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ACLASSIFICATIONOFQUANTUMPARTICLES

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I. Covariant Formulations Of Classical And Quantum Physics

In physics, the electromagnetic field has a dual character and plays a crucial role both in the formulation of relativity theory and quantum mechanics. However, since the electromagnetic field itself is regarded simply as a physical event whose dynamics can be described by mathematical methods therefore it is reasonable to suggest that it should be formulated in both forms of classical and quantum mathematical formulations. This amounts to suggesting that it should be derived from the same mathematical structure of classical theories, such as the gravitational field, and at the same time from the same mathematical structure of quantum theories, such as Dirac formulation of quantum mechanics. In this section we show that in fact this is the case. As shown in our works on spacetime structures of quantum particles [1], the three main dynamical descriptions of physical events in classical physics, namely Newton mechanics, Maxwell electromagnetism and Einstein gravitation, can be formulated in the same general covariant form and they can be represented by the general equation

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$$\nabla_\beta M = kJ \tag{1}$$

where M is a mathematical object that represents the corresponding physical system and ∇_β is a covariant derivative. For Newton mechanics, we have $M = E = \frac{1}{2}m\sum_{\mu=1}^{3}(dx^\mu/dt)^2 + V$ and $J = 0$. For Maxwell electromagnetism, $M = F^{\alpha\beta} = \partial^\mu A^\nu - \partial^\nu A^\mu$ with the four-vector potential $A^\mu \equiv (V, \mathbf{A})$ and J can be identified with the electric and magnetic currents. And for Einstein gravitation, $M = R^{\alpha\beta}$ and J can be defined in terms of a metric $g_{\alpha\beta}$ and the Ricci scalar curvature. It is shown in differential geometry that the Ricci tensor $R^{\alpha\beta}$ satisfies the Bianchi identities

$$\nabla_\beta R^{\alpha\beta} = \frac{1}{2}g^{\alpha\beta}\nabla_\beta R \tag{2}$$

where $R = g^{\alpha\beta}R_{\alpha\beta}$ is the Ricci scalar curvature [2]. Even though Equation (2) is purely geometrical, it has a covariant form similar to the electromagnetic tensor $\partial_\alpha F^{\alpha\beta} = \mu j^\beta$ defined in Euclidean space. If the quantity $\frac{1}{2}g^{\alpha\beta}\nabla_\beta R$ can be identified as a physical entity, such as a four-current of gravitational matter, then Equation (2) has the status of a dynamical law of a physical theory. In this case a four-current $j^\alpha = (\rho, \mathbf{j}_i)$ can be defined purely geometrical as

$$j^\alpha = \frac{1}{2}g^{\alpha\beta}\nabla_\beta R \tag{3}$$

If we use the Bianchi identities as field equations for the gravitational field then Einstein field equations, as in the case of the electromagnetic field, can be regarded as a definition for the energy-momentum tensor $T_{\mu\nu}$ for the gravitational field [3]

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\alpha\beta} = \kappa T_{\mu\nu} \tag{4}$$

For a purely gravitational field in which $\frac{1}{2}g^{\alpha\beta}\nabla_\beta R = 0$, the proposed field equations given in Equation (2) also give rise to the same results as those obtained from Einstein formulation of the gravitational field given in Equation (4). For a purely gravitational field, Equation (2) reduces to the equation

$$\nabla_\beta R^{\alpha\beta} = 0 \tag{5}$$

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From Equation (5), we can obtain solutions found from the original Einstein field equations, such as Schwarzschild solution, by observing that since $\nabla_\mu g^{\alpha\beta} \equiv 0$, Equation (5) implies

$$R_{\alpha\beta} = \Lambda g_{\alpha\beta} \tag{6}$$

where Λ is an undetermined constant. Furthermore, the intrinsic geometric Ricci flow that was introduced by Hamilton can also be derived from Equation (5) and given as follows

$$\frac{\partial g_{\alpha\beta}}{\partial t} = \kappa R_{\alpha\beta} \tag{7}$$

where κ is a scaling factor. Mathematically, the Ricci flow is a geometric process that can be employed to smooth out irregularities of a Riemannian manifold [4]. There is an interesting feature that can be derived from the definition of the four-current $j^\alpha = (\rho, \mathbf{j}_i)$ given in Equation (3). By comparing Equation (3) with the Poisson equation for a potential V in classical physics $\nabla^2 V = 4\pi\rho$, we can identify the scalar potential V with the Ricci scalar curvature R and then obtain a diffusion equation

$$\frac{\partial R}{\partial t} = k\nabla^2 R \tag{8}$$

where k is an undetermined dimensional constant. Solutions to Equation (8) can be found to take the form [5]

$$R(x,y,z,t) = \frac{M}{\left(\sqrt{4\pi kt}\right)^3} e^{-\frac{x^2+y^2+z^2}{4kt}} \tag{9}$$

Equation (9) determines the probabilistic distribution of an amount of geometrical substance M which is defined via the Ricci scalar curvature R and manifests as observable matter. It is interesting to note that in fact it is shown that a similar diffusion equation to Equation (8) can also be derived from the Ricci flow given in Equation (7) as follows [6]

$$\frac{\partial R}{\partial t} = \Delta R + 2|\mathrm{Ric}|^2 \tag{10}$$

where Δ is the Laplacian defined as $\Delta = g^{\alpha\beta}\nabla_\alpha\nabla_\beta$ and $|\mathrm{Ric}|$ is a shorthand for a mathematical expression. Therefore, the Bianchi field equations of general relativity in the covariant form given in Equation (2) can be used to formulate quantum particles as differentiable manifolds, in particular 3D differentiable manifolds.

On the other hand, we have also shown that Maxwell field equations of electromagnetism and Dirac relativistic equation of quantum mechanics can be formulated covariantly from a general system of linear first order partial differential equations [7,8,9]. An explicit form of a system of linear first order partial differential equations can be written as follows [10]

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^r \frac{\partial \psi_i}{\partial x_j} = k_1 \sum_{l=1}^{n} b_l^r \psi_l + k_2 c^r, \qquad r = 1,2,\dots,n \tag{11}$$

The system of equations given in Equation (11) can be rewritten in a matrix form as

$$\left(\sum_{i=1}^{n} A_i \frac{\partial}{\partial x_i}\right)\psi = k_1\sigma\psi + k_2 J \tag{12}$$

where $\psi = (\psi_1, \psi_2, \dots, \psi_n)^T$, $\partial\psi/\partial x_i = (\partial\psi_1/\partial x_i, \partial\psi_2/\partial x_i, \dots, \partial\psi_n/\partial x_i)^T$, A_i, σ and J are matrices representing the quantities a_{ij}^k, b_l^r and c^r and k_1 and k_2 are undetermined constants. Now, if we apply the operator $\sum_{i=1}^n A_i \frac{\partial}{\partial x_i}$ on the left on both sides of Equation (12) then we obtain

$$\left(\sum_{i=1}^{n} A_i \frac{\partial}{\partial x_i}\right)\left(\sum_{j=1}^{n} A_j \frac{\partial}{\partial x_j}\right)\psi = \left(\sum_{i=1}^{n} A_i \frac{\partial}{\partial x_i}\right)(k_1\sigma\psi + k_2 J) \tag{13}$$

If we assume further that the coefficients a_{ij}^k and b_l^r are constants and $A_i\sigma = \sigma A_i$, then Equation (13) can be rewritten in the following form

$$\left(\sum_{i=1}^{n} A_i^2 \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{n}\sum_{j>i}^{n}(A_iA_j + A_jA_i)\frac{\partial^2}{\partial x_i \partial x_j}\right)\psi = k_1^2\sigma^2\psi + k_1k_2\sigma J + k_2\sum_{i=1}^{n} A_i \frac{\partial J}{\partial x_i} \tag{14}$$

In order for the above systems of partial differential equations to be used to describe physical phenomena, the matrices A_i must be determined. We have shown that for both Dirac and Maxwell field equations, the matrices A_i must take a form so that Equation (14) reduces to the following equation

$$\left(\sum_{i=1}^{n} {A_i}^2 \frac{\partial^2}{\partial x_i^2}\right)\psi = k_1^2\sigma^2\psi + k_1 k_2 \sigma J + k_2 \sum_{i=1}^{n} A_i \frac{\partial J}{\partial x_i} \tag{15}$$

Therefore, the general system of linear first order partial differential equations can be used to formulate the dynamics of quantum fields that include the electromagnetic field and matter field of quantum particles. In Sections 3 and 5 we will give explicit forms for the matrices A_i for both Dirac and Maxwell field equations and show that these two systems of differential equations can be applied to classify quantum particles as fermions of half-integer spin and bosons of integer spin.

II. On The Dimensionality Of The Spatiotemporal Manifold

In classical physics, in order to formulate the dynamics of natural events that are observable we assume that spacetime is a continuum which consists of three spatial dimensions and one temporal dimension. At the macroscopic scale on which information about physical objects can be established with certainty the assumption seems to be reasonable because it can adequately be used for all dynamical formulations of physical theories. However, at the microscopic scale quantum responses of physical events have revealed that such simple picture of a four-dimensional spacetime continuum is in fact not adequate for physical descriptions, especially those that can be accounted for by observations that can only be set up within our perception of physical existence. This leads to a more fundamental problem in physical investigation of how we can justify the merit of a physical theory. From the perspective of scientific investigation, physical theories can only be evaluated on the subject of the accuracy to experimental results of their mathematical formulations that can be applied into the dynamical description of physical objects. But as far as we are concerned, the setup of a physical experiment is within the limit of three-dimensional domain, therefore, the dimensionality of the spatiotemporal manifold in fact still remains the most fundamental problem that needs to be addressed before any attempt to formulate physical theory can be justified. In our previous works on spacetime structures of quantum particles and geometric interactions we showed that it is possible to formulate quantum particles as three-dimensional differentiable manifolds which have further geometric and topological structure of a CW complex whose decomposed *n*-cells can be associated with physical fields that form the fundamental physical interactions between physical objects [11,12 ,13]. We also showed that it is possible to suggest that spacetime as a whole is a fiber bundle which admits different types of fibers for the same base space of spacetime and what we are able to observe are the dynamics of the fibers but not that of the base space itself [14]. Even though the fiber bundle formulation of the spatiotemporal manifold may provide a more feasible framework to deal with the dynamics of physical existence, the questions about the nature of the base space of the spatiotemporal fiber bundle, whether it can be observable and whether matter are physical entities or they are simply geometric and topological structures of the spacetime manifold still remain unanswered. We may also ask the question of how many dimensions the universe really has then even though the answer to this type of question will depend on our epistemological approach to the physical existence, within our geometric and topological formulation of spacetime we would say that it would depend on what is the highest dimension of the *n*-cells that are decomposed from the spacetime bundle that we can perceive. However, it seems natural that being apparently three-dimensional we perceive the physical existence in three spatial dimensions. It is also natural that due to our perception of the progress of physical events that occur in sequence that we recognise time as one-dimensional. In physics, practically, we describe the physical existence in terms of those that can be observed and measured. In classical physics, what we are observing are physical objects that move in a three-dimensional Euclidean space and the motion occurs in sequence that changes spatial position with respect to time, which itself can be measured by using the displacement of the physical objects. However, in quantum physics, the observation of physical objects itself is a new epistemological problem. The fundamental issue that is related to this epistemological problem of observation is the difficulty in knowing how quantum particles exist. In Einstein theory of special relativity the dimensionality of spacetime is assumed to be that of one-dimensional time and three-dimensional Euclidean space R^3, together they form a four-dimensional spacetime which has the Minkowski mathematical structure of pseudo-Euclidean geometry. This mathematical structure seems to be complete in itself if spacetime is not curved. However, in Einstein theory of general relativity, spacetime is assumed to be curved by matter and energy. As a consequence from the assumption of the Minkowski mathematical structure

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of a four-dimensional spacetime, the mathematical objects that are used to describe physical objects can only be described as two-dimensional manifolds embedded in the three-dimensional Euclidean space R^3. In fact, as shown in Section 3 below, this may be true for the case of massive quantum particles of half-integer spin. On the other hand, in order to describe physical objects that are assumed to possess the mathematical structures of three-dimensional manifolds an extra dimension of space must be used. For example, with Einstein field equations given as $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \quad g_{\mu\nu} = \kappa T_{\mu\nu}$ and the cosmological model that uses the Robertson-Walker metric of the form $ds^2 = c^2dt^2 - S^2(t)\big(dr^2/(1-kr^2) + r^2$, $(d\theta^2 + \sin^2\theta d\phi^2)\big)$, in order to derive the Robertson-Walker metric from Einstein field equations we assume that the quantity $S(t)$ is the radius of a 3-sphere embedded in a four-dimensional Euclidean space R^4. This raises the question of whether this extra spatial dimension is real or just for convenience. Furthermore, we may ask whether there are any other physical formulations of physics that also require an extra dimension of space. This is in fact also the case when we discussed the wave-particle duality in quantum mechanics in which quantum particles can be assumed to possess the geometric and topological structures of a three-dimensional differentiable manifold [15]. As a matter of fact, the anthropic cognition of spacetime with higher dimensions is a subject of scientific investigation and with open-mindedness there is no reason why we should avoid any attempt to formulate a physical theory that requires such perception with reasoning thinking. Even though the CW complex and fiber bundle formulations give a general description of the geometric structures of both quantum particles and the spatiotemporal manifold, the more important question that still remains is how to determine the specific structure of each quantum particle. For example, if quantum particles are considered as three-dimensional differentiable manifolds then it is reasonable to suggest that generally their geometric structures should be classified according to Thurston geometries [16]. However, even with a correct classification of quantum particles according to their intrinsic geometric structures, this type of geometric classification lacks the more important aspects of physical descriptions that are required for a physical theory which encompasses the dynamics and the interactions between them. With the assumption that quantum particles possess the intrinsic geometric structures of a CW complex and each geometric structure manifests a particular type of physical interactions, it is reasonable to assume that there is a close relationship between geometric structures in terms of decomposed n-cells from a CW complex and physical interactions. In general, we may consider physical objects of any scale as differentiable manifolds of dimension n which can emit submanifolds of dimension $m \leq n$ by decomposition. In order to formulate a physical theory we would need to devise a mathematical framework that allows us to account for the amount of subspaces that are emitted or absorbed by a differentiable manifold. This is the evolution of a geometric process that manifests as a physical interaction. We assume that an assembly of cells of a specified dimension will give rise to a certain form of physical interactions and the intermediate particles, which are the force carriers of physical fields decomposed during a geometric evolution, may possess a specified geometric structure, such as that of the n-spheres and the n-tori. Therefore, for observable physical phenomena, the study of physical dynamics reduces to the study of the geometric evolution of differentiable manifolds. In particular, if a physical object is considered to be a three-dimensional manifold then there are four different types of physical interactions that are resulted from the decomposition of 0-cells, 1-cells, 2-cells and 3-cells and these cells can be associated with the corresponding spatial forces $F_n = k_n r^n$ and temporal forces $F_n = h_n t^n$ with $-3 \leq n \leq 3$. In the case of $n = 0$, for a definite perception of a physical existence, we assume that space is occupied by mass points which interact with each other through the decomposition of 0-cells. However, since 0-cells have dimension zero therefore there are only contact forces between the mass points. When the mass points join together through the contact forces they form elementary particles. The 0-cells with contact forces can be arranged to form a particular topological structure [17]. Therefore, we can assume that a general spatiotemporal force which is a combination of the spatial and temporal forces resulted from the decomposition of spatio-temporal n-cells of all dimensions can take the form

$$F = \sum_{n=-3}^{3} (k_n r^n + h_n t^n) \tag{16}$$

where k_n and h_n are constants which can be determined from physical considerations. Using equations of motion from both the spatial and temporal Newton's second laws of motion

$$m\frac{d^2\mathbf{r}}{d\tau^2} = \mathbf{F} \tag{17}$$

$$D\frac{d^2\mathbf{t}}{ds^2} = \mathbf{F} \tag{18}$$

it is seen that a complete geometric structure would be the structure that is resulted from the relationship between space and time that satisfies the most general equation in the form

$$m\frac{d^2\mathbf{r}}{d\tau^2}+D\frac{d^2\mathbf{t}}{ds^2}=\sum_{n=-3}^{3}\left(k_n r^{n-1}\mathbf{r}+h_n t^{n-1}\mathbf{t}\right) \tag{19}$$

The above discussions suggest that the apparent geometric and topological structures of the total spatiotemporal manifold are due to the dynamics and the geometric interactions of the decomposed cells from the base space of the total spatiotemporal manifold, and the decomposed cells form different types of fibers which may also geometrically interact with each other and manifest as physical interactions. In this case we can only perceive the appearance of the intrinsic geometric structures that emerge on the base space of the total spatiotemporal manifold and the base space itself may not be observable with the reasonable assumption that a physical object is not observable if it does not have any form of geometric interactions. It could be that the base space of the spatiotemporal manifold at the beginning was only a six-dimensional Euclidean spatiotemporal continuum R^6 which had no non-trivial geometric structures therefore contained no physical objects. How could physical objects be formed from such a plain spacetime continuum? Even though we could suggest that physical objects could be formed as three-dimensional differentiable manifolds from mass points with contact forces associated with the decomposed 0-cells, it is hard to imagine how they can be formed from a plain continuum without assuming that there must be some form of spontaneous symmetry breaking of the vacuum. Since the apparent spacetime structures are formed by decomposed cells from the base spacetime and since there are many different relationships that arise from the geometric interactions of the decomposed cells of different dimensions, therefore there are different spactime structures each of which can represent a particular spacetime structure and all apparent spacetime structures can be viewed as parallel universes of a multiverse. If we assume that the spatiotemporal manifold is described by a six-dimensional differentiable manifold which is composed of a three-dimensional spatial manifold and a three-dimensional temporal manifold, in which all physical objects are embedded, then the manifold M can be decomposed in the form $M=M\#S_S^3\#S_T^3$, where S_S^3 and S_T^3 are spatial and temporal 3-spheres, respectively. Despite this form of decomposition can be used to describe gravity as a global structure it cannot be used as a medium for any other physical fields which possess a wave character. Therefore we would need to devise different types of decomposition to account for these physical fields that require a local geometric structure. For example, we may assume that *n*-cells can be decomposed from the spatiotemporal manifold at each point of the spatiotemporal continuum. This is equivalent to considering the spatiotemporal manifold as a fiber bundle $E=B\times F$, where B is the base space, which is the spatiotemporal continuum, and the fiber F, which is the *n*-cells. We will discuss in more details in Section 4 the local geometric and topological structure of the spatiotemporal manifold when we discuss the possibility to formulate a medium for the electromagnetic field in terms of geometric structures. From the above discussions on the dimensionality of spacetime it is clear that the observation of natural events needs to be addressed. It seems that due to our physical existence we do not have the ability to observe a complete picture of a physical object. We can only observe part of a physical object due to the fact that it may exist in a higher spatial dimension than ours. For example, if quantum particles exist as three-dimensional differentiable manifold embedded into a four-dimensional Euclidean space then we are unable to observe the physical object as a whole but only the cross-section of it. We can use mathematics to determine the whole structure of the object but we cannot measure what we can calculate. The seemingly strange behaviour of quantum particles may also be caused by bringing over their classical model into the quantum domain. For example, when interacting with a magnetic field an elementary particle shows that it has some form of dynamics that can only be represented by intrinsic angular momentum that is different from the angular momentum encountered in classical physics in which elementary particles are assumed to be simply mass points without any internal geometric structure. In the next section we will show that half-integer values of the intrinsic angular momentum of an elementary particle can be obtained by taking into account its possible internal geometric and topological structures.

III. Quantum Particles With Half Integer Spin

In this section we will discuss a possible physical structure possessed by a quantum particle of half-integer spin that exists in three-dimensional space. If quantum particles are considered as differentiable manifolds then they should have intrinsic geometric structures, therefore, in terms of physical formulations they are composite physical objects. As suggested in Section 2 on the geometric interactions, a composite physical object can be formed from mass points by contact forces associated with the 0-cells decomposed from the CW complex that represents the quantum particle. The intrinsic geometric structure can be subjected to a geometric evolutionary process which manifests as the dynamics of the mass points that form the quantum particle. The manifested physical process may be described as that of a fluid dynamics that can be formulated in terms of a potential, like the Coulomb

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potential of the electrostatic interaction in classical electrodynamics. Also discussed in Section 3 on the dimensionality of spacetime and the observability of quantum particles, physical objects can be observed completely if they can be described by a two-dimensional wave equation in which the solutions of the wave equation gives the description of the geometric structures of the physical object in a third spatial dimension. We now show how they can be obtained from a general two-dimensional wave equation, from two-dimensional Schrödinger wave equation and from Dirac equation in relativistic quantum mechanics. In particular, we will show that the two - dimensional Schrödinger wave equation does describe quantum particles with half-integer spin. Consider a quantum particle whose mass distribution is mainly on a two-dimensional membrane and whose charge is related to the vibration of a homotopy class of 2-spheres in which the charge can be described topologically in terms of surface density. The circular membrane is assumed to be made up of mass points that join together by contact forces which allow vibration. Without vibrating the membrane is a perfect two dimensional physical object, however when it vibrates it becomes a three dimensional physical object described as a two-dimensional manifold embedded in three-dimensional Euclidean space R^3. In this section we discuss the geometric structure of the quantum particle with regard to its distribution of mass and in the next section we will discuss the topological structure with regard to its distribution of charge density in terms of the homotopy fundamental group of surfaces. In this section we assume that a spacetime has three spatial dimensions and one temporal dimension. In general, the wave dynamics of a physical system in a two-dimensional space can be described by a wave equation written in the Cartesian coordinates (x, y) as

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} - \frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial y^2} = 0 \tag{20}$$

In particular, Equation (20) can be used to describe the dynamics of a vibrating membrane in the (x, y) - plane. If the membrane is a circular membrane of radius then the domain D is given as $D = \{x^2 + y^2 < a^2\}$. In the polar coordinates given in terms of the Cartesian coordinates (x, y) as $x = r\cos\theta$, $y = r\sin\theta$, the two-dimensi-as

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} - \frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} - \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} = 0 \tag{21}$$

The general solution to Equation (21) for the vibrating circular membrane with the condition $\psi=0$ on the boundary of can be found as [5]

$$\begin{aligned}\psi(r,\theta,t) = &\sum_{m=1} J_0\left(\sqrt{\lambda_{0m}}r\right)\left(C_{0m}\cos\sqrt{\lambda_{0m}}ct + D_{0m}\sin\sqrt{\lambda_{0m}}ct\right)\\ &+ \sum_{m,n=1} J_n\left(\sqrt{\lambda_{nm}}r\right)(A_{nm}\cos n\theta\\ &+ B_{nm}\sin n\theta)\left(\left(C_{nm}\cos\sqrt{\lambda_{nm}}ct + D_{nm}\sin\sqrt{\lambda_{nm}}ct\right)\right)\end{aligned} \tag{22}$$

where $J_n\left(\sqrt{\lambda_{nm}}r\right)$ is the Bessel function of order n and the quantities A_{nm}, B_{nm}, C_{nm} and D_{nm} can be specified by the initial and boundary conditions. It is also observed that at each moment of time the vibrating membrane appears as a 2D differentiable manifold which is a geometric object whose geometric structure can be constructed using the wavefunction given in Equation (22). We now show that the curvature of the surfaces obtained from the vibrating membrane at each moment of time can also be expressed in terms of the derivatives of the wavefunction given in Equation (22). In differential geometry, the Ricci scalar curvature is shown to be related to the Gaussian curvature K by the relation $R = 2K$, where K is expressed in terms of the principal radii k_1 and k_2 of a surface as $K = 1/k_1k_2$. Consider a surface defined by the relation $z = \psi(x, y)$ in Cartesian coordinates (x, y, z). The Ricci scalar curvature R can be found as [18]

$$R = \frac{2(\psi_{11}\psi_{22} - (\psi_{12})^2}{(1 + \psi_1^2 + \psi_2^2)^2} \tag{23}$$

where $\psi_\mu = \partial\psi/\partial x^\mu$ and $\psi_{\mu\nu} = \partial^2\psi/\partial x^\mu\partial x^\nu$. It is seen that the wavefunction $\psi(r,\theta,t)$ that is obtained from the wave equation given in Equation (22) can be used to determine the Ricci scalar curvature of a surface, which shows that the geometric structure of the vibrating membrane can be described by a classical wavefunction. In other words, wavefunctions that describe the wave motion of a vibrating membrane can be considered as a representation of physical objects. For the benefit of representation in the next section we now give a brief discussion on the geometric formation of quantum particles from a wave equation. We assumed that the circular membrane is made up of particles which are connected with each other by an elastic force. This assumption leads to a more general hypothesis that a vibrating object is made up of mass points that join together by contact forces. When the membrane vibrates it takes different shapes at each moment of time. Each shape is a 2D differentiable manifold that is embedded in the three-dimensional Euclidean space. Now, if we consider the whole vibrating membrane as a particle then its geometric structure is described by the wavefunction ψ. It is a time-dependent hypersurface embedded in a three-dimensional Euclidean space. Now imagine an observer who is a two-dimensional object living in the plane (x, y) and who wants to investigate the geometric structure of the vibrating membrane. Even though he or she would not be able to observe the shapes of the embedded 2D differentiable manifolds in the three-dimensional Euclidean space, he or she would still be able to calculate the value of the wavefunction ψ at each point (x, y) that belongs to the domain $x^2+y^2<a^2$. What would the observer think of the nature of the wavefunction ψ? Does it represent a mathematical object, such as a third dimension, or a physical one, such as fluid pressure? Firstly, because the wavefunction ψ is a solution of a wave equation therefore it must be a wave. Secondly, if the observer who is a 2D physical object and who does not believe in higher dimensions then he or she would conclude that the wavefunction ψ should only be used to describe events of physical existence other than space and time. In the next section we will show that this situation may in fact be that of the wave-particle duality that we are encountering in quantum physics when our view of the physical existence is restricted to that of a 3D observer. It is also observed that according to the 2D observer who is living on the (x, y)-plane, the vibrating membrane appears as an oscillating motion of a single string. If the vibrating string is set in motion in space then it can be seen as a particle. With a suitable experimental setup, the moving vibrating membrane may be detected as a wave. And furthermore, it can also generate a physical wave if the space is a medium. In fact, as shown in the following, a two-dimensional wave equation can be applied into quantum mechanics to describe the dynamics of a quantum system which is restricted to a two-dimensional space. This can be formulated either by the Schrödinger non-relativistic wave equation or Dirac relativistic wave equation. However, in order to obtain a classical picture of a quantum particle in two-dimensional space, let consider the classical dynamics of a particle moving in two spatial dimensions. In classical mechanics, expressed in plane polar coordinates, the Lagrangian of a particle of mass m under the influence of a conservative force with potential $V(r)$ is given as follows [19]

$$L=\frac{1}{2}m\left(\left(\frac{dr}{dt}\right)^2+r^2\left(\frac{d\theta}{dt}\right)^2\right)-V(r) \tag{24}$$

With the Lagrangian given in Equation (24), the canonical momentum p_θ is found as

$$p_\theta=\frac{\partial L}{\partial(d\theta/dt)}=mr^2\frac{d\theta}{dt} \tag{25}$$

The canonical momentum given in Equation (25) is the angular momentum of the system. By applying the Lagrange equation of motion

$$\frac{d}{dt}\frac{\partial L}{\partial(dq_i/dt)}-\frac{\partial L}{\partial q_i}=0,\ \ i=1,2,\dots,n \tag{26}$$

where q_i are the generalised coordinates, we obtain

$$\frac{dp_\theta}{dt}=\frac{d}{dt}\left(mr^2\frac{d\theta}{dt}\right)=0 \tag{27}$$

The areal velocity $d\mathbf{A}/dt$, which is the area swept out by the position vector of the particle per unit time, is found as

$$\frac{d\mathbf{A}}{dt}=\frac{\mathbf{r}(t)\times\mathbf{v}(t)}{2} \tag{28}$$

On the other hand, in classical dynamics, the angular momentum of the particle is defined by the relation

$$\mathbf{L}=\mathbf{r}(t)\times m\mathbf{v}(t) \tag{29}$$

From Equations (28) and (29), we obtain the following relationship between the angular momentum $\mathbf{L}$ of a particle and the areal velocity $d\mathbf{A}/dt$

$$\mathbf{L}=2m\frac{d\mathbf{A}}{dt} \tag{30}$$

It is seen from these results that the use of conservation of angular momentum for the description of the dynamics of a particle can be replaced by the conservation of areal velocity. For example, consider the circular motion of a particle under an inverse square field $F=kq^2/r^2$. Applying Newton's second law, we obtain

$$\frac{mv^2}{r}=\frac{kq^2}{r^2} \tag{31}$$

Using Equations (30) and (31) and the relation $L=mrv$, we obtain

$$r=\frac{4m}{kq^2}\left(\frac{dA}{dt}\right)^2 \tag{32}$$

The total energy E of the particle is

$$E = \frac{1}{2}mv^2 - \frac{kq^2}{r} = -\frac{kq^2}{2r} \quad (33)$$

Using Equations (32) and (33), the total energy can be rewritten as

$$E = -\frac{k^2q^4}{8m\left(\frac{dA}{dt}\right)^2} \quad (34)$$

It is seen from Equation (34) that the total energy of the particle depends on the rate of change of the area dA/dt. In the case of Bohr model of a hydrogen-like atom, from the quantisation condition $mrv = n/2\pi$, we have

$$\frac{dA}{dt} = n\left(\frac{}{4\pi m}\right) \quad (35)$$

Equation (35) shows that the rate of change of the area swept out by the electron is quantised in unit of $/4\pi m$. The two-dimensional Bohr model of a hydrogen atom has a classical configuration that provides a clear picture of the motion of the electron around a nucleus. As shown in our work on the quantization of angular momentum, the Schrödinger wave mechanics when applied to the two-dimensional model of the hydrogen atom also predicts that an intrinsic angular momentum of the electron must take half-integral values for the Bohr spectrum of energy to be retained [20]. Using the two-dimensional model of the hydrogen atom, in the following we will describe an elementary particle of half-integer spin as a differentiable manifold whose physical configuration is similar to that of a rotating membrane whose dynamics can be described in terms of the two-dimensional motion using the Schrödinger wave mechanics and Dirac relativistic quantum mechanics. First, if elementary particles are assumed to possess an internal structure that has the topological structure of a rotating membrane then it is possible to apply the Schrödinger wave equation to show that they can have spin of half-integral values. Consider an elementary particle whose physical arrangement can be viewed as a planar system whose configuration space is multiply connected. Since the system is invariant under rotations therefore we can invoke the Schrödinger wave equation for an analysis of the dynamics of a rotating membrane. In wave mechanics the time-independent Schrödinger wave equation is given as [21]

$$-\frac{^2}{2\mu}\nabla^2\psi(r) - V(r)\psi(r) = E\psi(r) \quad (36)$$

If we also assume that the overall potential $V(r)$ that holds the membrane together has the form $V(r) = A/r$, where A is a physical constant that is needed to be determined, then using the planar polar coordinates in two-dimensional space, the Schrödinger wave equation takes the form [22]

$$-\frac{^2}{2\mu}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}\right]\psi(r,\phi) - \frac{A}{r}\psi(r,\phi) = E\psi(r,\phi) \quad (37)$$

Solutions of the form $R(r)\Phi(\phi) = R(r)\Phi(\phi)$ reduce Equation (37) to two separate equations for the functions $\Phi(\phi)$ and $R(r)$ as follows

$$\frac{d^2\Phi}{d\phi^2} + m^2\Phi = 0 \quad (38)$$

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - \frac{m^2}{r^2}R + \frac{2\mu}{^2}\left(\frac{A}{r} + E\right)R = 0 \quad (39)$$

where m is identified as the intrinsic angular momentum of the membrane. Equation (38) has solutions of the form

$$\Phi(\phi) = \exp(im\phi) \quad (40)$$

Normally, the intrinsic angular momentum must take integral values for the single-valuedness condition to be satisfied. However, if we consider the configuration space of the membrane to be multiply connected and the polar coordinates have singularity at the origin then the use of multivalued wavefunctions is allowable. As shown below, in this case, the intrinsic angular momentum m can take half-integral values. If we define, for the case $E < 0$,

$$\rho = \left(\frac{8\mu(-E)}{^2}\right)^{1/2} r, \quad \lambda = \left(\frac{A\mu}{2\ ^2(-E)}\right)^{1/2} \quad (41)$$

then Equation (39) can be re-written in the following form

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} - \frac{m^2}{\rho^2}R + \frac{\lambda}{\rho}R - \frac{1}{4}R = 0 \quad (42)$$

If we seek solutions for $R(\rho)$ in the form $R(\rho) = \exp(-\rho/2)\rho^m S(\rho)$ then we obtain the following differential equation for the function $S(\rho)$

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$$\frac{d^2S}{d\rho^2}+\left(\frac{2m+1}{\rho}-1\right)\frac{dS}{d\rho}+\left(\frac{\lambda-m-\frac{1}{2}}{\rho}\right)S=0 \tag{43}$$

Equation (43) can be solved by a series expansion of $S(\rho)$ as , $S(\rho)=\sum_{n=0}a_n\rho^n$, with the coefficients a_n satisfying the recursion relation

$$a_{n+1}=\frac{n+m+\frac{1}{2}-\lambda}{(n+1)(n+2m+1)}a_n \tag{44}$$

Then the energy spectrum can be written explicitly in the form

$$E=\frac{A^2\mu}{2\ ^2\left(n+m+\frac{1}{2}\right)^2} \tag{45}$$

It is seen that if the result given in Equation (45) can also be applied to elementary particles which are assumed to behave like a hydrogen-like atom, which is viewed as a two-dimensional physical system, then the intrinsic angular momentum m must take half-integral values.

Now, we show that the wave equation for two-dimensional space given in Equation (20) can also be derived from Dirac equation that describes a quantum particle of half-integer spin. In our previous works [7,8,9], we have shown that both Dirac equation and Maxwell field equations can be formulated from a system of linear first order partial differential equations. Except for the dimensions that involve with the field equations, the formulations of Dirac and Maxwell field equations are remarkably similar and a prominent feature that arises from the formulations is that the equations are formed so that the components of the wavefunctions satisfy a wave equation. However, there are essential differences between the physical interpretations of Dirac and Maxwell physical fields. On the one hand, Maxwell electromagnetic field is a classical field which is composed of two different fields that have different physical properties even though they can be converted into each other. On the other hand, despite Dirac field was originally formulated to describe the dynamics of a single particle, such as the electron, it turned out that a solution to Dirac equation describes not only the dynamics of the electron with positive energy but it also describes the dynamics of the same electron with negative energy. The difficulty that is related to the negative energy can be resolved if the negative energy solutions can be identified as positive energy solutions that can be used to describe the dynamics of a positron. The seemingly confusing situation suggests that Dirac field of massive particles may actually be composed of two physical fields, similar to the case of the electromagnetic field which is composed of the electric field and the magnetic field. Dirac equation can be derived from Equation (12) by imposing the following conditions on the matrices A_i

$$A_i^2=\pm 1 \tag{46}$$

$$A_iA_j+A_jA_i=0 \quad \text{for} \quad i\neq j \tag{47}$$

For the case of $n=4$, the matrices A_i can be shown to take the form

$$A_1=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}\quad A_2=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&-1&0&0\\-1&0&0&0\end{pmatrix}\quad A_3=\begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&i&0&0\\-i&0&0&0\end{pmatrix}\quad A_4=\begin{pmatrix}0&0&1&0\\0&0&0&-1\\-1&0&0&0\\0&1&0&0\end{pmatrix} \tag{48}$$

With , $k_1=m$, $\sigma=1$ and $k_2=0$, the system of linear first order partial differential equations given in Equation (12) reduces to Dirac equation [23]

$$-\frac{\partial\psi_1}{\partial t}-im\psi_1=\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)\psi_4+\frac{\partial\psi_3}{\partial z} \tag{49}$$

$$-\frac{\partial\psi_2}{\partial t}-im\psi_2=\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)\psi_3-\frac{\partial\psi_4}{\partial z} \tag{50}$$

$$\frac{\partial\psi_3}{\partial t}-im\psi_3=\left(-\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)\psi_2-\frac{\partial\psi_1}{\partial z} \tag{51}$$

$$\frac{\partial\psi_4}{\partial t}-im\psi_4=\left(-\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)\psi_1+\frac{\partial\psi_2}{\partial z} \tag{52}$$

With the form of the field equations given in Equations (49-52), we may interpret that the change of the field (ψ_1,ψ_2) with respect to time generates the field (ψ_3,ψ_4), similar to the case of Maxwell field equations in which the change of the electric field generates the magnetic field. With this observation it may be suggested that, like the Maxwell electromagnetic field which is composed of two essentially different physical fields, the Dirac field of massive particles may also be viewed as being composed of two different physical fields, namely the field (ψ_1,ψ_2), which plays the role of the electric field in Maxwell field equations, and the field

(ψ_1, ψ_2), which plays the role of the magnetic field. The similarity between Maxwell field equations and Dirac field equations can be carried further by showing that it is possible to reformulate Dirac equation as a system of real equations. When we formulate Maxwell field equations from a system of linear first order partial differential equations we rewrite the original Maxwell field equations from a vector form to a system of first order partial differential equations by equating the corresponding terms of the vectorial equations. Now, since, in principle, a complex quantity is equivalent to a vector quantity therefore in order to form a system of real equations from Dirac complex field equations we equate the real parts with the real parts and the imaginary parts with the imaginary parts. In this case Dirac equation given in Equations (49-52) can be rewritten as a system of real equations as follows

$$\frac{\partial\psi_1}{\partial t} + \frac{\partial\psi_4}{\partial x} + \frac{\partial\psi_3}{\partial z} = 0 \tag{53}$$

$$\frac{\partial\psi_2}{\partial t} + \frac{\partial\psi_3}{\partial x} - \frac{\partial\psi_4}{\partial z} = 0 \tag{54}$$

$$\frac{\partial\psi_3}{\partial t} + \frac{\partial\psi_2}{\partial x} + \frac{\partial\psi_1}{\partial z} = 0 \tag{55}$$

$$\frac{\partial\psi_4}{\partial t} + \frac{\partial\psi_1}{\partial x} - \frac{\partial\psi_2}{\partial z} = 0 \tag{56}$$

$$\frac{\partial\psi_4}{\partial y} = m\psi_1 \tag{57}$$

$$-\frac{\partial\psi_3}{\partial y} = m\psi_2 \tag{58}$$

$$-\frac{\partial\psi_2}{\partial y} = m\psi_3 \tag{59}$$

$$\frac{\partial\psi_1}{\partial y} = m\psi_4 \tag{60}$$

The system of Dirac field equation given in Equations (53-60) can be considered as a particular case of a more general system of field equations written in the matrix form

$$\left(A_1\frac{\partial}{\partial t} + A_2\frac{\partial}{\partial x} + A_3\frac{\partial}{\partial y} + A_4\frac{\partial}{\partial z}\right)\psi = m\psi \tag{61}$$

where $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T$ and the real matrices A_i are given as

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad A_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad A_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \tag{62}$$

The matrices A_i satisfy the following commutation relations

$$A_i^2 = 1 \qquad \text{for} \qquad i = 1, 2, 3, 4 \tag{63}$$

$$A_1A_i + A_iA_1 = 2A_i \qquad \text{for} \qquad i = 2, 3, 4 \tag{64}$$

$$A_2A_3 + A_3A_2 = 2\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{65}$$

$$A_2A_4 + A_4A_2 = 0 \tag{66}$$

$$A_3A_4 + A_4A_3 = 2\begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{67}$$

By applying $\left(A_1\frac{\partial}{\partial t}+A_2\frac{\partial}{\partial x}+A_3\frac{\partial}{\partial y}+A_4\frac{\partial}{\partial z}\right)$ to Equation (61) and using the commutation relations given in Equations (63-67), then it can be shown that all components of the wavefunction $\psi=(\psi_1,\psi_2,\psi_3,\psi_4)^T$ satisfy the following equation

$$\frac{\partial^2\psi_i}{\partial t^2}-\frac{\partial^2\psi_i}{\partial x^2}+\frac{\partial^2\psi_i}{\partial y^2}-\frac{\partial^2\psi_i}{\partial z^2}=m^2\psi_i \quad \text{for} \quad i=1,2,3,4 \tag{68}$$

If the wavefunction ψ satisfies Dirac field equations given in Equations (53-60) then we obtain the following system of equations for all components

$$\frac{\partial^2\psi_i}{\partial^2 y}-m^2\psi_i=0 \tag{69}$$

$$\frac{\partial^2\psi_i}{\partial t^2}-\frac{\partial^2\psi_i}{\partial x^2}-\frac{\partial^2\psi_i}{\partial z^2}=0 \tag{70}$$

Solutions to Equation (69) are

$$\psi_i=c_1(x,z)e^{my}+c_2(x,z)e^{-my} \tag{71}$$

where c_1 and c_2 are undetermined functions of (x,z), which may be assumed to be constant. The solutions given in Equation (71) give a distribution of a physical quantity, such as the mass of a quantum particle, along the *y*-axis. On the other hand, Equation (70) can be used to describe the dynamics, for example, of a vibrating membrane in the (x,z)-plane. Solutions to Equation (70) can also be found in the form given in Equation (20). Even though elementary particles may have the geometric and topological structures of a 3D differentiable manifold, it is seen from the above descriptions via the Schrödinger wave equation and Dirac equation that they appear as 3D physical objects that embedded in three-dimensional Euclidean space. In Section 5 we will show that this may not be the case for elementary particles of integer spin, such as photons. However, in the next section we will show that the appearance of elementary particles of half-integer spin as 3D physical objects can be justified further by considering other physical properties that are associated with them, such as charge and magnetic monopole.

IV. On The Electric Charge And Magnetic Monopole

In Section 3 we show that massive quantum particles of half-integer spin can be described as 2D differentiable manifolds which are endowed with the geometric and topological structure similar to that of a gyroscope whose main component is a rotating and vibrating membrane that can be described by the solutions of a two-dimensional wave equation, in particular a 2D wave equation that is derived from the Dirac equation of relativistic quantum mechanics. However, the dynamics of the quantum particle is associated only with the distribution of mass of the particle but not other equally important physical matter, such as charge and magnetic monopole. In this section we will discuss further these physical properties of a quantum particle and show that they may be associated with the topological structure of the particle rather than physical quantities that form or are contained inside the particle. As shown in our works on the principle of least action and spacetime structures of quantum particles, the charge of a physical system may depend on the topological structure of the system and is classified by the homotopy group of closed surfaces [24]. In quantum mechanics, the Feynman's method of sum over random paths can be extended to higher-dimensional spaces to formulate physical theories in which the transition amplitude between states of a quantum mechanical system is the sum over random hypersurfaces [25]. This generalisation of the path integral method in quantum mechanics has been developed and applied to other areas of physics, such as condensed matter physics, quantum field theories and quantum gravity theories, mainly for the purpose of field quantisation. In the following, however, we focus attention on the general idea of a sum over random surfaces. This formulation is based on surface integral methods by generalising the differential formulation as discussed for the Bohr's model of a hydrogen-like atom. Consider a surface in R^3 defined by the relation $x^3=f(x^1,x^2)$. The Gaussian curvature K is given by the relation $K=(f_{11}f_{22}-(f_{12})^2)/(1+f_1^2+f_2^2)^2$, where $f_\mu=\partial f/\partial x^\mu$ and $f_{\mu\nu}=\partial^2 f/\partial x^\mu\partial x^\nu$, [18]. Let P be a three-dimensional physical quantity which plays the role of the momentum p in the two-dimensional space action integral. The quantity P can be identified with the surface density of a physical quantity, such as charge. Since the momentum p is proportional to the curvature κ, which determines the planar path of a particle, it is seen that in the three-dimensional space the quantity P should be proportional to the Gaussian curvature K, which is used to characterise a surface. If we consider a surface action integral of the form $S=\int PdA=\int(q_e/2\pi)KdA$, where q_e is a universal constant, which plays the role of Planck's constant, then we have

$$S=\frac{q_e}{2\pi}\int\frac{f_{11}f_{22}-(f_{12})^2}{(1+f_1^2+f_2^2)^{3/2}}dx^1dx^2 \tag{72}$$

According to the calculus of variations, similar to the case of path integral, to extremise the action integral $S = \int L(f, f_\mu, f_{\mu\nu}, x^\mu) dx^1 dx^2$, the functional $L(f, f_\mu, f_{\mu\nu}, x^\mu)$ must satisfy the differential equations [26]

$$\frac{\partial L}{\partial f} - \frac{\partial}{\partial x^\mu}\frac{\partial L}{\partial f_\mu} + \frac{\partial^2}{\partial x^\mu \partial x^\nu}\frac{\partial L}{\partial f_{\mu\nu}} = 0 \qquad (73)$$

It can be verified that with the functional of the form given in Equation (72) the differential equation given by Equation (73) is satisfied by any surface. Hence, we can generalise Feynman's postulate of random path to formulate a quantum theory in which the transition amplitude between states of a quantum mechanical system is a sum over random surfaces, provided the functional P in the action integral $S = \int PdA$ is taken to be proportional to the Gaussian curvature K of a surface. Consider a closed surface and assume that we have many such different surfaces which are described by the higher dimensional homotopy groups. As in the case of the fundamental homotopy group of paths, we choose from among the homotopy class a representative spherical surface, in which case we can write

$$\oint PdA = \frac{q_e}{4\pi}\oint d\Omega, \qquad (74)$$

where $d\Omega$ is an element of solid angle. Since $\oint d\Omega$ depends on the homotopy class of the spheres that it represents, we have $\oint d\Omega = 4\pi n$ where n is the topological winding number of the higher dimensional homotopy group. From this result we obtain a generalised Bohr quantum condition

$$\oint PdA = nq_e \qquad (75)$$

From the result obtained in Equation (75), as in the case of Bohr's theory of quantum mechanics, we may consider a quantum process in which a physical entity transits from one surface to another with some radiation-like quantum created in the process. Since this kind of physical process can be considered as a transition from one homotopy class to another, the radiation-like quantum may be the result of a change of the topological structure of the physical system, and so it can be regarded as a topological effect. Furthermore, it is interesting to note that the action integral $(q_e/4\pi)\oint KdA$ is identical to Gauss's law in electrodynamics [27]. In this case the constant q can be identified with the charge of a particle, which represents the topological structure of a physical system and the charge of a physical system must exist in multiples of q_e. Hence, the charge of a physical system may depend on the topological structure of the system and is classified by the homotopy group of closed surfaces. This result may shed some light on why charge is quantised even in classical physics. As a further remark, we want to mention here that in differential geometry, the Gaussian is related to the Ricci scalar curvature R by the relation $R = 2K$. And it has been shown that the Ricci scalar curvature can be identified with the potential of a physical system, therefore our assumption of the existence of a relationship between the Gaussian curvature and the surface density of a physical quantity can be justified [1]. Now, in order to establish a relationship between the electric charge q_e and the magnetic monopole associated with a quantum particle, similar to Dirac relation $\hbar c/q_e q_m = 2$, we need to extend Feynman's method of sum over random surfaces to temporal dynamics in which the magnetic monopole can also be considered as a topological structure of a temporal continuum. Even though the following results are similar to those obtained for the spatial Euclidean continuum, for clarity, we will give an abbreviated version by first defining a temporal Gaussian curvature in the temporal Euclidean continuum R^3 and then deriving a quantised magnetic charge from Feynman integral method. As in spatial dimensions, we consider a temporal surface defined by the relation $t^3 = f(t^1, t^2)$. Then, as shown in differential geometry, the temporal Gaussian curvature denoted by K_T can be determined by $f(t^1, t^2)$ and given as $K_T = (f_{11}f_{22} - (f_{12})^2)/(1+f_1^2+f_2^2)^2$, where $f_\mu = \partial f/\partial t^\mu$ and $f_{\mu\nu} = \partial^2 f/\partial t^\mu \partial t^\nu$. Let P_T be a 3-dimensional physical quantity which will be identified with the surface density of a magnetic substance, such as magnetic charge of an elementary particle. We therefore assume that an elementary particle is assigned not only with an electric charge q_e but also a magnetic charge q_m. We further assume that the quantity P_T is proportional to the temporal Gaussian curvature K_T. Now, as in the case with spatial dimensions, if we consider a surface action integral of the form $S = \int P_T dA_T = \int (q_m/2\pi) K_T dA_T$, then we have

$$S = \frac{q_m}{2\pi}\int \frac{f_{11}f_{22} - (f_{12})^2}{(1 + f_1^2 + f_2^2)^{3/2}} dt^1 dt^2 \qquad (76)$$

Similar to the case of the spatial integral, to extremise the action integral given in Equation (76), the functional $L(f, f_\mu, f_{\mu\nu}, t^\mu)$ must satisfy the differential equation given in Equation (73). Hence, we can also generalise Feynman's postulate of random surfaces to formulate a quantum theory in which the transition amplitude between states of a quantum mechanical system is a sum over random surfaces, provided the functional P_T in the action integral $S = \int P_T dA_T$ is taken to be proportional to the temporal Gaussian curvature K_T of a temporal surface. Similar to the random spatial surfaces, we obtain the following result

$$\oint P_T dA_T = \frac{q_m}{4\pi}\oint d\Omega = n_T q_m \tag{77}$$

The action integral $(q_m/4\pi)\oint K_T dA_T$ is similar to Gauss's law in electrodynamics. In this case the constant q_m can be identified with the magnetic charge of a particle. In particular, the magnetic charge q_m represents the topological structure of a physical system must exist in multiples of q_m Hence, the magnetic charge of a physical system, such as an elementary particle, may depend on the topological structure of the system and is classified by the homotopy group of closed surfaces. We are now in the position to show that it is possible to obtain the relationship between the electric charge q_e and the magnetic charge q_m derived by Dirac by considering a spatiotemporal Gaussian curvature K which is defined as a product of the temporal Gaussian curvature K_T and the spatial Gaussian curvature K_S as follows

$$K = K_T \times K_S \tag{78}$$

The spatiotemporal submanifold that gives rise to this form of curvature is homeomorphic to $S^2 \times S^2$ If K_T and K_S are independent from each other then we can write

$$\oint K\,dA = \oint K_T \times K_S\, dA_T dA_S = \oint K_T\, dA_T \times \oint K_S\, dA_S \tag{79}$$

If we assume further that $\oint K\,dA = k$, where k is an undetermined constant, then using the results $\oint K_S\, dA_S = \oint (q_e/2\pi) K_S\, dA_S$ and $\oint K_T\, dA_T = \oint (q_m/2\pi) K_T\, dA_T$, we obtain a general relationship between the electric charge q_e and the magnetic charge q_m

$$\frac{k}{q_e q_m} = n_S n_T \tag{80}$$

In particular, if $n_S = 1$, $n_T = 2$ and $k = \hbar c$, or $n_S = 2$ $n_T = 1$ and $k = \hbar c$, then we recover the relationship obtained by Dirac, $\hbar c/q_e q_m = 2$.

In the classical electromagnetic field, Maxwell field equations describe a conversion between the electric and magnetic field. If the electric field is associated with the electric charge, which is in turn associated with the spatial continuum, and the magnetic field with the magnetic charge, which is in turn associated with the temporal continuum, then we may speculate that the electromagnetic field is a manifestation of a conversion between the spatial and temporal manifolds. In the following we show that if we consider the spatiotemporal manifold as a spherical fiber bundle then it is possible to describe the electromagnetic field as a wave through a medium of fibers that are composed of 3-spheres [14, 28]. In classical physics, the formation of a wave requires a medium which is a collection of physical objects therefore with this classical picture in mind we may assume that the medium for the electromagnetic and matter waves is composed of quantum particles which have the geometric and topological structures of spatiotemporal *n*-cells that are decomposed from the spatiotemporal manifold at each point of the spatiotemporal continuum. This is equivalent to considering the spatiotemporal manifold as a fiber bundle $E = B \times F$, where B is the base space, which is the spatiotemporal continuum, and the fiber F, which is the *n*-cells. In the following we will only consider an *n*-cell as an *n*-sphere S^n and the total spatiotemporal manifold M will be regarded as an *n*-sphere bundle. It is reasonable to suggest that there may exist physical fields that are associated with different dimensions of the *n*-spheres, however, as an illustration, we will consider only the case with $n=6$ so that S^6 is homeomorphic to $S_S^3 \times S_T^3$, hence the medium of the electromagnetic and matter waves will be assumed to be composed of $S_S^3 \times S_T^3$ cells at each point of the spatiotemporal manifold. In other words, the 6-sphere fibers form the required medium for the electromagnetic and matter waves. Consequently, the problem that we want to address reduces to the problem of the conversion between the spatial and temporal manifolds S_S^3 and S_T^3. It is expected that the formulation of such conversion should be derived from a general line element $ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta$. As examples, we will show in the following that the conversion between the spatial and temporal manifolds S_S^3 and S_T^3 can be described by assuming the general line element to take the form of either a centrally symmetric metric or the Robertson-Walker metric [29]. A general six-dimensional centrally symmetric metric can be written as

$$ds^2 = e^{\psi} c^2 dt^2 + c^2 t^2\left(d\theta_T{}^2 + \sin^2\theta_T d\phi_T{}^2\right) - e^{\chi} dr^2 - r^2\left(d\theta_S{}^2 + \sin^2\theta_S d\phi_S{}^2\right) \tag{81}$$

If we rearrange the (θ, ϕ) directions of both the spatial and the temporal cells so that they coincide $\theta_S = \theta_T = \theta$, and $\phi_S = \phi_T = \phi$ then we have

$$ds^2 = e^{\psi} c^2 dt^2 - e^{\chi} dr^2 - (r^2 - c^2 t^2)(d\theta^2 + \sin^2\theta d\phi^2) \tag{82}$$

There are profound differences in the structure of the spatiotemporal manifold that arise from the line element given in Equation (82). The line element in Equation (82) can be rewritten in the form

$$ds^2 = e^{\psi}c^2dt^2 - e^{\chi}dr^2 - r^2\left(1 - \frac{c^2}{v^2}\right)(d\theta^2 + \sin^2\theta d\phi^2) \quad (83)$$

where we have defined the new quantity that has the dimension of speed as $v = r/t$. It is seen that if $v > c$ then the line element given in Equation (83) can lead to the conventional structure of spacetime in which, effectively, space has three dimensions and time has one dimension, and that if $v < c$ then the line element given in Equation (83) can lead to the conventional structure of spacetime in which time has three dimensions and space has one. However, for the purpose of discussing a conversion between the temporal manifold and the spatial manifold of spacetime we would need to consider possible relationship between space and time and how they change with respect to each other continuously. In order to fulfil this task we need to utilise the results obtained in our works on geometric interactions that show that there are various forces associated with the decomposed *n*-cells from which, by applying Newton's law of dynamics, different possible relationships between space and time could be derived [12,13]. For example, by applying the temporal Newton's second law for radial motion to the force that is associated with decomposed 1-cells we obtain

$$D\frac{d^2t}{dr^2} = {}_1t \quad (84)$$

General solutions to the equation given in Equation (84) are

$$t = c_1e^{\sqrt{h_1/D}r} + c_2e^{-\sqrt{h_1/D}r} \quad (85)$$

If $D = -m$ and $h_1 > 0$ then the following solution can be obtained

$$t = A\sin(\omega r) \quad (86)$$

where $\omega = \sqrt{h_1/D}$. By differentiation we have

$$\frac{dt}{dr} = A\omega\cos(\omega r) \quad (87)$$

If we assume a linear approximation between space and time for the values of $v \sim c$, i.e., $dr/dt \sim r/t = v$ then Equation (83) becomes

$$ds^2 = e^{\psi}c^2dt^2 - e^{\chi}dr^2 - r^2\left(1 - c^2A^2\omega^2\cos^2(\omega r)\right)(d\theta^2 + \sin^2\theta d\phi^2) \quad (88)$$

It is seen from Equation (88) that if $1 - c^2A^2\omega^2\cos^2(\omega r) > 0$ then effectively spacetime appears as a spatial manifold in which there are three spatial dimensions and one temporal dimension. Therefore it is expected that for $1 - c^2A^2\omega^2\cos^2(\omega r) < 0$ spacetime would appear as a temporal manifold. This is in fact the case as can be shown as follows. Instead of the metric form given in Equation (83), the line element given in Equation (82) can also be re-written in a different form as follows

$$ds^2 = e^{\psi}c^2dt^2 - e^{\chi}dr^2 + c^2t^2\left(1 - \frac{v^2}{c^2}\right)(d\theta^2 + \sin^2\theta d\phi^2) \quad (89)$$

Using Equation (87) we obtain

$$ds^2 = e^{\psi}c^2dt^2 + c^2t^2\left(1 - \frac{1}{c^2A^2\omega^2\cos^2(\omega r)}\right)(d\theta^2 + \sin^2\theta d\phi^2) - e^{\chi}dr^2 \quad (90)$$

Therefore, if the condition $1 - c^2A^2\omega^2\cos^2(\omega r) < 0$ is satisfied then Equation (90) is reduced to a line element for the spatiotemporal manifold which effectively has three temporal dimensions and one spatial dimension.

For the case $r^2 - c^2t^2 \neq 0$ the line element given in Equation (82) can be determined by applying Einstein field equations of general relativity

$$ds^2 = \left(1 - \frac{2GM}{c^2r}\right)c^2dt^2 - \left(1 - \frac{2GM}{c^2r}\right)^{-1}dr^2 - (r^2 - c^2t^2)(d\theta^2 + \sin^2\theta d\phi^2) \quad (91)$$

It should also be mentioned here that for the case $r^2 - c^2t^2 = 0$, the line element given in Equation (82) reduces to the simple form

$$ds^2 = e^{\psi}c^2dt^2 - e^{\chi}dr^2 \tag{92}$$

and as discussed in our previous works that spacetime that is endowed with this particular metric appears to behave as a wave where the functions ψ and χ satisfy the wave equation

$$\frac{\partial^2\psi}{c^2\partial t^2} - \frac{\partial^2\psi}{\partial r^2} = 0 \tag{93}$$

We can also obtain a conversion between the spatial and temporal manifolds similar to those that have been discussed above if we use the spatial Newton's second law instead. In this case the following results can be obtained

$$m\frac{d^2r}{dt^2} = k_1 r \tag{94}$$

$$r = c_1 e^{\sqrt{k_1/m}t} + c_2 e^{-\sqrt{k_1/m}t} \tag{95}$$

If we consider the case $m > 0$ and $k_1 < 0$ then we can obtain a simple solution

$$r = A\sin(\omega t) \tag{96}$$

where $\omega = \sqrt{-k_1/m}$. By differentiation, we obtain

$$\frac{dr}{dt} = A\omega\cos(\omega t) \tag{97}$$

If we also assume a linear approximation between space and time for the values of $v \sim c$, i.e., $dr/dt \sim r/t = v$ then then the line elements become

$$ds^2 = e^{\psi}c^2dt^2 - e^{\chi}dr^2 - r^2\left(1 - \frac{c^2}{A^2\omega^2\cos^2(\omega t)}\right)(d\theta^2 + \sin^2\theta d\phi^2) \tag{98}$$

$$ds^2 = e^{\psi}c^2dt^2 - e^{\chi}dr^2 + c^2t^2\left(1 - \frac{A^2\omega^2\cos^2(\omega t)}{c^2}\right)(d\theta^2 + \sin^2\theta d\phi^2) \tag{99}$$

It is seen from Equations (98) and (99) that there is also a conversion between the spatial and temporal submanifolds of the 6-spherical cells that are decomposed from the total spatiotemporal manifold.

Now, we consider the case when the decomposed $S_S^3 \times S_T^3$ cells from the spatiotemporal manifold are furnished with the Robertson-Walker metric. In the spatiotemporal manifold which has three spatial dimensions and one temporal dimension, the Robertson-Walker metric is given as

$$ds^2 = c^2dt^2 - S^2(t)\left(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right) \tag{100}$$

With the decomposition of $S_S^3 \times S_T^3$ cells from the spatiotemporal manifold which has the mathematical structure of an *n*-sphere bundle, the Robertson-Walker metric is assumed to be extended to a six-dimensional line element of the form

$$ds^2 = S^2(r)\left(\frac{dt^2}{1-k_T t^2} + t^2\left(d\theta_T{}^2 + \sin^2\theta_T d\phi_T{}^2\right)\right)$$

$$- S^2(t)\left(\frac{dr^2}{1-k_S r^2} + r^2\left(d\theta_S{}^2 + \sin^2\theta_S d\phi_S{}^2\right)\right) \tag{101}$$

If we also arrange the (θ, ϕ) directions of both spatial and the temporal manifolds so that $\theta_S = \theta_T = \theta$ and $\phi_S = \phi_T = \phi$ then the general space-time metric given in Equation (101) becomes

$$ds^2 = \frac{S^2(r)dt^2}{1-k_T t^2} - \frac{S^2(t)dr^2}{1-k_S r^2} - \left(r^2S^2(t) - t^2S^2(r)\right)(d\theta^2 + \sin^2\theta d\phi^2) \quad (102)$$

Equation (102) can be rewritten in the following form

$$ds^2 = \frac{S^2(r)dt^2}{1-k_T t^2} - \frac{S^2(t)dr^2}{1-k_S r^2} - \left(S^2(t) - \frac{1}{v^2}S^2(r)\right)r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (103)$$

where we have also defined $v = r/t$. Now, we need to look for possible relationships between space and time so that they can show a conversion between the temporal component S_T^3 and the spatial component S_S^3 of the decomposed spatiotemporal cells $S_S^3 \times S_T^3$. Even though the conditions that will be imposed are rather arbitrarily they do show that the temporal manifold S_T^3 and the spatial manifold S_S^3 can actually be converted into one another. It should also be mentioned that these are not the only conditions that can give rise to a conversion between space and time and, as shown in our works on Euclidean relativity, Euclidean special relativity also produces such conversion [30]. Now, if we impose the following condition

$$\frac{S^2(r)}{1-k_T t^2} = c^2 \quad (104)$$

then the line element given in Equation (103) reduces to

$$ds^2 = c^2dt^2 - \frac{S^2(t)dr^2}{1-k_S r^2} - \left(S^2(t) - \frac{c^2}{v^2}(1-k_T t^2)\right)r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (105)$$

Equation (105) describes particular structures of the temporal manifold with respect to the change of the spatial manifold. Using a linear approximation between space and time for the values of $v \sim c$, then from the relation $1/v^2 = A^2\omega^2\cos^2(\omega r)$, Equation (105) becomes

$$ds^2 = c^2dt^2 - \frac{S^2(t)dr^2}{1-k_S r^2} - S^2(t)\left(1 - \frac{c^2A^2\omega^2\cos^2(\omega r)(1-k_T t^2)}{S^2(t)}\right)r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (106)$$

If we further impose the condition

$$\frac{S^2(t)}{1-k_T t^2} = 1 \quad (107)$$

then we obtain

$$ds^2 = c^2dt^2 - \frac{S^2(t)dr^2}{1-k_S r^2} - S^2(t)\left(1 - c^2A^2\omega^2\cos^2(\omega r)\right)r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (108)$$

It is seen from the line element given in Equation (108) that if $\left(1 - c^2A^2\omega^2\cos^2(\omega r)\right) > 0$ then effectively the spatiotemporal manifold behaves as a spatial manifold endowed with the Robertson-Walker metric. On the other hand, the six-dimensional Robertson-Walker metric can also be written as

$$ds^2 = \frac{S^2(r)dt^2}{1-k_T t^2} - \frac{S^2(t)dr^2}{1-k_S r^2} - \left(v^2S^2(t) - S^2(r)\right)t^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (109)$$

If we impose the following condition

$$\frac{S^2(t)}{1-k_S r^2} = {c_T}^2 \quad (110)$$

then we obtain

$$ds^2 = \frac{S^2(r)dt^2}{1-k_T t^2} - c_T{}^2 dr^2 + S^2(r)\left(1 - \frac{v^2 c_T{}^2(1-k_S r^2)}{S^2(r)}\right) t^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{111}$$

From the linear approximation $1/v^2 \sim A^2\omega^2\cos^2(\omega r)$, Equation (111) becomes

$$ds^2 = \frac{S^2(r)dt^2}{1-k_T t^2} - c_T{}^2 dr^2 + S^2(r)\left(1 - \frac{c_T{}^2(1-k_S r^2)}{A^2\omega^2\cos^2(\omega r)S^2(r)}\right) t^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{112}$$

If we further impose the condition

$$\frac{S^2(r)}{1-k_S r^2} = c_T{}^2 c^2 \tag{113}$$

then we obtain

$$ds^2 = \frac{S^2(r)dt^2}{1-k_T t^2} - c_T{}^2 dr^2 + S^2(r)\left(1 - \frac{1}{c^2A^2\omega^2\cos^2(\omega r)}\right) t^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{114}$$

Therefore if $\left(1 - c^2A^2\omega^2\cos^2(\omega r)\right) < 0$ then effectively the spatiotemporal manifold behaves as a temporal manifold endowed with the temporal Robertson-Walker metric

$$ds^2 = S^2(r)\left(\frac{dt^2}{1-k_T t^2} + t^2(d\theta^2 + \sin^2\theta d\phi^2)\right) - c_T{}^2 dr^2 \tag{115}$$

It is also noted from the line element given in Equation (102) that when space and time satisfy the condition $r^2S^2(t) - t^2S^2(r) = 0$ then we have

$$ds^2 = \frac{S^2(r)dt^2}{1-k_T t^2} - \frac{S^2(t)dr^2}{1-k_S r^2} \tag{116}$$

The metric given in Equation (116) is a particular form of the general line element given in Equation (92) with $S^2(r)/(1-k_T t^2) = e^{\psi}c^2$ and $S^2(t)/(1-k_S r^2) = e^{\chi}$, therefore the wave motion of spacetime which is endowed with the Roberson-Walker metric also occurs at the position of conversion between the temporal and spatial manifolds.

V. Quantum Particles With Integer Spin

In Sections 3 and 4 we show that a complete picture of quantum particles can be visualised in the three-dimensional Euclidean space if their associated differentiable manifolds are solutions of a two-dimensional wave equation, and these massive quantum particles have half-integer spin therefore they can be identified with fermions. Actually, the energy spectrum obtained from the Schrödinger wave equation in a two-dimensional space given in Equation (45) also suggests that there may be massive quantum particles of integer spin associated with differentiable manifolds that are solutions of a two-dimensional wave equation. Nonetheless, it has been shown that quantum particles with integer spin, such as the massless quantum particles of the electromagnetic field, are described by a three-dimensional wave equation, therefore it is reasonable to suggest that the differentiable manifolds that are associated with these quantum particles, called bosons, not only should have different geometric and topological structures but also render different perceptions with regard to our observation of their physical behaviour. In classical physics, the dynamics of physical phenomena can be formulated based on the notion of elementary particles that exist as 3D solid balls containing all physical entities that are needed for physical formulations, such as mass and charge. It is then simply assumed that in order to interact these solid balls somehow generate physical fields, such as the gravitational field and the electromagnetic field, which can be derived from a three-dimensional wave equation. Despite with the fact that the existence of these physical fields is self-evident and they are widely applied their true natures are very much still unknown. However, in quantum physics bosons are quantum particles therefore as in the case of fermions considered in the previous sections we may suggest that bosons also possess the geometric and topological structures of differentiable manifolds which are solutions of a wave equation. Along the line of Einstein's perception of physical existence in which a 3-sphere can be constructed from a four-dimensional Euclidean space R^4, in this section we will discuss the possibility to extend the notion of wave motion into a fourth spatial dimension so that we can have a unified dynamical description in terms of wave equations for quantum particles of any spin. With this in mind, in this section we discuss a spacetime in which space has four dimensions and time has one dimension. Despite a spatial space with four dimensions is simply a mathematical extension of the concept of a spatial space with three dimensions it is still considered to be rather speculative in contrast to the three-dimensional space which is a direct application from the observation from physical existence that we can perceive. In classical physics, the three-dimensional wave equation written in Cartesian coordinates (x,y,z) of the form

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}-\frac{\partial^2\psi}{\partial x^2}-\frac{\partial^2\psi}{\partial y^2}-\frac{\partial^2\psi}{\partial z^2}=0 \qquad (117)$$

can be used to describe the wave motion of different physical fields. However, if we want to generalise the above discussions for 2D wave equations that describe a vibrating membrane then what geometrical characteristic should we assign to the wavefunction ψ? Since in 2D wave equations, the wavefunction are the actual height of the particles that form the medium which can be viewed in the third spatial dimension of the space in which they are embedded, therefore we may suggest that the wavefunction which is a solution to the wave equation given in Equation (117) should also be given the meaning of the height of the particles that form the medium. However, if we want to give the meaning of the height to the 3D wavefunction then the space in which the 3D vibrating object is embedded must be extended to a four-dimensional Euclidean space. Whether such extension can be justified is a subject that requires further investigation and in fact this can be shown to be related to the fundamental question of why we exist as 3D physical objects. Now, consider a region D which is embedded in a three-dimensional Euclidean space and bounded by a closed surface. As in the case of the membrane considered above, we assume that the D region is a physical object that is made up of mass points joined together by contact forces so that it can vibrate. In general, the region D can be any shape, however, as an illustration, we consider a simple case of which the region D is a solid ball embedded in the (x,y,z)-space defined by the relation $D=\{x^2+y^2+z^2<a^2\}$ with the condition $\psi=0$ on the boundary of D. In a three-dimensional Euclidean space, such physical objects can only be assumed to vibrate internally inside the solid ball and the mathematical object represented by the function ψ can only be assumed to be a physical entity, such as fluids and acoustics. However, as in the case of the membrane considered in Section 3 in which the mass points of the membrane can vibrate into the third dimension of the three-dimensional Euclidean space, we may assume that the mass points that form the physical object contained in the three-dimensional region $D=\{x^2+y^2+z^2<a^2\}$ can vibrate into the fourth dimension of a four-dimensional Euclidean space, therefore the mathematical object ψ represents a spatial dimension. When vibrating, at each moment of time, the solid ball becomes a three-dimensional differentiable manifold that is embedded in a four-dimensional Euclidean space. In this case, an observer who is a 3D physical object can only observe the cross-section which is the intersection of the time-dependent differentiable manifold and the three-dimensional Euclidean space into which that the observer is embedded. And the cross-section appears as a 3D wave to the 3D observer. Written in the spherical polar coordinates, which are defined in terms of the Cartesian coordinates (x,y,z) as $x=r\sin\theta\cos\phi$, $y=r\sin\theta\sin\phi$, $z=r\cos\theta$ the three-dimensional wave equation given in Equation (117) becomes

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}-\frac{\partial^2\psi}{\partial r^2}-\frac{2}{r}\frac{\partial\psi}{\partial r}-\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)-\frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}=0 \qquad (118)$$

The general solution to Equation (118) for the vibrating solid ball with a given initial condition can be found by separating the variables in the form $\psi(r,\theta,\phi,t)=S(r,\theta,\phi)T(t)$

$$\psi(r,\theta,\phi,t)=\sum_{l=0}^{\infty}\sum_{j=1}^{\infty}\sum_{m=-l}^{l}A_{lmj}e^{-k\lambda_{lj}t}\frac{J_{l+\frac{1}{2}}\left(\sqrt{\lambda_{lj}}r\right)}{\sqrt{r}}P_l^m(\cos\theta)\,e^{im\phi} \qquad (119)$$

where $P_l^m(\cos\theta)$ is the associated Legendre function and $J_{l+\frac{1}{2}}(\sqrt{\lambda_{lj}}r)$ is the Bessel function. The wavefunction given in Equation (119) is the general time-dependent shape of the vibrating solid ball embedded in the four-dimensional Euclidean space. Similar to the vibrating membrane, at each moment of time the vibrating solid ball appears as a 3D differentiable manifold which is a geometric object whose geometric structure can be constructed using the wavefunction given in Equation (119) and can be identified with a quantum particle. Therefore, what we observe as a wave may in fact be a particle and this kind of dual existence may be related to the problem of wave-particle duality we encounter in quantum mechanics. A simpler case is that of a quantum particle that appears as a spherical wave. In this case the wave equation given in Equation (119) reduces to

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}-\frac{\partial^2\psi}{\partial r^2}-\frac{2}{r}\frac{\partial\psi}{\partial r}=0 \qquad (120)$$

The general solution to Equation (120) can be found as

$$\psi(r,t)=\frac{c_1 f(r-ct)+c_2 g(r+ct)}{r} \qquad (121)$$

The above wavefunctions describe the geometric structures of quantum particles as differentiable manifolds embedded in a four-dimensional Euclidean space, therefore, if the Ricci scalar curvature of the vibrating solid ball can be formulated in terms of the wavefunction ψ then the geometric structure of the vibrating solid ball can be determined. Actually we can show how such relation can be realised for the case of the hydrogen atom when the Ricci scalar curvature can be constructed from the Schrödinger wavefunctions in wave machanics [1]. We showed that the scalar potential V can be identified with the Ricci scalar curvature as

$$V=kR \qquad (122)$$

where k is an undetermined dimensional constant. Using the relation between the scalar potential and the Ricci scalar curvature given in Equation (122), we can show that the Ricci scalar curvature can be constructed from the wavefunctions obtained from the Schrödinger wave equation in wave mechanics. In his original works, Schrödinger introduced a new function ψ which is real, single-valued and twice differentiable, through the relation $S=\hbar\ln\psi$, where the action S is defined by $S=\int Ldt$ and L is the Lagrangian defined by $L=T-\varphi$ with T is the kinetic energy and φ is the potential energy [21]. By applying the principle of least action defined in classical dynamics, Schrödinger arrived at the wave equation to describe the stationary state of the hydrogen atom

$$\nabla^2\psi+\frac{2m}{2}\left(E+\frac{kq^2}{r}\right)\psi=0 \qquad (123)$$

Now we show that Schrödinger wavefunction ψ can be used to construct the Ricci scalar curvature associated with the spacetime structures of the quantum states of the hydrogen atom. By using the defined relations $L=dS/dt$, $dS/dt=\partial_t S+\sum_{\mu=1}^{3}\partial_\mu S(dx^\mu/dt)$, $T=m\sum_{\mu=1}^{3}(dx^\mu/dt)^2$ and $\varphi=T-L$, the following relation can be obtained

$$\varphi=m\sum_{\mu=1}^{3}(dx^\mu/dt)^2-\frac{\partial_t\psi+\sum_{\mu=1}^{3}\partial_\mu\psi(dx^\mu/dt)}{\psi} \qquad (124)$$

Using the relations $V=k\mathrm{R}$ and $V=\varphi/m$, we obtain the following relationship between the Schrödinger wavefunction ψ and the Ricci scalar curvature R

$$\mathrm{R}=\frac{1}{k}\left(\sum_{\mu=1}^{3}(dx^\mu/dt)^2-\frac{1}{m}\frac{\partial_t\psi+\sum_{\mu=1}^{3}\partial_\mu\psi(dx^\mu/dt)}{\psi}\right) \qquad (125)$$

Finally, we would like to give more details how to formulate Maxwell field equations from the general system of linear first order partial differential equations given in Equation (12). In order to derive Maxwell field equations from Equation (12) we would need to identify the matrices A_i. For the case of Dirac equation, we simply impose the condition $A_iA_j+A_jA_i=0$ for $i\neq j$ and $A_i^2=1$. However, as shown below, for Maxwell field equations the identification of the matrices A_i is almost impossible without relying on the form of Maxwell field equations that have been established in classical electrodynamics. With the notation $\psi=(E_x,E_y,E_z,B_x,B_y,B_z)^T=(\psi_1,\psi_2,\psi_3,\psi_4,\psi_5,\psi_6)$, and $\epsilon\mu=1$, the most symmetric form of Maxwell field equations of the electromagnetic field that are derived from Faraday's law and Ampere's law can be written as

$$\frac{\partial\psi_1}{\partial t}+\mu j_1=\frac{\partial\psi_6}{\partial y}-\frac{\partial\psi_5}{\partial z} \tag{126}$$

$$\frac{\partial\psi_2}{\partial t}+\mu j_2=\frac{\partial\psi_4}{\partial z}-\frac{\partial\psi_6}{\partial x} \tag{127}$$

$$\frac{\partial\psi_3}{\partial t}+\mu j_3=\frac{\partial\psi_5}{\partial x}-\frac{\partial\psi_4}{\partial y} \tag{128}$$

$$\frac{\partial\psi_4}{\partial t}+j_4=\frac{\partial\psi_2}{\partial z}-\frac{\partial\psi_3}{\partial y} \tag{129}$$

$$\frac{\partial\psi_5}{\partial t}+j_5=\frac{\partial\psi_3}{\partial x}-\frac{\partial\psi_1}{\partial z} \tag{130}$$

$$\frac{\partial\psi_6}{\partial t}+j_6=\frac{\partial\psi_1}{\partial y}-\frac{\partial\psi_2}{\partial x} \tag{131}$$

where $J=(j_1,j_2,j_3,j_4,j_5,j_6)^T$ is the electromagnetic current in which the electric current is $j_e=(j_1,j_2,j_3)$ and the magnetic current is $j_m=(j_4,j_5,j_6)$. The system of equations given in Equations (126-131) can be written the following matrix form

$$\left(A_1\frac{\partial}{\partial t}+A_2\frac{\partial}{\partial x}+A_3\frac{\partial}{\partial y}+A_4\frac{\partial}{\partial z}\right)\psi=A_5J \tag{132}$$

with the matrices A_i are given as

$$A_1=\begin{pmatrix}-1&0&0&0&0&0\\0&-1&0&0&0&0\\0&0&-1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{pmatrix}\quad A_2=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&-1\\0&0&0&0&1&0\\0&0&0&0&0&0\\0&0&-1&0&0&0\\0&1&0&0&0&0\end{pmatrix}\quad A_3=\begin{pmatrix}0&0&0&0&0&1\\0&0&0&0&0&0\\0&0&0&-1&0&0\\0&0&1&0&0&0\\0&0&0&0&0&0\\-1&0&0&0&0&0\end{pmatrix}$$

$$A_4=\begin{pmatrix}0&0&0&0&-1&0\\0&0&0&1&0&0\\0&0&0&0&0&0\\0&-1&0&0&0&0\\1&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\quad A_5=\begin{pmatrix}\mu&0&0&0&0&0\\0&\mu&0&0&0&0\\0&0&\mu&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{pmatrix} \tag{133}$$

Furthermore, if an additional condition that imposes on the function ψ that requires that it also satisfies the wave equation given by Equation (15) then Gauss's laws will be recovered. From Equation (133) we obtain

$$A_1^2=\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{pmatrix}\quad A_2^2=\begin{pmatrix}0&0&0&0&0&0\\0&-1&0&0&0&0\\0&0&-1&0&0&0\\0&0&0&0&0&0\\0&0&0&0&-1&0\\0&0&0&0&0&-1\end{pmatrix}\quad A_3^2=\begin{pmatrix}-1&0&0&0&0&0\\0&0&0&0&0&0\\0&0&-1&0&0&0\\0&0&0&-1&0&0\\0&0&0&0&0&0\\0&0&0&0&0&-1\end{pmatrix}$$

$$A_4^2=\begin{pmatrix}-1&0&0&0&0&0\\0&-1&0&0&0&0\\0&0&0&0&0&0\\0&0&0&-1&0&0\\0&0&0&0&-1&0\\0&0&0&0&0&0\end{pmatrix}\quad A_5^2=\begin{pmatrix}\mu^2&0&0&0&0&0\\0&\mu^2&0&0&0&0\\0&0&\mu^2&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{pmatrix}$$

$$A_2A_3+A_3A_2=\begin{pmatrix}0&1&0&0&0&0\\1&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&1&0\\0&0&0&1&0&0\\0&0&0&0&0&0\end{pmatrix}\quad A_2A_4+A_4A_2=\begin{pmatrix}0&0&1&0&0&0\\0&0&0&0&0&0\\1&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&0&0\\0&0&0&1&0&0\end{pmatrix}\quad A_3A_4+A_4A_3=\begin{pmatrix}0&0&0&0&0&0\\0&0&1&0&0&0\\0&1&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&1&0\end{pmatrix}$$

$$A_1A_i+A_iA_1\quad\text{for}\quad i=2,3,4 \tag{134}$$

Now, if we apply the differential operator $(A_1\partial/\partial t+A_2\partial/\partial x+A_3\partial/\partial y+A_4\partial/\partial z)$ to Equation (132) then we arrive at

$$\left(\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{pmatrix}\frac{\partial^2}{\partial t^2}+\begin{pmatrix}0&0&0&0&0&0\\0&-1&0&0&0&0\\0&0&-1&0&0&0\\0&0&0&0&0&0\\0&0&0&0&-1&0\\0&0&0&0&0&-1\end{pmatrix}\frac{\partial^2}{\partial x^2}+\begin{pmatrix}-1&0&0&0&0&0\\0&0&0&0&0&0\\0&0&-1&0&0&0\\0&0&0&-1&0&0\\0&0&0&0&0&0\\0&0&0&0&0&-1\end{pmatrix}\frac{\partial^2}{\partial y^2}\right.$$
$$+\begin{pmatrix}-1&0&0&0&0&0\\0&-1&0&0&0&0\\0&0&0&0&0&0\\0&0&0&-1&0&0\\0&0&0&0&-1&0\\0&0&0&0&0&0\end{pmatrix}\frac{\partial^2}{\partial z^2}+\begin{pmatrix}0&1&0&0&0&0\\1&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&1&0\\0&0&0&1&0&0\\0&0&0&0&0&0\end{pmatrix}\frac{\partial^2}{\partial x\partial y}+\begin{pmatrix}0&0&1&0&0&0\\0&0&0&0&0&0\\1&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&0&0\\0&0&0&1&0&0\end{pmatrix}\frac{\partial^2}{\partial x\partial z}$$
$$\left.+\begin{pmatrix}0&0&0&0&0&0\\0&0&1&0&0&0\\0&1&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&1&0\end{pmatrix}\frac{\partial^2}{\partial y\partial z}\right)\psi=-\left(\begin{pmatrix}\mu&0&0&0&0&0\\0&\mu&0&0&0&0\\0&0&\mu&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{pmatrix}\frac{\partial}{\partial t}\right)J \tag{135}$$

From the equation given in Equation (135), using Gauss's law $\nabla.\mathbf{E}=\rho_e/\epsilon$ we obtain the following wave equations for the components of the electric field $\mathbf{E}=(E_x,E_y,E_z)=(\psi_1,\psi_2,\psi_3)$

$$\frac{\partial^2\psi_i}{\partial t^2}-\frac{\partial^2\psi_i}{\partial x^2}-\frac{\partial^2\psi_i}{\partial y^2}-\frac{\partial^2\psi_i}{\partial z^2}=-\mu\frac{\partial j_i}{\partial t}\quad \text{for}\quad i=1,2,3 \tag{136}$$

Similarly for the magnetic field $\mathbf{B}=(B_x,B_y,B_z)=(\psi_4,\psi_5,\psi_6)$ we can also obtain the following wave equations for the components of the magnetic field $\mathbf{B}=(B_x,B_y,B_z)=(\psi_4,\psi_5,\psi_6)$

$$\frac{\partial^2\psi_i}{\partial t^2}-\frac{\partial^2\psi_i}{\partial x^2}-\frac{\partial^2\psi_i}{\partial y^2}-\frac{\partial^2\psi_i}{\partial z^2}=-\frac{\partial j_i}{\partial t}\quad \text{for}\quad i=4,5,6 \tag{137}$$

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