Quark Condensate and Confining Potentials

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ABSTRACT – Two seminal ideas are considered in this paper. One of them was introduced by Tryon [Nature **246**, 396(1973)], dealing with the possibility of the universe being created from nothing. The other one was proposed by Thompson [J. Phys. **A9**, L25(1976)], in order to study the critical behavior of a cooperative system. Both ideas are implemented conjointly with the use of linear and quadratic confining potentials as a means to make estimates of the quark condensate of the QCD. In accomplishing this task, the MIT bag model by Chodos et al. [Phys. Rev. **D9**, 3471(1974)] is also taken in account.

## 1 – Introduction

QCD (Quantum Chromodynamics) is the most fundamental theory of the strong interactions, where quarks endowed with color-charges interact through the exchange of gluons. This non-abelian theory exhibits an SU(3) internal symmetry [1,2].

The QCD Lagrangian can be written in the form [3]

$$L_{QCD} = \Sigma_j \underline{\Psi}_j \,(i\gamma_\mu D^\mu + m_j)\, \Psi_j - ¼\, G_{\mu\upsilon}{}^a\, G^{\mu\upsilon}{}_a \,, \qquad (1A)$$

where

$$D^\mu = \partial^\mu + ½\, ig\, \lambda_a A^\mu{}_a \,, \qquad G^{\mu\upsilon}{}_a = \partial^\mu A^\upsilon{}_a - \partial^\upsilon A^\mu{}_a - g\, f_{abc} A^\mu{}_b A^\upsilon{}_c \,. \qquad (1B)$$

In (1), $m_j$ and $\Psi_j$ are the mass and the fermionic field of the quark of flavor j, A is the gluonic field, μ and υ are space-times indexes and a, b, c are color indexes.

At high energies, due to the asymptotic freedom behavior of the strong coupling constant, perturbation theory is a convenient tool to deal with in

QCD calculations. However its infrared sector deserves other techniques of calculations.

The aim of this work is evaluate alternatives ways of estimating the quark condensate and relating them to different potentials leading to quark confinement.

2 – Quark Condensate from Thompson's Approach

Thompson [4] has proposed heuristic approach as a means to evaluate the critical exponent of a $\Phi^4$-theory [5], which belongs to the same universality class of the Ising Model.

According one of the Thompson's prescriptions, when the lagrangian of the $\Phi^4$-theory is integrated in a certain "coherence volume", in a four-volume ($V_4$) for instance, each term of this action is separately of the order of the unity.

Let us apply Thompson's recipe to the "kinetic" contribution of the fermionic field $\Psi$ of the QCD lagrangian. We write (here we will take this prescription in a more restrictive form)

$$\left| \int (\underline{\Psi}_j \, i\gamma_\mu \, \partial^\mu \, \Psi_j) \, dV_4 \right| = 1. \tag{2}$$

In analyzing (2) we take in account that in the MIT bag model [6,7,8], quarks are confined inside a bag, subjected to the influence of the vacuum pressure at its boundary. Therefore based in (2) we have

$$\left| < \underline{\Psi} \, \Psi > \right| L^{-1} \, V_4 = 1. \tag{3}$$

We have extracted from the integral in (2) the average value of the quark condensate, and next we pass to the analysis of the L and $V_4$ terms which appear in (3).

Let us pay attention on the MIT bag model of the nucleon: three quarks with the flavors up (u) or down (d), namely uud representing the proton and ddu for the neutron, acquire their constituent masses from the QCD dynamics. We notice that the bare masses of these up and down quarks are only a few MeVs in magnitude, as compare with their constituent mass of the order of hundred of MeVs. An analogy can be established [9] with the process of crystal growing from water solutions, for instance. We put crystals seeds in contact with its saturated water solution and wait for the growing of monocrystals. Here the chromodynamical vacuum stems for the solution, and seeds are represented by the bare masses of the up and down quarks. The completion of the process is reached, when each of the constituent quarks acquires approximately one third of the nucleon mass. It seems that a length which characterizes this process can be defined as the "Compton" wavelength of a particle of mass

$$m_q = M/3, \tag{4A}$$

where $m_q$ and M are respectively the quark constituent and the nucleon masses. Thus we have

$$L = \hbar/(m_q c) \qquad \text{and} \qquad V_4 = L^4. \tag{4B}$$

Inserting the information given by (4) in (3), we get by taking $\hbar = c = 1$,

$$< \underline{\Psi}\, \Psi > = - L^{-3} = - m_q^{\,3} = - M^{3}/27. \tag{5}$$

The minus signal we adopted in (5) will be understood in the next section.

3 – The Newtonian-like Approach

Let us consider two particles of mass $m_q$ separated by a distance L, and interacting "gravitationally" with a coupling $G_s$. The potential energy associated to this attractive interaction reads

$$U_g = - G_s\, m_q^{\,2}/L. \tag{6}$$

Now, we define a very special system of units, such that besides $\hbar = c = 1$, we also will take $G_s = 1$. In this new system of units we have

$$U_{sg} = - m_q^{\,3} \equiv < \underline{\Psi}\, \Psi >. \tag{7}$$

Alternatively, we may consider

$$U_{g,\,new} = -\,G_s\, m_q^2 / R. \qquad (8)$$

In (8), R is the nucleon radius as estimated by X. Ji [10] using the MIT bag model. As quoted in [10], it is given by

$$R = 4 / M. \qquad (9)$$

Putting (4) into (8), we obtain in the new units

$$U_{sg,\,new} = -\,(3/4)\, m_q^3 = -\,M^3/36\,. \qquad (10)$$

4 – The Tryon Idea

In a paper entitled: "Is the universe a vacuum fluctuation?", E. P. Tryon [11] proposes that the creation of a certain amount of mass-energy plus the potential energy of this mass interacting with the rest of the universe sums up to zero. We can put relation (10) of this paper in a Tryon-like form. We have

$$m_q^3 + \langle \underline{\Psi}\, \Psi \rangle = 0. \qquad (11)$$

In (11) we identify $\langle \underline{\Psi}\, \Psi \rangle$ as the potential energy in the new system of units ($G_s = \hbar = c = 1$) and, $m_q^3$ as the mass-energy or kinetic energy, also expressed in this new system of units. We rewrite (11) in the form

$$K_s + U_{sg} = 0. \qquad (12)$$

At the same token, we can also write

$$K_{s,\,new} + U_{sg,\,new} = 0. \qquad (13)$$

Relation (13) implies

$$K_{sg,\,new} = (3/4)\, m_q^{\,3} = M^{\,3}/36\,. \qquad (14)$$

5 – The Linear Confining Potential

In the case of confining potentials it is possible to write an action describing the behavior of the fermionic field Ψ, namely

$$A = \int [(\, \underline{\Psi}\, i\, \partial/\partial r\, )\, \Psi + P(r)\, \underline{\Psi}\Psi]\, dV_4\,. \qquad (15)$$

Now consider the case of the linear confining potential, given by

$$P(r) = \sigma\, r, \qquad (16)$$

where σ is the string constant .

Applying Thompson's recipe [11] to the action (15), remember: "The absolute value of each term of the action A is separately equal to the unity", we have

$$\left| \int [(\, \underline{\Psi}\, i\, \partial/\partial r\, )\, \Psi]\, dV_4 \right| = \left| \int (\sigma\, r\, \underline{\Psi}\Psi\, )\, dV_4 \right| = 1. \qquad (17)$$

Assuming that the equality between integrals, corresponds to the equality between integrands, leads to the differential equation

$$i\, (d\Psi/dr) = \pm\, \sigma\, r\, \Psi. \qquad (18)$$

Performing the integration of (18), we get

$$\Psi = \Psi_0\, \exp(\, \pm\, ½\, i\, \sigma\, r^{\,2}\, ). \qquad (19)$$

Now, let us impose a boundary condition on the Ψ-function. We write

$$\pm \tfrac{1}{2}\, i\, \sigma\, r_n^{\,2} = \pm\, i\, 2\pi\, n, \qquad \text{with} \qquad n = 1,2,3,... \tag{20}$$

Relation (20) yields

$$r_n = 2\, (\pi n / \sigma)^{1/2}, \qquad \text{and} \qquad p_n = 1 / r_n = \tfrac{1}{2}\, [\sigma / (\pi n)]^{\,1/2}. \tag{21}$$

WE notice that $p_1$ corresponds to the maximum value of the momentum, and we are interested in evaluate the value of a certain volume in the momentum space, namely

$$V_{p1} = 2\, (4/3)\, \pi\, p_1^{\,3}. \tag{22}$$

The factor 2 in (21) stems for the degree of degenerescence of the spin-½ fermionic field Ψ.

Inserting the value of $p_1$, the value of p for n = 1obtained from (21) into (22), we get

$$V_{p1} = [1 / (3\sqrt{\pi})]\, \sigma^{\,3/2}. \tag{23}$$

The value of σ can be obtained from the paper dealing with quark confinement [12], and by considering the strong coupling ($\alpha_s$) value of 4 ⁄ 9, as estimated in [9]. We have from reference [12]

$$\sigma = m_q^{\,2} / \alpha_s = 9\, m_q^{\,2} / 4 = M^{\,2} / 4. \tag{24}$$

Putting the value of σ estimated in (24) into (23), we finally obtain

$$V_{p1} = M^{\,3} / (24\sqrt{\pi}) \approx M^{\,3} / 42.5. \tag{25}$$

Now we are led to interpret $V_{p1}$ as the "kinetic energy" in the new system of units ($G_s = \hbar = c = 1$) and making use of Tryon's idea we can write

$$K_s + U_{sg} = V_{p1} + < \underline{\Psi}\, \Psi > = 0. \qquad (26)$$

Relations (24) and (25) permit us estimate the value of the quark condensate in a linear confining potential (LCP) as

$$< \underline{\Psi}\, \Psi >_{LCP} = - M^3 / (24\sqrt{\pi}) . \qquad (27)$$

## 6 – The Quadratic Confining Potential

For the quadratic confining potential, being k the spring constant, we can write

$$P(r) = ½\, k\, r^2. \qquad (28)$$

Working in an analogous way we have done in the previous section, we get

$$\Psi = \Psi_0 \exp(\pm\, i\, k\, r^3 / 6). \qquad (29)$$

The requirement on boundary conditions implies

$$\pm\, i\, k\, r_n^3 / 6 \ = \pm\, i\, 2\pi n, \qquad n = 1, 2, 3, \ldots \qquad (30)$$

Pursuing further we obtain

$$r_1^3 = 12\pi / k \qquad \text{and} \qquad p_1^3 = k / (12\pi). \qquad (31)$$

The volume in the momenta space, in the case of the quadratic potential, will be given by

$$V_{p1} = (8\pi / 3)\, p_1^3 = 8\, k / 36 . \qquad (32)$$

Now let us estimate the spring constant k. We assume the mass of the nucleon is given by the energy of the ground state of a four-dimensional harmonic oscillator (by taking space and time et equal footing) and write

$$4(\tfrac{1}{2}\hbar\omega) = Mc^2, \qquad \text{or} \qquad 2\omega = M, \qquad (\hbar = c = 1). \tag{33}$$

We consider $\mu = M/2$, the reduced mass of the self-interacting hadron (nucleon), which leads to

$$\omega = (2k/M)^{1/2} . \tag{34}$$

Combining relations (33) and (34), we get

$$k = M^3/8 . \tag{35}$$

Finally inserting (35) into (32), we obtain

$$V_{p1} = M^3/36 . \tag{36}$$

Again, we make use of the Tryon idea [11], and we get for the quadratic confining potential (QCP), the volume in the space of momenta

$$V_{p1} + \langle \underline{\Psi}\, \Psi \rangle_{QCP} = 0. \tag{37}$$

From (36) and (37) we get

$$\langle \underline{\Psi}\, \Psi \rangle_{QCP} = - M^3/36 . \tag{38}$$

We notice that $K_{s,new}$ (please see (14)) as estimated in section 4, agrees with the Tryon idea in the QCP case.

7 – Comparison with other results of the literature

As a means to make comparison of the present estimates of the quark condensate with other results of the literature, let us put numbers in the expressions here obtained. We have

$$< \underline{\Psi}\, \Psi >_{\text{Thompson}} = - M^3 / 27 = - (313\ \text{MeV})^3. \qquad (39)$$

$$< \underline{\Psi}\, \Psi >_{\text{QCP}} = - M^3 / 36 = - (284\ \text{MeV})^3. \qquad (40)$$

$$< \underline{\Psi}\, \Psi >_{\text{LCP}} = - M^3 / (24\sqrt{\pi}) = - (269\ \text{MeV})^3. \qquad (41)$$

Meanwhile, other results of the literature are for instance quoted in table 2 of a paper by Mota et al. [13] and go from: $- (265\ \text{MeV})^3$ to $- (313\ \text{MeV})^3$. We also may compare with the experimental result of reference [14], namely $[- (296 \pm 25)\ \text{MeV}]^3$. On exhibiting the numbers from relations (39) to (41), we have taken M = 939 MeV.

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