An Identity for Horadam Sequences

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Abstract

We derive an identity connecting any two Horadam sequences having the same recurrence relation but whose initial terms may be different. Binomial and ordinary summation identities arising from the identity are developed.

1 Introduction

This paper is concerned with establishing an identity connecting any two Horadam sequences, $(G_n)_{n\in\mathbb{Z}}$ and $(H_n)_{n\in\mathbb{Z}}$, having the same recurrence relation but whose initial terms may be different. Thus, for $n \ge 2$ and with p and q arbitrary fixed non-zero complex constants, we define

$$G_n = pG_{n-1} + qG_{n-2}\,, \tag{1.1}$$

where the initial terms G_0 and G_1 are given arbitrary integers, not both zero; and

$$H_n = pH_{n-1} + qH_{n-2}\,, \tag{1.2}$$

with initial terms H_0 and H_1 given arbitrary integers, not both zero.

Extension of the definition of G_n and H_n to negative subscripts is provided by writing the recurrence relation as

$$G_{-n} = (G_{-n+2} - pG_{-n+1})/q \tag{1.3}$$

and

$$H_{-n} = (H_{-n+2} - pH_{-n+1})/q\,. \tag{1.4}$$

In section 2, we will derive an identity connecting (G_n) and (H_n), for arbitrary integers. We will illustrate the results by deriving identities for six well-known Horadam sequences, namely, Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal and Jacobsthal-Lucas sequences.

1.1 Notation and definitions

The Fibonacci numbers, F_n, and the Lucas numbers, L_n, are defined, for $n \in \mathbb{Z}$, as usual, through the recurrence relations $F_n = F_{n-1} + F_{n-2}$ $(n \geq 2)$, $F_0 = 0$, $F_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$ $(n \geq 2)$, $L_0 = 2$, $L_1 = 1$, with $F_{-n} = (-1)^{n-1} F_n$ and $L_{-n} = (-1)^n L_n$. Exhaustive discussion of the properties of Fibonacci and Lucas numbers can be found in Vajda [9] and in Koshy [6]. Generalized Fibonacci numbers having the same recurrence as the Fibonacci and Lucas numbers but with arbitrary initial values will be denoted $\mathcal{F}_n$.

The Jacobsthal numbers, J_n, and the Jacobsthal-Lucas numbers, j_n, are defined, for $n \in \mathbb{Z}$, through the recurrence relations $J_n = J_{n-1} + 2J_{n-2}$ $(n \geq 2)$, $J_0 = 0$, $J_1 = 1$ and $j_n = j_{n-1} + 2j_{n-2}$ $(n \geq 2)$, $j_0 = 2$, $j_1 = 1$, with $J_{-n} = (-1)^{n-1} 2^{-n} J_n$ and $j_{-n} = (-1)^n 2^{-n} j_n$. Horadam [5] and Aydin [2] are good reference materials on the Jacobsthal and associated sequences. Generalized Jacobsthal numbers having the same recurrence as the Jacobsthal and Jacbsthal-Lucas numbers but with arbitrary initial values will be denoted $\mathcal{J}_n$.

The Pell numbers, P_n, and Pell-Lucas numbers, Q_n, are defined, for $n \in \mathbb{Z}$, through the recurrence relations $P_n = 2P_{n-1} + P_{n-2}$ $(n \geq 2)$, $P_0 = 0$, $P_1 = 1$ and $Q_n = 2Q_{n-1} + Q_{n-2}$ $(n \geq 2)$, $Q_0 = 2$, $Q_1 = 2$, with $P_{-n} = (-1)^{n-1} P_n$ and $Q_{-n} = (-1)^n Q_n$. Koshy [7], Horadam [4] and Patel and Shrivastava [8] are useful source materials on Pell and Pell-Lucas numbers. Generalized Pell numbers having the same recurrence as the Pell and Pell-Lucas numbers but with arbitrary initial values will be denoted $\mathcal{P}_n$.

For reference, the first few values of the six sequences are given below:

n:	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8
F_n:	5	-3	2	-1	1	0	1	1	2	3	5	8	13	21
L_n:	-11	7	-4	3	-1	2	1	3	4	7	11	18	29	47
P_n:	29	-12	5	-2	1	0	1	2	5	12	29	70	169	408
Q_n:	-82	34	-14	6	-2	2	2	6	14	34	82	198	478	1154
J_n:	11/32	$-5/16$	3/8	$-1/4$	1/2	0	1	1	3	5	11	21	43	85
j_n:	$-31/32$	17/16	$-7/8$	5/4	$-1/2$	2	1	5	7	17	31	65	127	257

2 Main results

2.1 Recurrence relations

Theorem 1. *Let $(G_n)_{n\in\mathbb{Z}}$ and $(H_n)_{n\in\mathbb{Z}}$ be any two Horadam sequences having the same recurrence relation. Then, the following identity holds for arbitrary integers n, m, a, b, c and d:*

$$\begin{aligned}(G_{d-b}G_{c-a} - G_{d-a}G_{c-b})H_{n+m} &\\ = (G_{d-b}G_{m-a} - G_{d-a}G_{m-b})H_{n+c} &\\ + (G_{c-a}G_{m-b} - G_{c-b}G_{m-a})H_{n+d}\,. &\end{aligned}$$

In particular, we have

$$\begin{aligned}(G_{d-b}G_{c-a} - G_{d-a}G_{c-b})G_{n+m} &\\ = (G_{d-b}G_{m-a} - G_{d-a}G_{m-b})G_{n+c} &\\ + (G_{c-a}G_{m-b} - G_{c-b}G_{m-a})G_{n+d}\,, &\end{aligned} \tag{2.1}$$

for any Horadam sequence, $(G_n)_{n\in\mathbb{Z}}$.

Proof. Since both sequences (G_n) and (H_n) have the same recurrence relation we choose a basis set in one and express the other in this basis. We write

$$H_{n+m} = \lambda_1 G_{m-a} + \lambda_2 G_{m-b}\,, \tag{2.2}$$

where a, b, n and m are arbitrary integers and the coefficients λ_1 and λ_2 are to be determined. Setting $m = c$ and $m = d$, in turn, produces two simultaneous equations:

$$H_{n+c} = \lambda_1 G_{c-a} + \lambda_2 G_{c-b}, \quad H_{n+d} = \lambda_1 G_{d-a} + \lambda_2 G_{d-b}\,.$$

The identity of Theorem 1 is established by solving these equations for λ_1 and λ_2 and substituting the solutions into identity (2.2). □

Corollary 2. *The following identity holds for integers a, b, n and m:*

$$\begin{aligned}&(G_{a-b}G_{b-a} - G_0^2)H_{n+m}\\&\quad= (G_{b-a}G_{m-b} - G_0G_{m-a})H_{n+a}\\&\qquad+ (G_{a-b}G_{m-a} - G_0G_{m-b})H_{n+b}\,.\end{aligned}$$

In particular,

$$\begin{aligned}&(G_{a-b}G_{b-a} - G_0^2)G_{n+m}\\&\quad= (G_{b-a}G_{m-b} - G_0G_{m-a})G_{n+a}\\&\qquad+ (G_{a-b}G_{m-a} - G_0G_{m-b})G_{n+b}\,.\end{aligned} \tag{2.3}$$

2.2 Summation identities

2.2.1 Summation identities not involving binomial coefficients

Lemma 1 ([1, Lemma 1]). *Let $\{X_n\}$ and $\{Y_n\}$ be any two sequences such that X_n and Y_n, $n \in \mathbb{Z}$, are connected by a three-term recurrence relation $X_n = f_1X_{n-a} + f_2Y_{n-b}$, where f_1 and f_2 are arbitrary non-vanishing complex functions, not dependent on r, and a and b are integers. Then,*

$$f_2\sum_{j=0}^{k}\frac{Y_{n-ka-b+aj}}{{f_1}^j} = \frac{X_n}{{f_1}^k} - f_1X_{n-(k+1)a}\,,$$

for k a non-negative integer.

Lemma 2 ([1, Lemma 2]). *Let $\{X_n\}$ be any arbitrary sequence, where X_n, $n \in \mathbb{Z}$, satisfies a three-term recurrence relation $X_n = f_1X_{n-a} + f_2X_{n-b}$, where f_1 and f_2 are arbitrary non-vanishing complex functions, not dependent on r, and a and b are integers. Then, the following identities hold for integer k:*

$$f_2\sum_{j=0}^{k}\frac{X_{n-ka-b+aj}}{f_1^j} = \frac{X_n}{f_1^k} - f_1X_{n-(k+1)a}\,, \tag{2.4}$$

$$f_1\sum_{j=0}^{k}\frac{X_{n-kb-a+bj}}{f_2^j} = \frac{X_n}{f_2^k} - f_2X_{n-(k+1)b} \tag{2.5}$$

and

$$\sum_{j=0}^{k}\frac{X_{n-(a-b)k+b+(a-b)j}}{(-f_1/f_2)^j} = \frac{f_2X_n}{(-f_1/f_2)^k} + f_1X_{n-(k+1)(a-b)}\,. \tag{2.6}$$

In order to state the next two theorems in a compact form, we introduce the following notation:

$$f_G(u,v;s,t) = G_{u-s}G_{v-t} - G_{u-t}G_{v-s}\,,$$

with the following symmetry properties:

$$f_G(u,v;t,s) = -f_G(u,v;s,t)\,, \quad f_G(v,u;s,t) = -f_G(u,v;s,t)\,,$$

$$f_G(v,u;t,s) = f_G(u,v;s,t)\,,$$

and

$$f_G(u,u;s,t) = 0\,, \quad f_G(u,v;s,s) = 0\,.$$

In this notation, the identity of Theorem 1 becomes

$$f_G(d,c;b,a)H_{n+m} = f_G(d,m;b,a)H_{n+c} + f_G(c,m;a,b)H_{n+d}\,. \tag{2.7}$$

The results in the next theorem follow from direct substitutions from identity (2.7) into Lemma 2.

Theorem 3. *The following identities hold for arbitrary integers a, b, c, d and m for which $f_G(d,m;b,a) \neq 0$ and $f_G(c,m;b,a) \neq 0$:*

$$\begin{aligned} f_G(c,m;a,b)\sum_{j=0}^{k}\left(\frac{f_G(d,c;b,a)}{f_G(d,m;b,a)}\right)^j H_{n-(m-c)k-(m-d)+(m-c)j} \\ = \frac{f_G(d,c;b,a)^{k+1}}{f_G(d,m;b,a)^k}H_n - f_G(d,m;b,a)H_{n-(m-c)(k+1)}\,, \end{aligned} \tag{2.8}$$

$$\begin{aligned} f_G(d,m;b,a)\sum_{j=0}^{k}\left(\frac{f_G(d,c;b,a)}{f_G(c,m;a,b)}\right)^j H_{n-(m-d)k-(m-c)+(m-d)j} \\ = \frac{f_G(d,c;b,a)^{k+1}}{f_G(c,m;a,b)^k}H_n - f_G(c,m;a,b)H_{n-(m-d)(k+1)} \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} f_G(d,c;b,a)\sum_{j=0}^{k}\left(-\frac{f_G(d,m;b,a)}{f_G(c,m;a,b)}\right)^j H_{n-(c-d)k+(m-c)+(c-d)j} \\ = (-1)^k\frac{f_G(d,m;b,a)^{k+1}}{f_G(c,m;a,b)^k}H_n + f_G(c,m;a,b)H_{n-(c-d)(k+1)}\,. \end{aligned} \tag{2.10}$$

2.2.2 Binomial summation identities

Lemma 3 ([1, Lemma 3])**.** *Let $\{X_n\}$ be any arbitrary sequence. Let X_n, $n \in \mathbb{Z}$, satisfy a three-term recurrence relation $X_n = f_1X_{n-a} + f_2X_{n-b}$, where f_1 and f_2 are non-vanishing complex functions, not dependent on n, and a and b are integers. Then,*

$$\sum_{j=0}^{k}\binom{k}{j}\left(\frac{f_1}{f_2}\right)^j X_{n-bk+(b-a)j} = \frac{X_n}{f_2^k}\,, \tag{2.11}$$

$$\sum_{j=0}^{k}\binom{k}{j}\frac{X_{n+(a-b)k+bj}}{(-f_2)^j}=\left(-\frac{f_1}{f_2}\right)^k X_n \tag{2.12}$$

and

$$\sum_{j=0}^{k}\binom{k}{j}\frac{X_{n+(b-a)k+aj}}{(-f_1)^j}=\left(-\frac{f_2}{f_1}\right)^k X_n\,, \tag{2.13}$$

for k a non-negative integer.

Substituting from identity (2.7) into Lemma 3, we have the results stated in the next theorem.

Theorem 4. *The following identities hold for positive integer k and arbitrary integers a, b, c, d and m for which $f_G(d,m;b,a)\neq 0$ and $f_G(c,m;b,a)\neq 0$:*

$$\sum_{j=0}^{k}\binom{k}{j}\left(\frac{f_G(d,m;b,a)}{f_G(c,m;a,b)}\right)^j H_{n-(m-d)k+(c-d)j}=\left(\frac{f_G(d,c;b,a)}{f_G(c,m;a,b)}\right)^k H_n\,, \tag{2.14}$$

$$\sum_{j=0}^{k}\binom{k}{j}\left(-\frac{f_G(d,c;b,a)}{f_G(c,m;a,b)}\right)^j H_{n-(c-d)k+(m-d)j}=\left(-\frac{f_G(d,m;b,a)}{f_G(c,m;a,b)}\right)^k H_n \tag{2.15}$$

and

$$\sum_{j=0}^{k}\binom{k}{j}\left(-\frac{f_G(d,c;b,a)}{f_G(d,m;b,a)}\right)^j H_{n+(c-d)k+(m-c)j}=\left(-\frac{f_G(c,m;a,b)}{f_G(d,m;b,a)}\right)^k H_n\,. \tag{2.16}$$

3 Application

3.1 Identities involving generalized Fibonacci numbers

In Corollary 2, let $(G_n)\equiv(F_n)$ be the Fibonacci sequence and let $(H_n)\equiv(\mathcal{F}_n)$ be the generalized Fibonacci sequence. Then, the identity of Corollary 2 reduces to

$$F_{a-b}\mathcal{F}_{n+m}=F_{m-b}\mathcal{F}_{n+a}-(-1)^{a-b}F_{m-a}\mathcal{F}_{n+b}\,. \tag{3.1}$$

The presumably new identity (3.1) subsumes most known three term recurrence relations involving Fibonacci numbers, Lucas numbers and the generalized Fibonacci numbers. We will give a couple of examples to illustrate this point.

Incidentally, identity (3.1) can also be written as

$$F_{a-b}\mathcal{F}_{n+m}=\mathcal{F}_{m-b}F_{n+a}-(-1)^{a-b}\mathcal{F}_{m-a}F_{n+b}\,. \tag{3.2}$$

Setting $a=0$, $b=m-n$ in identity (3.1) gives

$$F_{n-m}\mathcal{F}_{n+m}=F_n\mathcal{F}_n-(-1)^{n-m}F_m\mathcal{F}_m\,, \tag{3.3}$$

which is a generalization of Catalan's identity:

$$F_{n-m}F_{n+m}=F_n^2+(-1)^{n+m+1}F_m^2\,. \tag{3.4}$$

Setting $b=-a$ in identity (3.1) gives

$$F_{2a}\mathcal{F}_{n+m}=F_{m+a}\mathcal{F}_{n+a}-F_{m-a}\mathcal{F}_{n-a}\,,\tag{3.5}$$

with the special case

$$\mathcal{F}_{n+m}=F_{m+1}\mathcal{F}_{n+1}-F_{m-1}\mathcal{F}_{n-1}\,,\tag{3.6}$$

which is a generalization of the following known identity (Halton [3, Identity (63)]):

$$F_{n+m}=F_{m+1}F_{n+1}-F_{m-1}F_{n-1}\,.\tag{3.7}$$

Setting $b=2k$, $a=1$ and $b=2k$, $a=0$, in turn, in identity (3.1) produces

$$F_{2k-1}\mathcal{F}_{n+m}=F_{m-2k}\mathcal{F}_{n+1}+F_{m-1}\mathcal{F}_{n+2k}\tag{3.8}$$

and

$$F_{2k}\mathcal{F}_{n+m}=F_{m}\mathcal{F}_{n+2k}-F_{m-2k}\mathcal{F}_{n}\,.\tag{3.9}$$

Identity (3.9) is a generalization of the known identity (Vajda [9, Formula (8)]):

$$\mathcal{F}_{n+m}=F_{m-1}\mathcal{F}_{n}+F_{m}\mathcal{F}_{n+1}\,.\tag{3.10}$$

Setting $a=n$ and $b=-m$ in (3.1) produces

$$F_{2m}\mathcal{F}_{2n}=F_{n+m}\mathcal{F}_{n+m}-F_{n-m}\mathcal{F}_{n-m}\,.\tag{3.11}$$

3.2 Identities involving generalized Pell numbers

Since $P_0=0$ and $P_{b-a}=(-1)^{a-b-1}P_{a-b}$ for all $a,b\in\mathbb{Z}$ just like in the Fibonacci case; we find that the Pell relations derived from the identity of Corollary 2, (with $(G_n)\equiv(P_n)$, the Pell sequence, and $(H_n)\equiv(\mathcal{P}_n)$, the generalized Pell sequence), are identical to those derived in section 3.1. Thus, we have

$$P_{a-b}\mathcal{P}_{n+m}=P_{m-b}\mathcal{P}_{n+a}-(-1)^{a-b}P_{m-a}\mathcal{P}_{n+b}\,,\tag{3.12}$$

and a couple of special instances:

$$P_{n-m}\mathcal{P}_{n+m}=P_{n}\mathcal{P}_{n}-(-1)^{n-m}P_{m}\mathcal{P}_{m}\,,\tag{3.13}$$

$$P_{2a}\mathcal{P}_{n+m}=P_{m+a}\mathcal{P}_{n+a}-P_{m-a}\mathcal{P}_{n-a}\,,\tag{3.14}$$

$$P_{2k-1}\mathcal{P}_{n+m}=P_{m-2k}\mathcal{P}_{n+1}+P_{m-1}\mathcal{P}_{n+2k}\,,\tag{3.15}$$

$$P_{2k}\mathcal{P}_{n+m}=P_{m}\mathcal{P}_{n+2k}-P_{m-2k}\mathcal{P}_{n}\tag{3.16}$$

and

$$P_{2m}\mathcal{P}_{2n}=P_{n+m}\mathcal{P}_{n+m}-P_{n-m}\mathcal{P}_{n-m}\,.\tag{3.17}$$

From identity (3.13), we see that Pell numbers also obey Catalan's identity:

$$P_{n-m}P_{n+m}=P_n^2+(-1)^{n+m+1}P_m^2\,.\tag{3.18}$$

We have the following particular cases of identity (3.14):

$$P_{2a}P_{n+m}=P_{m+a}P_{n+a}-P_{m-a}P_{n-a}\tag{3.19}$$

and

$$P_{2a}Q_{n+m}=P_{m+a}Q_{n+a}-P_{m-a}Q_{n-a}\,,\tag{3.20}$$

with the special evaluations:

$$2P_{n+m}=P_{m+1}P_{n+1}-P_{m-1}P_{n-1}\tag{3.21}$$

and

$$2Q_{n+m}=P_{m+1}Q_{n+1}-P_{m-1}Q_{n-1}\,.\tag{3.22}$$

3.3 Identities involving generalized Jacobsthal numbers

With $(G_n) \equiv (J_n)$, the Jacobsthal sequence, and $(H_n) \equiv (\mathcal{J}_n)$, the generalized Jacobsthal sequence, the identity of Corollary 2 reduces to

$$J_{a-b}\mathcal{J}_{n+m} = J_{m-b}\mathcal{J}_{n+a} - (-1)^{a-b}2^{a-b}J_{m-a}\mathcal{J}_{n+b}\,. \tag{3.23}$$

Proceeding as in section 3.1, we have the following particular instances of identity (3.23):

$$J_{n-m}\mathcal{J}_{n+m} = J_n\mathcal{J}_n - (-1)^{n-m}2^{n-m}J_m\mathcal{J}_m\,, \tag{3.24}$$

$$J_{2a}\mathcal{J}_{n+m} = J_{m+a}\mathcal{J}_{n+a} - J_{m-a}\mathcal{J}_{n-a}\,, \tag{3.25}$$

$$J_{2k-1}\mathcal{J}_{n+m} = J_{m-2k}\mathcal{J}_{n+1} + J_{m-1}\mathcal{J}_{n+2k}\,, \tag{3.26}$$

$$J_{2k}\mathcal{J}_{n+m} = J_m\mathcal{J}_{n+2k} - J_{m-2k}\mathcal{J}_n \tag{3.27}$$

and

$$J_{2m}\mathcal{J}_{2n} = J_{n+m}\mathcal{J}_{n+m} - J_{n-m}\mathcal{J}_{n-m}\,. \tag{3.28}$$

Identity (3.24) is a generalization of

$$J_{n-m}J_{n+m} = J_n^2 + (-1)^{n+m+1}2^{n-m}J_m^2\,, \tag{3.29}$$

which is the Jacobsthal version of Catalan's identity.

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