The complete Solution for existence and smoothness of Navier-Stokes equation

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Abstract: *The motive of this paper is to put forward a general solution to Navier-stokes equation which describes the motion of viscous fluid substances, derived by applying Newton's second law to fluid motion. These equations are the set of coupled differential equations, which are too difficult to solve analytically.*

Introduction: The Navier-Stokes equation was proposed by Claude-Louis Navier and George Gabriel Stokes in the 19th century, which described the motion of viscous fluids. In mathematics, these equations are a system of nonlinear partial differential equations for vector fields of any size. Since, in three dimensions, there are three equations and four unknowns (three scalar velocities and the pressure) so, an extra equation is needed in order to completely solve the Navier-Stokes equations. So far, continuity equations are used as the fourth equation for incompressible fluids that describes the conservation of mass of the fluid.In this paper, I intend to define an exact and smooth solution without using the continuity equation.

General Solution to the problem: In order to define the existence and smoothness of the equation, let us first consider the navier-stokes equation given by equation (1) and, the R.H.S of equation (1) represents the net force on the body shown by equation (2). Hence,

$$.\rho \frac{d\overline{V}}{dt} = -\nabla P + \rho\overline{g} + \mu\nabla^2\overline{V} \tag{1}$$

Where,

$\overline{V} = u, v, w \; in \; x, y, z \; direction \; respectively$ with initial conditions,

$V_{initial} = V_0 = V_x^0, \; V_y^0, \; V_z^0 \; in \; x, y, z \; directions \; respectively$

$$\overline{F} = -\nabla P + \rho\overline{g} + \mu\nabla^2\overline{V} \tag{2}$$

Now, according to newton's second law, In an inertial reference frame, the vector sum of the forces F on an object is equal to the mass *m* of that object multiplied by the acceleration a of the object and is given by:

$$\overline{F} = M.\overline{a} \tag{3}$$

Since mass and acceleration is given by $\rho\, dx.dy.dz$, $\frac{dV}{dt}$ substituting in equation (3), we have:

$$F = (\rho\, dx.dy.dz)\,\frac{dV}{dt} \qquad (4)$$

Also, the energy of the system is given by $dE = F.ds$, substituting F from equation (4) yields,

$$dE = (\rho.dx.dy.dz)\frac{dV}{dt}.ds$$

On rearranging the terms,

$$\frac{dE}{dx.dy.dz} = \rho.V\,dV \qquad (5)$$

The term on the L.H.S of equation (5) represents the value of force in x, y, z directions respectively, hence solving the above equation for x direction gives:

$$df_x = \rho.V\,dV \qquad (6)$$

Integrating both sides, with force ranging from 0 to $(f_x)^2$ with initial and final velocities being u, V_x^0 respectively,

$$\int_0^{(f_x)^2} df_x = \rho \int_{V_x^0}^{u} V\,dV$$

$$(f_x)^2 = \frac{\rho}{2}\left[u^2 - (V_x^0)^2\right] \qquad (7)$$

The force for y and z components can be determined similarly and is given by

$$(f_y)^2 = \frac{\rho}{2}\left[v^2 - (V_y^0)^2\right] \qquad (8)$$

$$(f_z)^2 = \frac{\rho}{2}\left[u^2 - (V_z^0)^2\right] \qquad (9)$$

Adding equation (7), (8) and (9) respectively

$$(f_x)^2 + (f_y)^2 + (f_z)^2 = \frac{\rho}{2}\left[u^2 - (V_x^0)^2\right] + \frac{\rho}{2}\left[v^2 - (V_y^0)^2\right] + \frac{\rho}{2}\left[w^2 - (V_z^0)^2\right]$$

On rearranging the terms

$$\left[\sqrt{(f_x)^2 + (f_y)^2 + (f_z)^2}\right]^2 = \frac{\rho}{2}\left[\left(\sqrt{u^2 + v^2 + w^2}\right)^2 - \left(\sqrt{(V_x^0)^2 + (V_y^0)^2 + (V_z^0)^2}\right)^2\right] \qquad (10)$$

Here, $[\sqrt{(f_x)^2 + (f_y)^2 + (f_z)^2}]$, $(\sqrt{u^2 + v^2 + w^2})$ and $(\sqrt{(V_x^0)^2 + (V_y^0)^2 + (V_z^0)^2}$ is net force ($\overline{F}$) final velocity ($\overline{V}$) and initial velocity (V^0) respectively. Hence substituting these values in equation (10) gives

$$F^2 = \frac{\rho}{2}.(V^2 - (V^0)^2)$$

On rearranging,

$$V = \sqrt{\frac{2}{\rho}F^2 + (V^0)^2} \tag{11}$$

Now, in order to solve navier-stokes equation we need to consider two cases in equation (11). One in which the initial velocity is equal to zero and the other in which initial velocity is not equal to zero.

Case 1: When initial velocity (V^0) is equal to zero.

When initial velocity is equal to zero the term V^0 in equation (11) becomes equal to zero and it reduces down to the equation shown below:

$$V = \sqrt{\frac{2}{\rho}}\mathrm{F} \tag{12}$$

Substituting the value of equation (12) in equation (1)

$$\rho\frac{d(\sqrt{\frac{2}{\rho}}F)}{dt} = -\nabla P + \rho\overline{g} + \mu\nabla^2\overline{V} \tag{13}$$

The term on the R.H.S of equation (13) is nothing but force as shown in equation (2), thus substituting F in place of $-\nabla P + \rho\overline{g} + \mu\nabla^2\overline{V}$ gives

$$\rho\frac{d(\sqrt{\frac{2}{\rho}}F)}{dt} = \mathrm{F}$$

$$\sqrt{2\rho}\,\frac{dF}{dt} = F$$

$$\frac{dF}{F} = \frac{dt}{\sqrt{2\rho}}$$

Integrating both sides

$$\int_0^F \frac{dF}{F} = \int_0^t \frac{dt}{\sqrt{2\rho}} \Rightarrow\!\!> \log F = \frac{t}{\sqrt{2\rho}} \Rightarrow\!\!> F = e^{\frac{t}{\sqrt{2\rho}}} \qquad (14)$$

Substituting the value of equation (14) in equation (12) and (2) respectively yields,

$$V = \sqrt{\frac{2}{\rho}}\, e^{\frac{t}{\sqrt{2\rho}}} \qquad (15)$$

$$\nabla P = \rho\overline{g} + \mu\nabla^2\overline{V} - e^{\frac{t}{\sqrt{2\rho}}} \qquad (16)$$

It can be clearly seen that, at any particular time, there always exist a defined value of velocity and pressure gradient thus completing the solution.

Case 2 : When initial velocity is not equal to zero

When initial velocity is not equal to zero, then equation (11) remains as it is:

$$V = \sqrt{\frac{2}{\rho}F^2 + (V^0)^2}$$

Substituting the value of V in equation (1) yields,

$$.\rho\frac{d\sqrt{\frac{2}{\rho}F^2 + (V^0)^2}}{dt} = -\nabla P + \rho\overline{g} + \mu\nabla^2\overline{V} \qquad (17)$$

Again, the term on the R.H.S of equation (13) is nothing but force as shown in equation (2), thus substituting F in place of $-\nabla P + \rho\overline{g} + \mu\nabla^2\overline{V}$ gives

$$.\rho\frac{d\sqrt{\frac{2}{\rho}F^2 + (V^0)^2}}{dt} = F \qquad (18)$$

Since, for a given motion, initial velocity remains the same, it can be treated as a constant. So,

$$.\rho\frac{d\sqrt{\frac{2}{\rho}F^2 + (V^0)^2}}{d(\frac{2}{\rho}F^2 + (V^0)^2)}.\frac{d(\frac{2}{\rho}F^2 + (V^0)^2)}{dt} = F$$

$$\rho \frac{1}{2\sqrt{\frac{2}{\rho}F^2 + (V^0)^2}} \cdot \frac{4}{\rho} F \cdot \frac{dF}{dt} = F$$

On rearranging,

$$\frac{dF}{\sqrt{\frac{2}{\rho}F^2 + (V^0)^2}} = \frac{dt}{2\rho}$$

Integrating both side, with Force (F) limiting from 0 to F for time 0 to t which yields,

$$F = \frac{(V^0)\rho}{2} \tan\left(\frac{t}{\sqrt{2}\,\rho^{\frac{3}{2}}}\right) \tag{19}$$

Substituting the value of F in equation (11) and equation (2) respectively, yields:

$$V = \sqrt{\left(V^0 \tan\left(\frac{t}{\sqrt{2}\,\rho^{\frac{3}{2}}}\right)\right)^2 + (V^0)^2}$$

$$= V^0 \sqrt{\left(1 + \tan^2\left(\frac{t}{\sqrt{2}\,\rho^{\frac{3}{2}}}\right)\right)}$$

$$V = V^0 \sec\left(\frac{t}{\sqrt{2}\,\rho^{\frac{3}{2}}}\right) \tag{20}$$

And,

$$\nabla P = \rho\overline{g} + \mu\nabla^2\overline{V} - \frac{(V^0)\rho}{2} \tan\left(\frac{t}{\sqrt{2}\,\rho^{\frac{3}{2}}}\right) \tag{21}$$

Again, It can be clearly seen that, at any particular time, there always exist a defined value of velocity and pressure gradient thus completing the solution.

Conclusion:

It can be clearly seen from equation (15) and (16) (*when initial velocity is equal to zero*) and from equation (20) and (21) (*when initial velocity is not equal to zero*) that there always exist a smooth solutions at a given time t. In other words, In three space dimensions and time, given an initial velocity field, there always exists a vector velocity and a scalar pressure field, which are both smooth and globally defined, thus completing the solution.

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