A PROOF TO BENDORD'S LAW

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Abstract. Since the first digit phenomena was discovered by [2] and discovered again many years after [1], its still like an empirical law, but [3] got a mathematical proof. We show in this paper another proof of Benford's Law. The idea starts with the problem of to find the first digit of a power. Then we deduced a function to calculate the first digit of any power a^j called L_f function. The theorem 1.2 its a consequence of the periodicity of the L_f function.

1. Introduction

Definition 1.1. We define the constant σ_a as

$$\sigma_a = \log_a 10$$

For example, $\sigma_2 = \log_2 10 = 3.3219280$. By the way, we have:

$$a^{\sigma_a} = 10$$

Theorem 1.2. *The first digit of a^n is $\lfloor a^{n \text{ mod } \sigma_a} \rfloor$*

2. Proof of theorem 1.2

To proof the theorem 1.2 we start with the number of digits problem of any integer number N, after we construct a function to find the first digit of the power a^n. Finally, the periodic property of L_f function on lemma 2.2 completes this proof.

To find the number of digits of any natural number N we use:

$$1 + \lfloor \log N \rfloor$$

Then the first digit of N will be:

$$\left\lfloor \frac{N}{10^{\lfloor \log N \rfloor}} \right\rfloor$$

Doing N a power of base a and expoent j, ($N = a^j$):

$$\left\lfloor \frac{a^j}{10^{\lfloor \log a^j \rfloor}} \right\rfloor$$

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Now, using the change of base property in the logarithm $\log a^j$ to the base a:

$$\log a^j = \frac{\log_a a^j}{\log_a 10} = \frac{j}{\log_a 10}$$

By the definition 1.1 : $\sigma_a = \log_a 10$ and $10 = a^{\sigma_a}$, i.e., $10^{\Upsilon} = (a^{\sigma})^{\Upsilon}$. Thus,

$$\left\lfloor \frac{a^j}{10^{\lfloor \log a^j \rfloor}} \right\rfloor = \left\lfloor \frac{a^j}{10^{\lfloor \frac{j}{\sigma} \rfloor}} \right\rfloor = \left\lfloor \frac{a^j}{a^{\sigma \lfloor \frac{j}{\sigma} \rfloor}} \right\rfloor = \left\lfloor a^{j-\sigma \lfloor \frac{j}{\sigma} \rfloor} \right\rfloor = \lfloor a^{j-\sigma\Upsilon} \rfloor$$

Where, $\Upsilon = \lfloor j/\sigma \rfloor$. So the first digit of a^j is given by the function $L_f(j)$ (Last is first):

$$L_f(j) = \lfloor a^{j-\sigma\Upsilon} \rfloor \qquad 2.1$$

Lemma 2.1. *Let the notation $\sigma_a = \log_a 10$, then the first digit of a^j, $j \in \mathbb{N}$ and $a \in \mathbb{R}$, is obtained by the L_f function:*

$$L_f(j) = \left\lfloor a^{j-\sigma \lfloor \frac{j}{\sigma} \rfloor} \right\rfloor \qquad 2.2$$

Lemma 2.2. *The L_f function is periodic in its domain, with domain equal σ :*

$$L_f(j+\sigma) = L_f(j) \qquad 2.3$$

Proof. Replace $j = j + \sigma$ in 2.2

$$\begin{aligned}
\left\lfloor a^{j-\sigma \lfloor \frac{j}{\sigma} \rfloor} \right\rfloor &= \left\lfloor a^{(j+\sigma)-\sigma \lfloor \frac{j+\sigma}{\sigma} \rfloor} \right\rfloor \\
&= \left\lfloor a^{(j+\sigma)-\sigma \lfloor \frac{j}{\sigma}+1 \rfloor} \right\rfloor \\
&= \left\lfloor a^{(j+\sigma)-\sigma \lfloor \frac{j}{\sigma} \rfloor - \sigma} \right\rfloor \\
&= \left\lfloor a^{j-\sigma \lfloor \frac{j}{\sigma} \rfloor} \right\rfloor \\
&= \lfloor a^{j-\sigma\Upsilon} \rfloor
\end{aligned}$$

□

Remark 2.3. For all expoent n (such $n > \sigma$) the first digit of a^n is given by the L_f function. But, using 2.2 the first digit can be obtained from the remaind division of n by σ. It's complete the proof of 1.2.

3. The conection between the L function and The Benford's Law

Theorem 3.1. *Let the set $A = \{a^1, a^2, ..., a^k\}$ with $a, k \in \mathbb{N}$. Let $d \in \mathbb{N}$ $(1 \leq d \leq 9)$. The density, ρ_d, of numbers of elements in A such the first digit n is*

$$\rho_d = \left\lfloor k \log\left(1 + \frac{1}{d}\right)\right\rfloor$$

$\forall\ a \in \mathbb{R}$. *In another words, the set A has a Benford's distribuition.*

To proof theorem 3.1 we show the general graph of the $L_f(x)$, if we prove that the distribution on interval $[0, \sigma]$ is a Benford's distribution then will be valid across the domain of L_f, since the function is periodic.

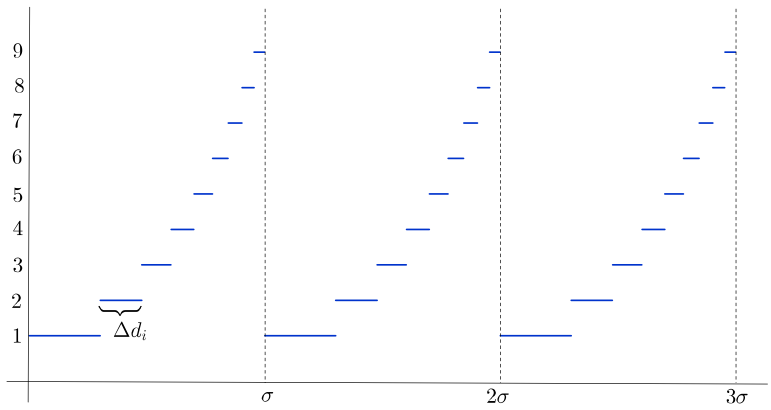

Figure 1. The L_f graph function

Proof. First let the inequality

$$d \leq a^x < d + 1 \qquad 3.1$$

$d = 1,\ 2, \ldots,\ 9$. Solving we find an interval to values of x such satisfies the inequality. The lenght of this interval is the same at Δd_i on L_f graph. Applying the logarithm on both sides of the inequality:

$$x = \log_a (d + 1) - \log_a (d) = \log_a \left(1 + \frac{1}{d}\right)$$

$$\Delta d_i = \log_a \left(1 + \frac{1}{d}\right) \qquad 3.2$$

Then, the distribution $p(d_i)$ of the elements in A with first digit d_i is given:

$$p(d_i) = \frac{\Delta d_i}{\sigma_a} \qquad 3.3$$

Replace $\sigma_a = \log_a 10$ we find:

$$p(d_i) = \frac{\log_a\left(1+\frac{1}{d_i}\right)}{\log_a 10} = \log\left(1+\frac{1}{d_i}\right) \qquad 3.4$$

The result implies that the distribution is invariant of the base. So, for all exponential phenomena has a Bendord's distribution and we showed that the L_f function is behind the first digit phenomenon. □

References

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