The 2*n*-2 Lines Proof of the *n* x *n* Points Puzzle

Marco Ripà

sPIqr Society, World Intelligence Network
Rome, Italy
e-mail: marcokrt1984@yahoo.it

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Abstract: This paper offers the first proof that the minimal solution of the *n* x *n* dots puzzle, for any $n \geq 3$, counts 2*n*-2 lines. Furthermore, a general criterion to solve any *n* x *n* grid is given.

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1 Introduction

The classic *nine dots problem* by Samuel Loyd [2] states that we have to "(…) draw a continuous line through the center of all the eggs so as to mark them off in the fewest number of strokes" (see [1, 3]).

We can naturally extend the puzzle to a generic *n* x *n* points grid, asking: "How many straight lines connected at their end-points we need to join *n* x *n* points arranged in a regular grid, formed by *n* equidistant rows and *n* equidistant columns?".

Since 2013, Ripà and others have investigated this problem and its extensions to a multidimensional space, but a strict proof of the best solution, for any $n \in \mathbb{N} - \{0\}$, have not been given yet [4, 5].

The aim of this paper is to show that, for any $n > 2$, the *n* x *n* points problem cannot be solved using less than 2*n*-2 lines, providing a fixed pattern to solve every *n* x *n* puzzle with exactly 2*n*-2 lines [6].

2 Trivial cases: *n* < 5

Let $h(n)$ represent the minimum number of straight lines connected at their end-points to join the n x n points of the grid, it is trivial to see that $h(1) = 1$ and $h(2) = 3$. In fact, we need one line to fit a single point and 3 lines to join two pairs of points laying on two parallel segments.

In this section, our goal is to prove that $h(n) = 2 \cdot n - 1$, for $n = 3$ and $n = 4$.

We know [4] that it is impossible to join more than

$$n + (n - 1) \cdot (h(n) - 1) \quad (1)$$

points using h straight lines connected at their end-points. Since the points are n^2, it follows that

$$n + (n - 1) \cdot (h(n) - 1) \leq n^2 \quad (2)$$

Thus,

$$h(n) = \left\lceil \frac{n^2 - n}{n - 1} \right\rceil + 1 \quad (3)$$

Assume that $n \neq 1$, the (3) can be simplified as

$$h(n) = n + 1 \quad (4)$$

The (3) ensures that we cannot solve the 3 x 3 puzzle using less than 4 lines: $h(3) = 3 + 1$.

In order to show that $h(4) > 5$, let we assume $r \geq 2$: we count how many patterns of $r + 1$ consecutive lines fit a total of $n + r \cdot (n - 1)$ points. It is trivial to observe how exist only 3 patterns such that $r \geq 2$, because a perfect 2-dimensional pattern cannot include segments which do not lay on the four sides of the perimeter of the square, since (at least one of) them would join $n - 2$ unvisited points (at most), and this is unacceptable if we want to join $n + (h - 1) \cdot (n - 1)$ points with $r + 1$ lines connected at their end-points.

The same rule implies that we cannot use "diagonal" lines different from the 6 shown in Figure 1.

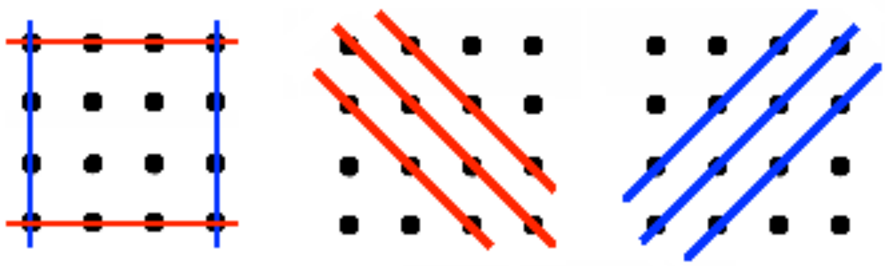

Figure 1. Let $r \geq 2$, there are only 10 lines that are able to join $n + r \cdot (n - 1)$ unvisited points.

It follows that, for any $n \geq 3$, $\exists!$ pattern such that $r = 3$ (out of 3 patterns with $r \geq 2$). The only exception to this rule implies $n = 5$, as shown by Figure 2.

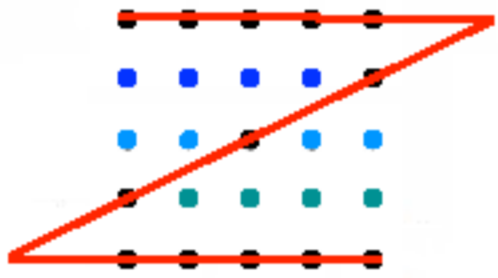

Figure 2. The *special case* for $n = 5, r = 2$: it is impossible to spend less than 5 connected lines to join 12 points which are equally distributed along 3 parallel segments contained in a 2x4 box.

$r = 3$ only if n is an odd value (formally, $n = 2 \cdot m + 1,\ m \in \mathbb{N} - \{0\}$). In this case, the related pattern is the same one we use to solve the classic 3 x 3 puzzle (see Figure 3).

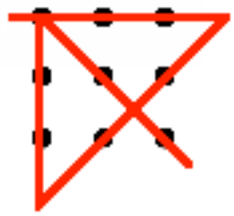

Figure 3. The only pattern with $r = 3$ implies $n = 2 \cdot m + 1,\ m \in \mathbb{N} - \{0\}$.

The consideration above ($r_{max} = 3$) is sufficient to prove that $h(4) = 6$.

Let $r = 3$ and $n = 4$ (even if we already know that this cannot happen, since 4 is an even number), there is no way to join more than $n + r \cdot (n - 1) + n - 2 = 15$ points with 5 straight lines connected at their end-points. Thus, $h_{min}(4) = h(4) = 6$, and a solution is shown in Figure 4.

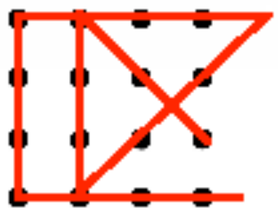

Figure 4. Solving the 4 x 4 points problem with 6 lines. $\square$

3 Proof for any $n \geq 5$

We are finally ready to prove the general case: $n \geq 5$.

First of all, we provide a general criterion to solve any $n\text{x}n$ grid ($n > 2$) using exactly $2n - 2$ lines [4]. The pattern to achieve our goal (solving the puzzle *inside the box* too) is shown in Figure 5.

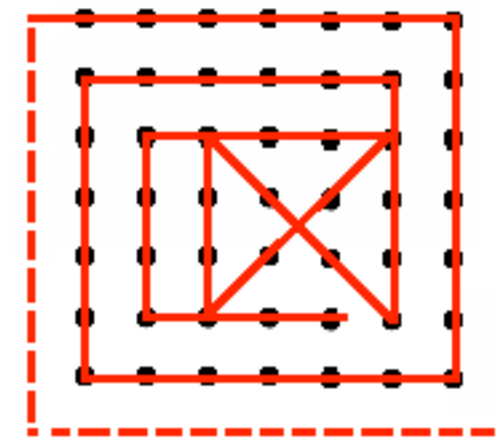

Figure 5. The square-spiral pattern solves any $n\text{x}n$ grid with $2n - 2$ lines, for any $n \geq 5$.

Taking a look at Figure 6, we study the 3 basic patterns with $r \geq 2$ starting from the worst-case scenario: $n = 2 \cdot m + 1,\ m \in \mathbb{N} - \{0, 1\}$.

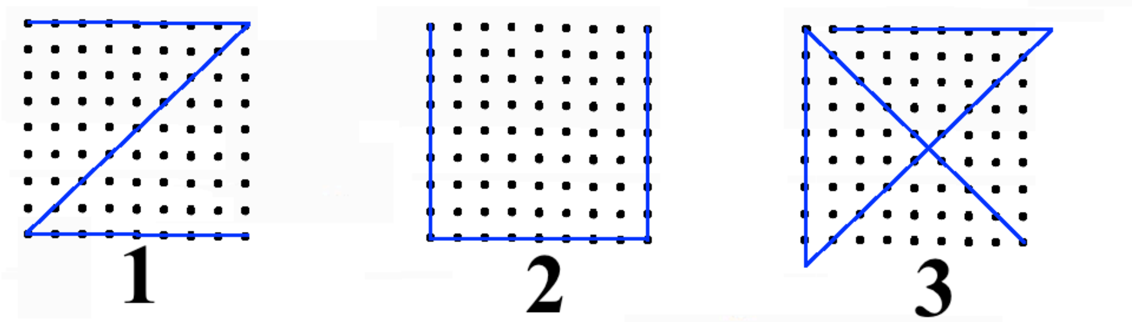

Figure 6. These are the only 3 patterns with $r \geq 2$, for any $n = 2 \cdot m + 1,\ m \in \mathbb{N} - \{0, 1, 2\}$. If $n = 5$, there would be also the pattern shown in Figure 2 (already discharged).

The basic idea is to start from each one of the 3 patterns above, extending them from the ends, with the aim to maximize the number of new dots visited by small sets of connected lines. In this way, by comparison, it is possible to write down the corresponding series which maximize the number of visited points after $2n$-3 lines, showing that this value is $< n^2$, for any $n \geq 5$.

Taking into account the pattern 1 by Figure 6, we have the following scheme:

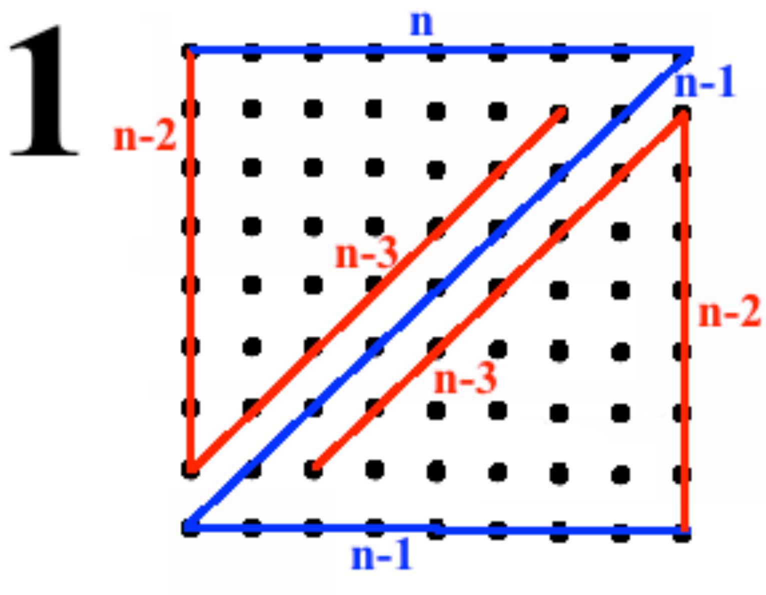

Figure 7. Best development of the first pattern shown in Figure 6: new visited dots are indicated.

Following the pattern above, we have the series:

$$n + 2 \cdot (n-1) + 2 \cdot (n-2) + 2 \cdot (n-3) + 2 \cdot (n-4) + \cdots \tag{5}$$

Thus, in order to prove that the square-spiral pattern (spending $2 \cdot n - 2$ lines) is the best approach at all, we have to show that $\sum_{i=1}^{h(n)-1}(n-i) < n^2 \leftrightarrow \sum_{i=1}^{2n-3}(n-i) < n^2$, for any $n \geq 5$. Since $\sum_{i=1}^{2n-3}(n-i) = n^2 - n - 2$, this is true for any $n \geq 5$, the first pattern cannot improve the general $2 \cdot n - 2$ lines solution based on the square-spiral pattern (e.g., $h(5) = 7$ iff $7 \cdot 5 - 12 < 5^2$ is false, hence $h(5) = 8$).

Considering the second scheme by Figure 6, we derive a few equivalent patterns.

Lemma 1: Let $n = 2 \cdot m + 1$, $\nexists$ pattern with at least 10 straight lines connected at their end-points which improves the result described in (7), as shown in Figures 7, 8 and 9.

It easy to understand the Lemma 1 looking at the comparison between the two best patterns available: the first one is the best pattern for $7 \leq h(n) \leq 9$, while the second one cannot be improved for any $h(n) \geq 10$. Thus, it is sufficient to calculate how many "new" points it is possible to visit with the first 7 lines by the pattern 2a (given that n is odd by hypothesis): if the value is < 5^2 we are allowed to consider the pattern 2b only.

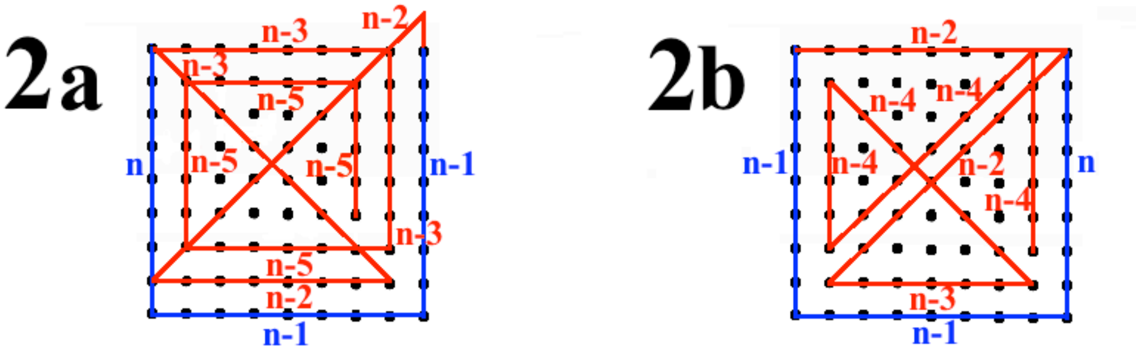

Figure 8. Best developments of the pattern 2 by Figure 6: new visited dots are indicated.

The series related to the pattern 2a is as follows:

$$n + 2 \cdot (n-1) + 2 \cdot (n-2) + 3 \cdot (n-3) + 4 \cdot (n-5) + \cdots \tag{6}$$

The series related to the pattern 2b is as follows:

$$n + 2 \cdot (n-1) + 2 \cdot (n-2) + n - 3 + 4 \cdot (n-4) + 2 \cdot (n-5) + \cdots \tag{7}$$

Let $n = 5$, from the (6) we get $h(5) = 7$ iff $7 \cdot 5 - 12 \geq 5^2$.
As (6) $\leq$ (7), it is sufficient to verify that

$$n + 2 \cdot (n-1) + 2 \cdot (n-2) + n - 3 + 4 \cdot (n-4) + 2 \cdot (n-5) + \cdots < n^2 \tag{8}$$

For any $n \geq 7$, the (8) is true since $1 + \sum_{i=1}^{n-2}(n-i) < n^2 \leftrightarrow 1 + n^2 - n - 2 + n < n^2$ (e.g., $h(7) = 11$ iff $1 + 11 \cdot 7 - 30 < 7^2$ is false, hence $h(7) = 12$), therefore neither the pattern 2a nor the pattern 2b improves the square-spiral solution.

Now, let us consider the last pattern introduced in Figure 6; the final scheme is as follows (Figure 9):

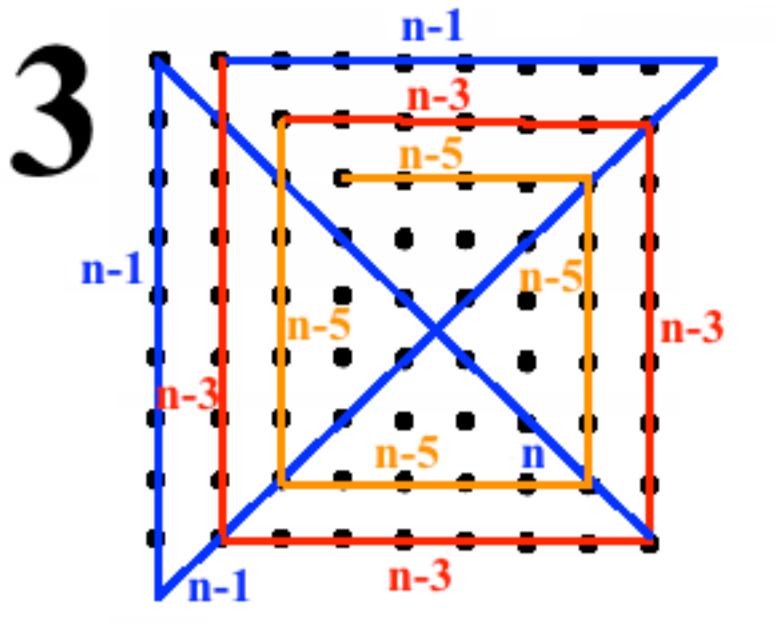

Figure 9. Best development of the pattern 3 by Figure 6: new visited dots are indicated.

The related series is:

$$n + 3 \cdot (n-1) + 4 \cdot (n-3) + 4 \cdot (n-5) + \cdots =$$
$$= 1 + 4 \cdot (n-1) + 4 \cdot (n-3) + 4 \cdot (n-5) + \cdots \quad (9)$$

It is easy to note that, for any $h(n) \geq 10$, (9) $\leq$ (8), while the first 7 segments joins only $7 \cdot 5 - 12 < 25$ points.

This concludes the proof that, for any odd value of n, $\nexists$ a pattern such that $h(n) < 2 \cdot (n-1)$.

It is necessary to study also the 3 patterns with $r = 2$ (see Figure 6), for any $n = 2 \cdot m,\ m \in \mathbb{N} - \{0, 1, 2\}$ (since $r < 3$ for any even value of n), in order to prove that $h_{min}(n) = h(n) = 2 \cdot n - 2$ for any $n > 2$.

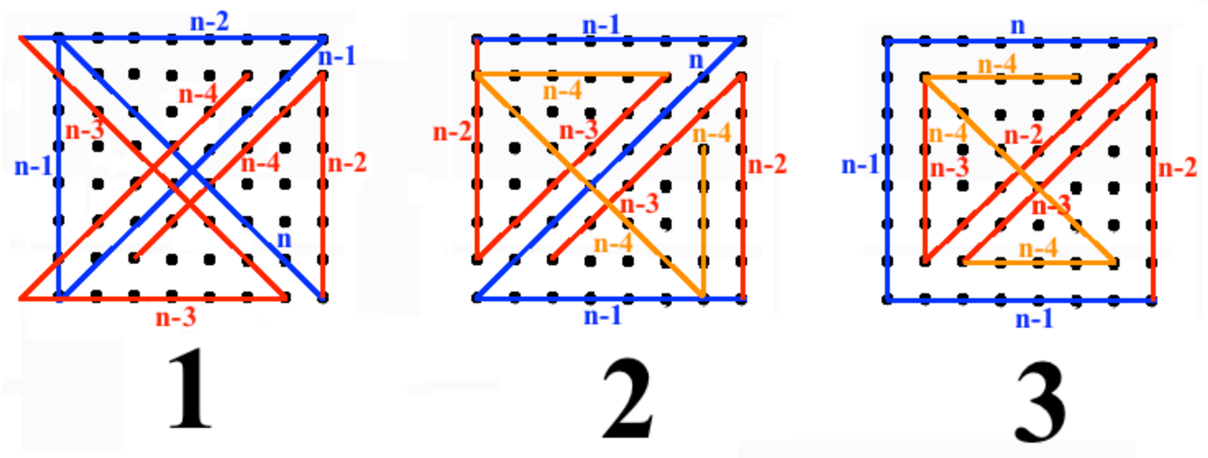

Figure 10. Best developments of the 3 patterns with $r > 1$, for any $n = 2 \cdot m, \forall m > 2$.

Following the 3 patterns represented in Figure 10, we get the series (10), (11), (12).

Pattern 1: $n + 2 \cdot (n-1) + 2 \cdot (n-2) + 2 \cdot (n-3) + 2 \cdot (n-4) + 2 \cdot (n-5) + 2 \cdot (n-6) + \cdots$ (10)

Pattern 2: $n + 2 \cdot (n-1) + 2 \cdot (n-2) + 2 \cdot (n-3) + 3 \cdot (n-4) + 4 \cdot (n-6) + 4 \cdot (n-8) \ldots$ (11)

Pattern 3: $n + 2 \cdot (n-1) + 2 \cdot (n-2) + 2 \cdot (n-3) + 3 \cdot (n-4) + n - 5 + 2 \cdot (n-6) + \cdots$ (12)

Let $h(n) \geq 11$, by comparison, we see that (11) $\leq$ (12) and (10) $\leq$ (12) $\leq$ (7), for any $n = 2 \cdot m,\ m \in \mathbb{N} - \{0, 1, 2, 3\}$, since we assume $h(n) \coloneqq 2 \cdot n - 3$.

In order to complete the proof, it is necessary to check that $h(6) > 9$. This is clear, since $9 \cdot 6 - 20 < 36$, by the (10), (11) and (12).

Finally, we can conclude that, for any $n \in \mathbb{N} - \{0\}$, $\nexists$ a pattern such that $h(n) < 2 \cdot n - 2$. Therefore, assuming $n \geq 5$, the spiral-square solution uses the minimum number of straight lines connected at their end-points to solve any n x n grid. □

In conclusion, we observe that the best pattern shown on these pages (the pattern 2b associated to the (7)) is isomorphic to the square-spiral scheme, represented in Figure 5.

These two patters are indeed the same one, as shown in Figure 11.

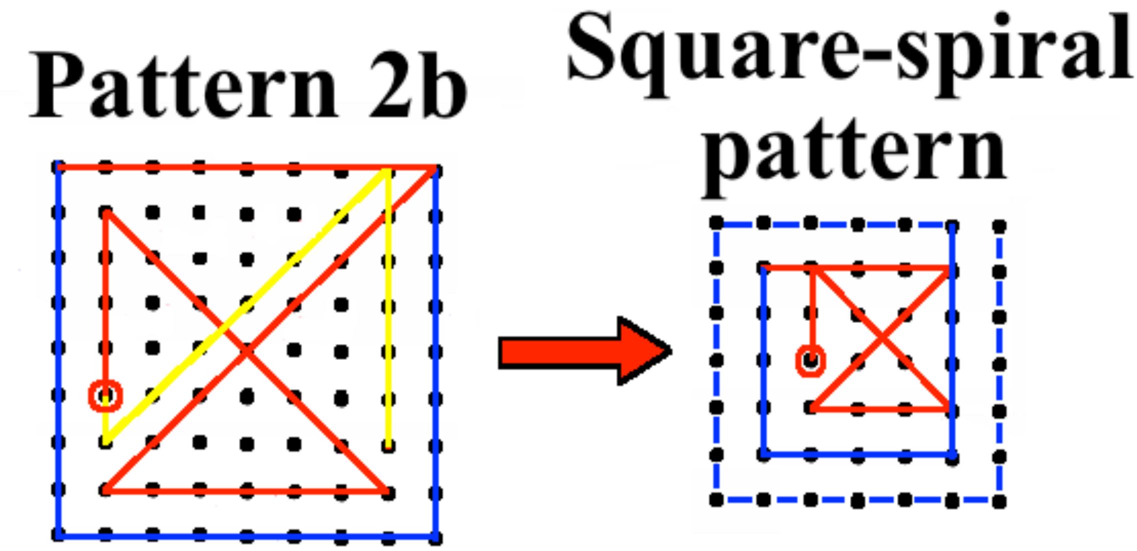

Figure 11. The best pattern we have considered in these pages corresponds to the well known square-spiral scheme: the solution cannot involve less than $2 \cdot n - 2$ connected lines.

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