A fully relativistic description of stellar or planetary objects orbiting a very heavy central mass

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Abstract

A fully relativistic numerical program is used to calculate the advance of the peri-helium of Mercury or the deflection of light by the Sun is here used also to discuss the case of S2, a star orbiting a very heavy central mass of the order of $4.3\,10^6$ solar masses.

1 Equations of motion

Given an space-time metric:

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta \tag{1}$$

it is usual to refer to the equations of motion of free bodies as the the geodesic equations:

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{ds} \frac{dx^\gamma}{ds} = 0 \tag{2}$$

but it is important to keep in mind that this is only the case if the space-time trajectories of free particles are parameterized by the proper space-time s or any other affine parameter $s' = k_1 s + k_2$. This is not always the best choice to make.

I shall use here a different parametrization that was introduced by Eisenhart ([1]) when dealing with general linear connections, but can be used

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also when dealing with Riemannian ones. This consists in using instead the equations of motion:

$$\frac{d^2x^\alpha}{ds^2}+\Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds}=a\frac{dx^\alpha}{ds} \tag{3}$$

where a depends on the parameter that is used to describe the solutions. by The purpose of this paper is to look for models of test objects orbiting a central mass, described by the Schwarzschild solution in some of the more used coordinates. I consider two cases. The first case is that of the isotropic coordinates:

$$ds^2=-\frac{\left(1-\frac{m}{2r}\right)^2}{\left(1+\frac{m}{2r}\right)^2}c^2dt^2+\left(1+\frac{m}{2r}\right)^4(dr^2+r^2(d\theta^2+\sin^2\theta d\varphi^2) \tag{4}$$

where $m=GM/c^2$.

The second case is that of of a family of coordinates depending on a parameter λ

$$ds^2=-\frac{(r-\lambda m)c^2dt^2}{r+(2-\lambda)m}+\frac{r+(2-\lambda)m}{r-\lambda m}dr^2+(r+(2-\lambda)m)^2(d\theta^2+\sin^2\theta d\varphi^2) \tag{5}$$

$\lambda=0$ corresponds to Brillouin coordinates; $\lambda=1$ corresponds to Fock coordinates, and $\lambda=2$ corresponds to Droste-Hilbert coordinates

Considering that the space trajectory of the object lies on the plane $\theta=\pi/2$ the only Christoffel symbols to consider are the following:

$$\Gamma^1_{33}=\frac{r(-2r+m)}{2r+m},\ \Gamma^3_{13}=-\frac{-2r+m}{r(2r+m)}, \tag{6}$$

$$\Gamma^1_{44}=-\frac{64r^4(-2r+m)c^2m}{(2r+m)^7},\ \Gamma^4_{14}=\frac{4m}{(2r+m)(-2r+m)},\Gamma^1_{11}=-\frac{2m}{r2r+m}, \tag{7}$$

in Isotropic coordinates. And

$$\Gamma^1_{33}=\lambda m-r,\ \Gamma^3_{13}=-\frac{1}{\lambda m-2m-r},\ \Gamma^1_{11}=-\frac{1}{(\lambda m-r)(\lambda m-2m-r)} \tag{8}$$

$$\Gamma^1_{44}=\frac{(\lambda m-r)c^2m}{(\lambda m-2m-r)^3},\ \Gamma^4_{14}=\frac{m}{(\lambda m-r)(\lambda m-2m-r)} \tag{9}$$

in parameterized coordinates.

My second choice is to choose the coordinate φ as the evolution parameter and therefore describe the motion of the object with the functions $r(\phi)$ and $t(\phi)$. In which case we have:

$$a = -\frac{2(-2r+m)vr}{r(2r+m)} \text{ or } \ a = -\frac{2}{\lambda m - 2m - r)} \tag{10}$$

where here and later:

$$vr = \frac{dr}{d\varphi}, \ vt = \frac{dt}{d\varphi}, \tag{11}$$

The equations to solve are then the following

$$\frac{dvr}{d\varphi} + \Gamma^1_{33} + \Gamma^1_{44}vt^2 + \Gamma^1_{11}vr^2 = a\,vr \tag{12}$$

$$\frac{dvt}{d\varphi} + 2\Gamma^4_{14}vr\,vt = a\,vt \tag{13}$$

or more explicitly:

$$\frac{dvr}{d\varphi} = \frac{2vr^2m}{r(2r+m)} + \frac{64vt^2r^4(-2r+m))mc^2}{(2r+m)^7} \tag{14}$$

$$-\frac{2(-2r+m)vr^2}{r(2r+m)} - \frac{(-2r+m)r}{2r+m} \tag{15}$$

$$\frac{dvt}{d\varphi} = \frac{8vrm\,vt}{(2r+m)(-2r+m)} - \frac{2(-2r+m)vr\,vt}{r(2r+m)} \tag{16}$$

when using isotropic coordinates and:

$$\frac{dvr}{d\phi} = +\frac{m\,vr^2}{(\lambda m - r)(\lambda m - 2m - r)} - \frac{(\lambda m - r)c^2m\,vt^2}{(\lambda m - 2m - r)^3} - \lambda m + r - \frac{-2vr^2}{\lambda m - 2m - r} \tag{17}$$

$$\frac{dvt}{d\phi} = -\frac{2m\,vt\,vr}{(\lambda m - r)(\lambda m - 2m - r)} - \frac{2vr^2}{\lambda m - 2m - r} \tag{18}$$

$$ds^2 = -\frac{(r-\lambda m)c^2dt^2}{r+(2-\lambda)m} + \frac{r+(2-\lambda)m}{r-\lambda m}dr^2 + (r+(2-\lambda)m)^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{19}$$

when using λ coordinates.

An important feature of this system of differential equations is its scale invariance. This meaning that the substitutions:

$$r,\ t,\ m \to kr,\ kt,\ km \tag{20}$$

leaves the system invariant.

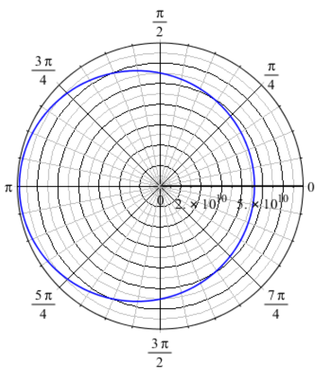
From now on isotropic coordinates will be used, but any of the numerical results to be mentioned have been checked, to the required precision, to coincide with the corresponding result using Brillouin, Fock or Droste-Hilbert coordinates. This would not be the case at a much higher precision of the observational data since the covariance of the theory does not imply the invariance of the numerical results when using different systems of coordinates.

2 The planet Mercury

In this case we know the mass m of the Sun, the length of the perihelion rp and the maximal orbital velocity $vmax$:

$$m = 1484.851528,\ rp = 46\,10^9,\ vmax = 58.98\,10^3 \tag{21}$$

With these data the numerical integration proceeds smoothly, wherefrom we can obtain known Mercury's data including its sidereal orbit period: 86.64 days (NASA's value is 87.969). The polar plot of the trajectory and the plot of the linear velocity $vl = r/vt$ around it are:



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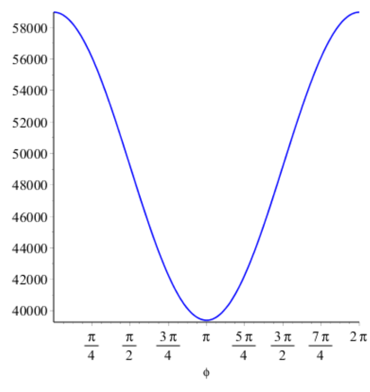

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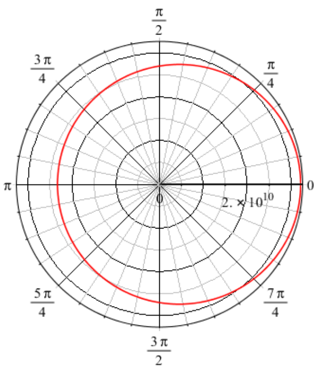
The maximum discrepancy with the third law of Kepler, is:

$$\frac{DA(0) - DA(\pi)}{DA(0) + DA(\pi)} \approx 10^{-8}, \; with \; DA = \frac{dA}{d\varphi} \tag{22}$$

The relativistic advance of the peri-helium per century comes out as 43.46 arcseconds per century derived from the first value of $\phi = 2\pi + \delta\phi$ such that $vr = 0$. A very simple interpolation program is useful.

Let us consider a planet that differs from Mercury in the sense that its distance to the Sun $r0$ is 0.7 times the perihelion of Mercury, and has equal linear velocity at this distance. This would suffice to lead to an orbit whose polar plot trajectory and linear velocity around it would be:



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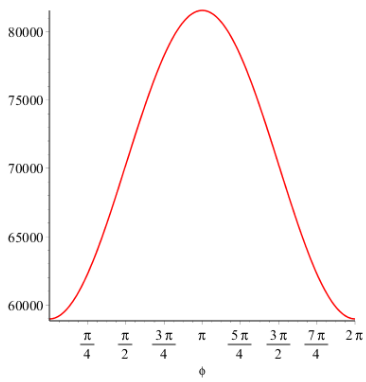

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showing that the initial value of $r0$ is the apo-helium of this fictitious planet.

3 The deflection of light by the Sun

In this case $ds^2 = 0$ along the path of light and the system of differential equations has to be integrated with the inial conditions:

$$r(0) = 6.95\,10^8, \; vr(0) = 0, \; vt(0) = \frac{1}{4}\frac{(2r0 + m)^3}{r0(2r0 - m)c} \tag{23}$$

($vt(0)$ follows from $vr(0) = 0$ and $ds^2 = 0$) and the integration has to proceed until the tangent to the trajectory:

$$\delta\varphi_\infty \equiv \frac{dy}{dx} = \frac{r\sin\varphi - vr\cos\varphi}{r\cos\varphi + vr\sin\varphi} \tag{24}$$

reaches a stationary value. Twice this value:

$$\delta\varphi_\infty = 1.76\, arcseconds \tag{25}$$

is the value of of the angular deviation of a light ray skimming the surface of the Sun. and:

$$\delta\varphi_\infty = 705\, arcseconds \tag{26}$$

is the corresponding deviation for a light-ray approaching Sagittarius A^* to a distance of $50\, au$.

4 The Star S2

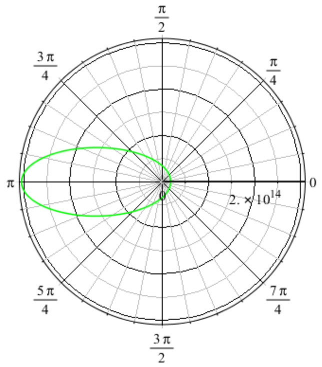
S2 is a star circling Sagittarius A^* with a mass estimated to $4.3\, 10^6$, solar masses, along an ellipse with eccentricity 0.87 and a period of 15.2 years. Also known is its peri-center length of $120\, au$.

Mass and peri-center length are not sufficient initial conditions to derive a unique model integrating the system of differential equations (15)-(16). The value of vt0 or equivalently the value of the linear velocity $rp/vt0$ must be also known.

A trial and error method leads easily to the value $vt0 = rp(38.6/c)$ and to the observed value of the period: 15.2 years. This result has been celebrated ([3]) as a confirmation of General relativity. This is incorrect because the same result holds if instead of integrating the system of equations (15)-(16), we integrate its first quasi Newtonian approximation.

The precession of the peri-center of S2 can now be calculated easily by solving the equation $r(\pi + \delta) = rp$ or $vr(\pi + \delta) = 0$. The result is $\delta\varphi = 0.3393$ arcdegrees per revolution

The polar plot of the trajectory and the plot of the linear velocity around it are:



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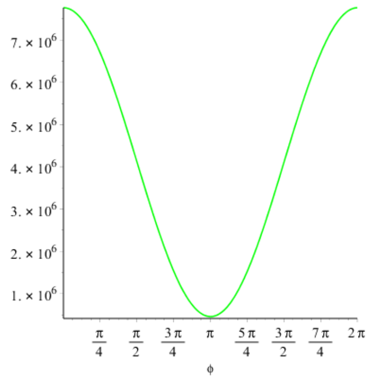

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5 Appendix: Radial motion

Of course, in this case the coordinate φ can not be used to describe the dynamics of a particle moving in a radial direction $\varphi = const.$, and the variable t has to be used as parameter. The single equation derived from the geodesic equations 2 is

$$\frac{dvr}{dt} = \frac{2mvr^2}{r(2r+m)} + \frac{64r^4(-2r+m)c^2m}{(2r+m)^7} + \frac{8mvr^2}{(2r+m)(-2r+m)} \tag{27}$$

where the following value of a has been used.

$$a = \frac{2mvr}{r(r+2m)}, \text{with } vr = \frac{dr}{dt} \tag{28}$$

And solving the partial differential equation:

$$\frac{\partial f(r,vr)}{\partial r}vr + \frac{\partial f(r,vr)}{\partial vr}\frac{dvr}{dt} = 0 \tag{29}$$

we get the Energy function per unit mass function:

$$E = \frac{1}{32}\frac{(2r+m)^8m^2vr^2}{(-2r+m)^4r^4} - \frac{4mr}{(-2r+m)^2} \tag{30}$$

where an appropriate arbitrary multiplicative constant 1/32 has been chosen to obtain the correct non relativistic limit.

$$E = \frac{1}{2}vr^2 - \frac{GM}{r} \tag{31}$$

References

[1] L. P. Eisenhart, *Non Riemannian geometry* , American Mathematical Society (1929)

[2] Nature, **419**, October 2002

[3] Astronomy and Astrophysics, **615** July 2018