Fourier Series - Number Pi - Lerch Function

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abstract

In this note we recall a formula for pi.The distinctive feature of these formula is that π is expressed in terms of the Lerch Transcendent Function.

keywords: fourier series,number pi,Lerch function

The Number Pi

- The number pi is defined by

$$\pi = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{2\,n+1} = 3.1415926535\ldots \tag{1}$$

Fourier Series

- Example of fourier series

$$\tan^{-1}\left(\frac{a \sin x}{1 - a \cos x}\right) = \sum_{n=1}^{\infty} \frac{a^n}{n} \sin(n\,x) \quad , \quad -\pi < x < \pi \; , \; |a| < 1 \tag{2}$$

- Particular case

$$x = \frac{\pi}{16} \; , \quad a = \frac{1}{\cos(\pi/16) + \sin(\pi/16)} = \frac{2}{\sqrt{2+\sqrt{2+\sqrt{2}}} + \sqrt{2-\sqrt{2+\sqrt{2}}}} \tag{3}$$

$$\pi = 4 \sum_{n=1}^{\infty} \frac{a^n}{n} \sin\left(\frac{n\,\pi}{16}\right) \tag{4}$$

The Function s(n)

- The function $s(n)$ is defined by

$$s(n) = \sin\left(\frac{n\,\pi}{16}\right) \; , \; n = 1, 2, 3, \ldots \tag{5}$$

- Some relations

$$s(16\,n) = 0\;,\;\; n = 1,\, 2,\, 3,\; \ldots \tag{6}$$

$$s(n) = s(n+32)\;,\;\; n = 1,\, 2,\, 3,\; \ldots \tag{7}$$

$$s(8\,n - 8 + k) = s(k)\sin\left(\frac{n\,\pi}{2}\right) - s(8-k)\cos\left(\frac{n\,\pi}{2}\right)\;,\;\; n = 1,\, 2,\, 3,\; \ldots;\; k = 1,\; \ldots,\; 8. \tag{8}$$

$$(-1)^{n-1}\, s(16-k) = s(16\,n - 16 + k)\;,\;\; n = 1,\, 2,\, 3,\; \ldots;\; k = 1,\; \ldots,\; 16. \tag{9}$$

$$s(k) = s(16-k)\;,\;\; k = 1,\; \ldots,\; 16. \tag{10}$$

$$s(k) = -s(32-k)\;,\;\; k = 1,\; \ldots,\; 16. \tag{11}$$

- Fundamental values

$$s(1) = \frac{1}{2}\sqrt{2 - \sqrt{2 + \sqrt{2}}} \tag{12}$$

$$s(2) = \frac{1}{2}\sqrt{2 - \sqrt{2}} \tag{13}$$

$$s(3) = \frac{1}{4}\left(\sqrt{2 - \sqrt{2}}\,\sqrt{2 + \sqrt{2 + \sqrt{2}}} + \sqrt{2 + \sqrt{2}}\,\sqrt{2 - \sqrt{2 + \sqrt{2}}}\right) \tag{14}$$

$$s(4) = \frac{1}{2}\sqrt{2} \tag{15}$$

$$s(5) = \frac{1}{4}\sqrt{2}\left(\sqrt{2 - \sqrt{2 + \sqrt{2}}} + \sqrt{2 + \sqrt{2 + \sqrt{2}}}\right) \tag{16}$$

$$s(6) = \frac{1}{4}\sqrt{2}\left(\sqrt{2 - \sqrt{2}} + \sqrt{2 + \sqrt{2}}\right) \tag{17}$$

$$s(7) = \frac{1}{2}\sqrt{2 + \sqrt{2 + \sqrt{2}}} \tag{18}$$

$$s(8) = 1 \tag{19}$$

Series for Pi

$$\pi = 4\sum_{n=1}^{\infty}\frac{a^n}{n}\sin\left(\frac{n\,\pi}{16}\right) = 4\sum_{n=1}^{\infty}\frac{a^n}{n}\,s(n) \tag{20}$$

$$\pi = 4\sum_{n=1}^{\infty}\sum_{k=1}^{8}\frac{a^{8\,n-8+k}}{8\,n-8+k}\left(\sin\left(\frac{n\,\pi}{2}\right)s(k) - \cos\left(\frac{n\,\pi}{2}\right)s(8-k)\right) \tag{21}$$

The Lerch Function

The Lerch transcendent Φ is the analytic continuation of the series

$$\Phi(z,\ s,\ u) = \frac{1}{u^s} + \frac{z}{(u+1)^s} + \frac{z^2}{(u+2)^s} + \ldots = \sum_{n=0}^{\infty} \frac{z^n}{(u+n)^s} \tag{22}$$

which converges for any real number $u > 0$ if z and s are any complex numbers with either $|z| < 1$, or $|z| = 1$ and $\mathrm{Re}(s) > 1$.

Special cases include the analytic continuations

of the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \Phi(1,\ s,\ 1) \tag{23}$$

the Hurwitz zeta function

$$\zeta(s,\ u) = \sum_{n=0}^{\infty} \frac{1}{(u+n)^s} = \Phi(1,\ s,\ u) \tag{24}$$

the alternating zeta function (also know as Dirichlet' s eta function $\eta(s)$)

$$\zeta^{*}(s) = \eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = \Phi(-1,\ s,\ 1) \tag{25}$$

the Dirichlet beta function

$$\beta(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2\,n+1)^s} = 2^{-s}\,\Phi\left(-1,\ s,\ \frac{1}{2}\right) \tag{26}$$

the Legendre chi function

$$\chi_s(z) = \sum_{n=0}^{\infty} \frac{z^n}{(2\,n+1)^s} = 2^{-s}\,z\,\Phi\left(z^2,\ s,\ \frac{1}{2}\right) \tag{27}$$

the polylogarithm

$$\mathrm{Li}_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k} = z\,\Phi(z,\ k,\ 1) \tag{28}$$

and the Lerch zeta function

$$L(\lambda,\ \alpha,\ s) = \Phi(\exp(2\,\pi\,i\,\lambda),\ s,\ \alpha) \tag{29}$$

- The Lerch transcendent satisfies the following identities :

$$\Phi(z,\ s,\ u+1) = \frac{1}{z}\left(\Phi(z,\ s,\ u) - \frac{1}{u^s}\right) \tag{30}$$

$$\Phi(z,\ s-1,\ u) = \left(u + z\,\frac{\partial}{\partial z}\right)\Phi(z,\ s,\ u) \tag{31}$$

$$\Phi(z,\ s+1,\ u) = -\frac{1}{s}\,\frac{\partial \Phi}{\partial u}(z,\ s,\ u) \tag{32}$$

If either $|z| < 1$ and $\mathrm{Re}(s) > 0$, or $z = 1$ and $\mathrm{Re}(s) > 1$, then

$$\Phi(z,\ s,\ u) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{e^{-(u-1)\,t}}{e^t - z}\,t^{s-1}\,dt \tag{33}$$

For all complex s , and complex z with $\mathrm{Re}(z) < 1/2$,

$$(1-z)\,\Phi(z,s,u)=\sum_{n=0}^{\infty}\left(\frac{-z}{1-z}\right)^{n}\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}(u+k)^{-s} \tag{34}$$

If either $z \in \mathbb{C}-\{(-\infty,-1]\cup[1,\infty)\}$ and $s \in \mathbb{C}$, or $z=\pm 1$ and $s \in \mathbb{C}-\{1\}$, then

$$\Phi(z,s,u)=2^{-s}\left[\Phi\left(z^{2},s,\frac{u}{2}\right)+z\,\Phi\left(z^{2},s,\frac{u+1}{2}\right)\right] \tag{35}$$

Pi Formula

- If $a=\dfrac{2}{\sqrt{2+\sqrt{2+\sqrt{2}}}+\sqrt{2-\sqrt{2+\sqrt{2}}}}$, $s(n)=\sin\left(\dfrac{n\pi}{16}\right)$, then

$$\pi=\frac{1}{4}\sum_{n=1}^{15}a^{n}\,s(n)\,\Phi\left(-a^{16},1,\frac{n}{16}\right) \tag{36}$$

$$\pi=\left\{\frac{1}{4}\sum_{n=1}^{7}s(n)\left(a^{n}\,\Phi\left(-a^{16},1,\frac{n}{16}\right)+a^{16-n}\,\Phi\left(-a^{16},1,\frac{16-n}{16}\right)\right)\right\}+\frac{1}{4}s(8)\,a^{8}\,\Phi\left(-a^{16},1,\frac{1}{2}\right) \tag{37}$$

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