Maxwell Equations Re-derived Mathematically

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Abstract

In this article, by applying Gauss/Coulomb's law, we re-derive mathematically the whole set of Maxwell equations, which means that, as long as a charge satisfies Gauss/Coulomb's law, the uniform motion of the charge generates inevitably a magnetic field and an axial electric field. This fact is predetermined mathematically.

Key words: Maxwell equations, Faraday's law, Ampere's law, Ampere-Maxwell's law, electromagnetic fields, uniformly moving electric charge

1. Introduction

Maxwell equations govern electromagnetic fields generated by stationary charges and steady current. There is an interesting phenomenon: When a charge is at rest, it generates an electric field determined by the Gauss/Coulomb's law $\nabla \cdot \mathbf{E} = 4\pi Q$, and doesn't generate a magnetic field. However, an observer moving uniformly relative to the same charge observes not only an electric field but also a magnetic field. The magnetic field is completely different from electric field in the following senses: (1) the way the fields generated; (2) the nature of the fields; (3) effects of the fields on charges.

A physics student may ask questions: Why a uniformly moving observer experiences such different magnetic field? Or why the motion of charges generates magnetic fields? A teacher's answer is that magnetism is the combination of electric field with special relativity.

Now we ask further questions: Does the motion of charges generate magnetic fields inevitably? The answer is yes. A famous sentence is that "The world is built mathematically". We argue that the equations describing magnetic fields must be able to be derived mathematically, i.e., Maxwell equations must be able to be derived mathematically.

Historically, Maxwell equations, except the displacement current, were established based on series experiments. Up to now, as I know of, there is no approach to re-derive Maxwell equations mathematically.

In this article we propose an approach to re-derive mathematically Maxwell equations and show that a moving charge generates inevitably axial magnetic field and axial electric field.

2. Mathematic Preparation

We will utilize Gauss/Coulomb's law, which relates the divergences of a vector electric field to a charge,

$$\nabla \cdot \mathbf{E} = 4\pi Q. \tag{1}$$

The vector analysis formulas we will utilize are shown below,

$$\nabla \times (\mathbf{S} \times \mathbf{T}) = \mathbf{S}(\nabla \cdot \mathbf{T}) - \mathbf{T}(\nabla \cdot \mathbf{S}) + (\mathbf{T} \cdot \nabla)\mathbf{S} - (\mathbf{S} \cdot \nabla)\mathbf{T}, \tag{2}$$

$$\boldsymbol{\nabla}(\mathbf{S} \cdot \mathbf{T}) = (\mathbf{S} \cdot \boldsymbol{\nabla})\mathbf{T} + (\mathbf{T} \cdot \boldsymbol{\nabla})\mathbf{S} + \mathbf{S} \times (\boldsymbol{\nabla} \times \mathbf{T}) + \mathbf{T} \times (\boldsymbol{\nabla} \times \mathbf{S}), \tag{3}$$

$$\mathbf{S} \cdot (\mathbf{T} \times \mathbf{Z}) = \mathbf{T} \cdot (\mathbf{Z} \times \mathbf{S}) = \mathbf{Z} \cdot (\mathbf{S} \times \mathbf{T}). \tag{4}$$

where **S**, **T** and **Z** are arbitrary vectors.

Eq. (2) is the most noteworthy formula that it indicates that the combination of gradient and divergence operations of two arbitrary vectors generates inevitably an axial vector. The Coulomb's law is a divergence law describing charges. The basic concept of this article is that the combination of the Coulomb's law, the uniform motion of charges, and Eq. (2) must generate an axial field, magnetic field.

We only consider the situations of charges moving with non-spatially-varying and uniform velocity $\mathbf{v}$, thus, $\dot{\mathbf{v}} = 0$, $\nabla \cdot \mathbf{v} = (\mathbf{Y} \cdot \nabla)\mathbf{v} = \boldsymbol{\nabla} \times \mathbf{v} = 0$, where **Y** represents either **S** or **T** vector. Therefore, let $\mathbf{S} = \mathbf{v}$, Then Eq. (2) and (3) become,

$$\nabla \times (\mathbf{v} \times \mathbf{T}) = \mathbf{v}(\nabla \cdot \mathbf{T}) - (\mathbf{v} \cdot \nabla)\mathbf{T}, \tag{5}$$

$$\boldsymbol{\nabla}(\mathbf{v} \cdot \mathbf{T}) = (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{T} + \mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{T}). \tag{6}$$

Let $\mathbf{S} = \mathbf{T} = \mathbf{v}$, Eq. (4) gives

$$\mathbf{v} \cdot (\mathbf{v} \times \mathbf{Z}) = \mathbf{Z} \cdot (\mathbf{v} \times \mathbf{v}) = 0. \tag{7}$$

3. Magnetic Field induced Inevitably by Velocity of Charges

Let's apply this approach to re-derive Ampere's law mathematically. The combination of the vector analysis formula and Coulomb's law leads us to let $\mathbf{T} = \mathbf{E}$. Then Eq. (5) gives

$$\nabla\times(\mathbf{v}\times\mathbf{E}) = \mathbf{v}(\nabla\cdot\mathbf{E}) - (\mathbf{v}\cdot\nabla)\mathbf{E}. \quad (8)$$

Define a field, **Y,**

$$\mathbf{Y} \equiv \mathbf{v}\times\mathbf{E}, \quad (9)$$

Combining Eq. (8) and (9), we obtain,

$$\nabla\times\mathbf{Y} = 4\pi Q\mathbf{v} - (\mathbf{v}\cdot\nabla)\mathbf{E}, \quad (10)$$

where Coulomb's law has been applied. Eq. (10) shows that the velocity of a charge indeed generates inevitably an axial field **Y**. For an axial field, we have mathematically,

$$\nabla\cdot\mathbf{Y} = 0. \quad (11)$$

The Ampere's law is,

$$\nabla\times\mathbf{B} = 4\pi Q\mathbf{v}. \quad (12)$$

Comparing Eq. (10) and (12), we assume that the same current will generates the same fields. Without the term, $(\mathbf{v}\cdot\nabla)\mathbf{E}$, Eq. (10) is identical to the Ampere's law. Therefore, we identify the axial field **Y** as magnetic field **B**. Eq. (10) and (11) can be rewritten as,

$$\nabla\times\mathbf{B} = 4\pi Q\mathbf{v} - (\mathbf{v}\cdot\nabla)\mathbf{E}, \quad (13)$$

$$\nabla\cdot\mathbf{B} = 0. \quad (14)$$

We will discuss the term, $(\mathbf{v}\cdot\nabla)\mathbf{E}$, later.

In the derivation above, the only physics law we applied is the Coulomb's law, which indicates that, for a charge satisfying Coulomb's law, the uniform motion of the charge inevitable generates an axial magnetic **B** field. The generation of the magnetic **B** field is predetermined by the wonderful mathematical formula Eq. (2 and 5). Namely, the combination of gradient and divergence operations of a vector electric field and vector velocity of charges must generate an axial vector magnetic field.

4. Faraday's Law

Based on symmetry, it is nature to expect that the velocity of charges generate an axial electric field as well. For this aim, let's apply the formula, Eq. (5), again, and let $\mathbf{T} = \mathbf{B}$, where **B** is magnetic field derived in last section, we obtain,

$$\nabla\times(\mathbf{v}\times\mathbf{B}) = -(\mathbf{v}\cdot\nabla)\mathbf{B}. \quad (15)$$

Define an axial field, **Z,**

$$\mathbf{Z} \equiv -\mathbf{v}\times\mathbf{B}. \quad (16)$$

Note the **Z** field is an axial field related with velocity **v** and magnetic field **B**. Substituting Eq. (16), into Eq. (15), we obtain,

$$\nabla\times\mathbf{Z} = (\mathbf{v}\cdot\nabla)\mathbf{B}. \quad (17)$$

The magnetic field **B** and the velocity of a charge indeed generates an axial field **Z**. Eq. (17) relate the **Z** field, velocity **v**, and magnetic field **B.**

We introduce vector potential **A** defined below,

$$\mathbf{B} = \nabla\times\mathbf{A}, \quad (18)$$

$$\mathbf{Z} = -\frac{\partial \mathrm{A}}{\partial \mathrm{t}}. \quad (19)$$

Taking time derivative of Eq. (18) or taking curl of Eq. (19), we obtain,

$$\frac{\partial \mathbf{B}}{\partial \mathbf{t}} = \nabla\times\frac{\partial \mathbf{A}}{\partial \mathrm{t}} = -\nabla\times\mathbf{Z},$$

or

$$\nabla\times\mathbf{Z} = -\frac{\partial\nabla\times\mathbf{A}}{\partial \mathbf{t}} = -\frac{\partial \mathbf{B}}{\partial \mathrm{t}}. \tag{20}$$

Comparing Eq. (20) with the Faraday's law,

$$\nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial \mathrm{t}},$$

and assuming that the term $\frac{\partial \mathbf{B}}{\partial \mathbf{t}}$ will generate the same axial field, we identify the axial vector field **Z** as the axial electric field **E**. Eq. (20) is identical to the Faraday's law.

We re-derive the Faraday's law again mathematically.

5. Interpretation of axial B and Axial E Fields

Consider a charge Q and an observer. When the charge is at rest relative to the observer, it generates a vector **E** field; when the charge is moving with a constant velocity relative to the observer, beside the vector **E** field, the observer also experiences a axial **B** and axial **E** fields generated by the velocity of the charge, $\mathbf{B} = \mathbf{v}\times\mathbf{E}$, and $\mathbf{E} = -\mathbf{v}\times\mathbf{B}$. Namely, the constant velocity of a charge generates two axial fields (Table B1).

Table B1: Fields generated by stationary charge and the motion of charges

	Generated Field
Stationary Charge: $\mathbf{v} = \mathbf{0}$	$\mathbf{E}$
Uniform motion of charge: $\mathbf{v} \neq \mathbf{0},\ \dot{\mathbf{v}} = \mathbf{0}$	$\mathbf{B} = \mathbf{v}\times\mathbf{E}$ $\mathbf{Z} = -\mathbf{v}\times\mathbf{B}$

6. Ampere-Maxwell law

Now let's study the term, $(\mathbf{v}\cdot\nabla)\mathbf{E}$, of Eq. (13). Applying the vector analysis formula, Eq. (6). Let $\mathbf{T} = \mathbf{E}$, Eq. (6) gives

$$\boldsymbol{\nabla}(\mathbf{v}\cdot\mathbf{E}) = (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{E} + \mathbf{v}\times(\boldsymbol{\nabla}\times\mathbf{E}).$$

Or rewrite as

$$(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{E} = \boldsymbol{\nabla}(\mathbf{v}\cdot\mathbf{E}) - \mathbf{v}\times(\boldsymbol{\nabla}\times\mathbf{E}). \tag{21}$$

Substituting Eq. (21) into Eq. (10), we obtain

$$\nabla\times\mathbf{Y} = 4\pi\mathrm{Q}\mathbf{v} - \boldsymbol{\nabla}(\mathbf{v}\cdot\mathbf{E}) + \mathbf{v}\times(\boldsymbol{\nabla}\times\mathbf{E}). \tag{22}$$

Next let's find $\mathbf{v}\times(\boldsymbol{\nabla}\times\mathbf{E})$. For this aim, utilizing Eq. (16) and (20), we obtain

$$\mathbf{v}\times(\boldsymbol{\nabla}\times\mathbf{E}) = -\mathbf{v}\times\frac{\partial\mathbf{B}}{\partial \mathrm{t}} = -\frac{\partial\mathbf{v}\times\mathbf{B}}{\partial \mathrm{t}} = \frac{\partial\mathbf{E}}{\partial \mathrm{t}}. \tag{23}$$

Now calculate the term $\boldsymbol{\nabla}(\mathbf{v}\cdot\mathbf{E})$. From Eq. (7) and (16) with $\mathbf{E} = \mathbf{Z}$, we have

$$\mathbf{v}\cdot\mathbf{E} = -\mathbf{v}\cdot(\mathbf{v}\times\mathbf{B}) = 0. \tag{24}$$

Then, substituting Eq. (23 and 24) into Eq. (22), we obtain,

$$\nabla\times\mathbf{Y} = 4\pi\mathrm{Q}\mathbf{v} + \frac{\partial\mathbf{E}}{\partial \mathrm{t}}. \tag{25}$$

Comparing Eq. (25) with the Ampere-Maxwell's equation,

$$\nabla\times\mathbf{B} = 4\pi\mathrm{Q}\mathbf{v} + \frac{\partial\mathbf{E}}{\partial \mathrm{t}}$$

and assuming that the term, $4\pi\mathrm{Q}\mathbf{v} + \frac{\partial\mathbf{E}}{\partial \mathrm{t}}$, will generate the same axial vector field, thus we identify the axial field **Y** as the magnetic field **B**. We derive mathematically the Ampere-Maxwell's law.

7. Summary

Based on the Gauss/Coulomb's law, for a uniformly moving charge, we derive mathematically the rest of Maxwell equations,

Gauss/Coulomb' law: $\nabla \cdot \mathbf{E} = 4\pi Q$

Ampere-Maxwell's law: $\nabla \times \mathbf{B} = 4\pi Q\mathbf{v} + \frac{\partial \mathbf{E}}{\partial t}$. (23)

Faraday's law: $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial \mathbf{t}}$. (18)

Gauss's law for B field: $\nabla \cdot \mathbf{B} = 0$. (12)

The conclusion is that the uniform motion of a charge must generate an axial vector magnetic field. This fact is predetermined mathematically by Eq. (1) and (2).