2} \right] \quad , \tag{41}$$

where the second equation requires

$$y_0'(\rho) - \frac{E'}{3E} G_1(z,\rho) f(z,\rho) = -2 \sum_{n=3}^{\infty} E' \left(\frac{d\phi_n}{dE} - \frac{\phi_n}{3E} \right) z^{-(n-2)/2} \quad , \tag{42}$$

with $y_0(1) = 0$, for all $z \geq 1$. By comparison with the recurrence relation(36), the right side of this equation may be identified term by term with

$$\left(\rho E + \frac{1}{3} \rho^2 E' \right) E^{-1/3} - \frac{E'}{3E} G_1(z,\rho) f(z,\rho) \quad ,$$

so we deduce that

$$y_0'(\rho) = \left(\rho E + \frac{1}{3} \rho^2 E' \right) E^{-1/3} \quad , \tag{43}$$

and hence

$$y_0(\rho) = \frac{1}{2} \left(\rho^2 E^{2/3} - 1 \right) \quad , \tag{44}$$

so that finally

$$y(\rho,z) = E^{-1/3} \left[z + \frac{1}{2} \left(\rho^2 E - E^{1/3} \right) - \sum_{n=3}^{\infty} \frac{2}{n-2} \phi_n z^{-(n-2)/2} \right] \quad . \tag{45}$$

In the OS case, $E = 1$, this reduces to $y = z + (\rho^2 - 1)/2$.

4 Formation of the shell

The radius r at cotime y, on a world line specified by the parameter R, is

$$r(R,y) = \left(y + \frac{1}{2} \right) R E^{1/3} - R^3 E + \sum_{n=3}^{\infty} \frac{2R}{n-2} \phi_n(R) \left(\frac{r}{R} \right)^{-(n-2)/2} \quad , \tag{46}$$

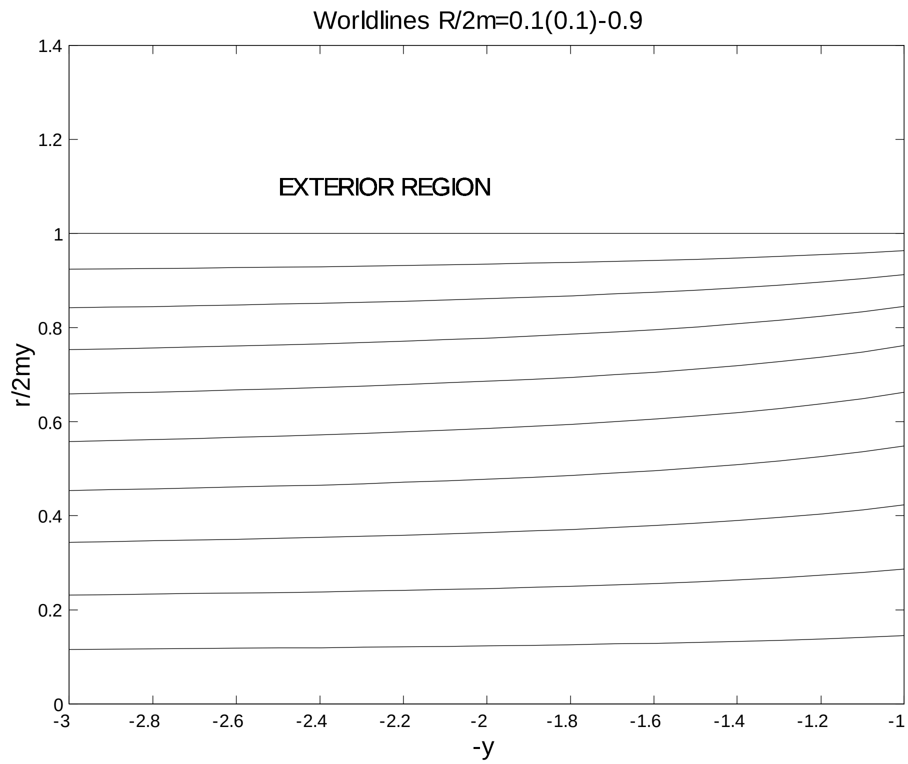

Figure 1: Interior world lines, with comoving parameters in $0<R<2m$, showing crowding towards the surface at $R=2m$. The initial cumulative density is $M(R)=R^3(1-0.3+0.3R)$.

where the inversion of $z(R,y) = r(R,y)/R$ from $y(R,z)$, being monotonic in y, presents no difficulty, and has been displayed in Figure 1, with the variable $r(R,y)/r(1,y) = r(R,y)/2my$ plotted against y at $\lambda = -0.3$. This choice for λ gives an initial density maximum at the centre.

The figure displays the same property of the world lines as was noted in [6]; they show a tendency to crowd toward the surface $R = R_b$. From the general expressions for ϕ_3, ϕ_5 and ϕ_6, given by equations (32,34,33), we obtain their derivatives at $R = 1$ as

$$\phi_3'(1) = -\frac{F'(1)}{6}, \quad \phi_5'(1) = \frac{F'(1)}{2}, \quad \phi_6'(1) = 0 \quad , \tag{47}$$

and subsequent recurrences give, with equal generality,

$$\phi_n'(1) = 0, \quad (n \geq 6) \quad . \tag{48}$$

We now define

$$\psi(R,y) = \left(\frac{\partial r}{\partial R}\right)_y \quad , \tag{49}$$

Substituting in (46) with $R = 1$, we find that

$$\psi(1,y) = (y-1)\left(1 + \frac{F'(1)}{3}\right) - \frac{F'(1)}{6}(z^{-1/2} - z^{-3/2}) \quad , \tag{50}$$

and since $z(1,y) = y$, it follows that, at $R = 1$, $(\partial r/\partial R)_y \to 0$ as $y \to 1$, thereby extending the OS limiting property of infinite shell density as $t \to +\infty$.

The latter property displays, in extreme form, the crowding of world lines at $R = R_b$. It is a consequence of the boundary condition there, having required the time dilation of the interior world lines to be continuous with the same property in the exterior region. It should be regarded as a property of dust stars in general, that is of matter with zero pressure, and we may expect that a more realistic equation of state will evolve into a collapse with finite density everywhere.

5 Discussion

We have shown how complete matching of the interior with the exterior metric, leading to a proper understanding of the limits of the time variables, is intimately linked with the shell structure of the end state. This matching, requiring the construction of our integrating factor $f(R,z)$, was an essential part of the analysis of the original OS article, and also of Weinberg[9], but was not considered by Penrose[5] or Choquet-Bruhat[8].

Of course, we should recognize that only a limited set of inferences may be drawn from these highly idealized dust models, but, since they are the only complete time-dependent solutions for collapse, they should properly be taken account of in trying to describe collapsars with more realistic equations of state. A contemporary statement from Christodoulou[15] along these lines is

> An important remark at this point is that it is not a priori obvious that closed trapped surfaces are evolutionary. That is, it is not obvious whether closed trapped surfaces can form in evolution starting from initial conditions in which no such surfaces are present. What is more important, the physically interesting problem is the problem where the initial conditions are of arbitrarily low compactness, that is, arbitrarily far from already containing closed trapped surfaces, and we are asked to follow the long time evolution and show that, under suitable circumstances, closed trapped surfaces eventually form. Only an analysis of the dynamics of gravitational collapse can achieve this aim.

The alleged evolution of a collapsar towards a trapped surface inside the horizon is, of course a necessary step in the black-hole conjecture[5]. The argument of the previous sections was almost entirely geometrical, but the latter quotation emphasizes the need for a field-theoretic content. A first step in that direction was already taken in [6], by considering the stress tensor, but a more ambitious and revealing programme will necessitate studying, for example, the Landau energy pseudotensor[16]; this is an area which we investigated in a previous publication[10].

Although we sometimes like to pretend we have outgrown the notion of gravitational force, it has also become commonplace to argue that,"once collapse has occurred inside the event horizon, no force can override the gravitational attraction, so further collapse to a black hole is inevitable". Such an argument leaves us permanently entrapped in the Newtonian theory. In the tensor theory of General Relativity, it is not implausible to consider[6][17] the possibility of *gravitational repulsion* from within an inner core. Equally plausible is the recognition that the shell may exert a *gravitational attraction whose source is in the outer part of the shell itself.* Such a description is ruled out in Newtonian gravity, because of the inverse square law and the consequent Shell Theory, but may properly be investigated with the Landau pseudotensor.

The latter feature of the dust collapsar also indicates the need to take account of the nonzero pressures inside real collapsars. In dense bodies like neutron stars and galactic centres there is a core of negative gravitational energy with negative mass, which not only produces repulsion but also cancels out a proportion of the "proper mass"[9][18] contained in the stellar material. A nonzero pressure is what prevents repulsive gravity effects going to the extremes of the dust models. The pioneering work in this area was the article of Oppenheimer and Volkoff[19] (OV), published a few months prior to the OS article. To the extent that such an investigation has been carried through to black hole related models, it has been dominated by what may be considered a Newtonian insistence that gravity can only be attractive, with the consequence that nuclear material is squeezed to very high central densities, giving birth to exotic material such as hyperons and quarks. Note, however, that Oppenheimer and Volkoff, in their footnote 10, conceded the possibility of varying the central boundary condition, but did not investigate it further in the light of the OS article. In our articles cited above we indicated the profound way in which the incorporation of repulsive gravity changes such theories, through changed boundary conditions at the centre. It should be borne in mind that the pressures required to prevent infinite density at the surface are less by many orders of magnitude than would be required to prevent an infinite central density. It should also be noted that collapsars with shell-like density profiles, and with a realistic equation of state, have been proposed in the context of metrics having an empty de Sitter metric[14] at the centre.

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