The Complexity of Student-Project-Resource Matching-Allocation Problems

Anisse Ismaili[1,2]
[1] RIKEN AIP Center, Tokyo, Japan
[2] Yokoo-Laboratory, Kyushu University, Fukuoka, Japan

Abstract

In this technical note, I settle the computational complexity of nonwastefulness and stability in student-project-resource matching-allocation problems, a model that was first proposed by (Yamaguchi and Yokoo 2017). I show that computing a nonwasteful matching is complete for class FP^{NP}[poly] and computing a stable matching is complete for class Σ_2^P. These results involve the creation of two fundamental problems: PARETOPARTITION, shown complete for FP^{NP}[poly], and $\forall\exists$-4-PARTITION, shown complete for Σ_2^P. Both are number problems that are hard in the strong sense.

Model

Definition 1 (Student-Project-Resource (SPR) Instance)**.** An SPR instance is a tuple $(S, P, R, X, \succsim_S, \succsim_P, T_R, q_R)$.

- $S = \{s_1, \ldots, s_n\}$ is a set of students.
- $P = \{p_1, \ldots, p_m\}$ is a set of projects.
- $R = \{r_1, \ldots, r_k\}$ is a set of resources.
- $X \subseteq S \times P$ is a finite set of contracts.
- $\succsim_S = (\succsim_s)_{s \in S}$ are students' preferences over projects.
- $\succsim_P = (\succsim_p)_{p \in P}$ are projects' preferences over students.
- Resource r fits projects $T_r \subseteq P$, and $T_R = (T_r)_{r \in R}$.
- Resource r has capacity $q_r \in \mathbb{N}_{>0}$, and $q_R = (q_r)_{r \in R}$.

A contract $x = (s,p) \in X$ means that student s is matched to project p. For each student $s \in S$, strict order $\succ_s$ represents her preference over set $P \cup \{\emptyset\}$. For each project $p \in P$, weak order $\succsim_p$ represents its preference over set $S \cup \{\emptyset\}$. Preferences $\succsim_p$ extend to 2^S in a non-specified manner that is both responsive and separable: For every pair of students $s, s' \in S$ and subset $S' \subseteq S \setminus \{s, s'\}$,

$$s \succsim_p s' \Leftrightarrow S' \cup \{s\} \succsim_p S' \cup \{s'\}.$$

For each $s \in S$ and $S' \subseteq S \setminus \{s\}$, $s \succsim_p \emptyset \Leftrightarrow S' \cup \{s\} \succsim_p S'$. Contract (s,p) is acceptable for student s if $p \succsim_s \emptyset$ holds. Contract (s,p) is acceptable for project p if $s \succsim_p \emptyset$ holds. Without loss of generality, we assume that every contract $(s,p) \in X$ is acceptable for student s and project p. (Any contract which is non-acceptable for either side is discarded from set X.) $\lrcorner$

Given preference $\succsim$, let $\sim$ (resp. $\succ$) be its symmetric (resp. asymmetric) part. Given subset of contracts $Y \subseteq X$, student s and project p, set Y_s denotes $\{(s,p) \in Y \mid p \in P\}$ and set Y_p denotes $\{(s,p) \in Y \mid s \in S\}$. Preferences naturally extend over contracts. When no misunderstanding is possible, we omit the subscript and just write $\succ$ or $\succsim$.

Definition 2 (Matching)**.** A (many students to one project) matching is a subset of contracts $Y \subseteq X$ such that for every student s, $|Y_s| \leq 1$. We can then abuse shorthand Y in a functional manner:

- Student s is mapped to project $Y(s) \in P \cup \{\emptyset\}$.
- Project p hires students $Y(p) \subseteq S$. $\lrcorner$

Definition 3 (Feasibility)**.** Matching $Y \subseteq X$ is feasible if there exists an allocation function $\mu : R \to P$ that maps each resource r to one compatible project $\mu(r) \in T_r$, and that satisfies for every project $p \in P$ that:[1]

$$|Y_p| \leq \sum_{r \in \mu^{-1}(p)} q_r.$$

A feasible matching (Y, μ) is a couple of a matching and an allocation as above. Let $q_\mu(p) = \sum_{r \in \mu^{-1}(p)} q_r$ be the total of capacities allocated to project p. $\lrcorner$

Definition 4 (Nonwastefulness)**.** For feasible matching (Y, μ), a contract $(s,p) \in X \setminus Y$ is an improving pair if and only if:

- student s has preference $p \succ_s Y(s)$,
- project p has preference $s \succ_p \emptyset$,
- and matching $(Y \setminus Y_s) \cup \{(s,p)\}$ is feasible.

A feasible matching (Y, μ) is nonwasteful if it admits no improving pair. $\lrcorner$

Definition 5 (Fairness)**.** For feasible matching (Y, μ), contract $(s,p) \in X \setminus Y$ is an envious pair if and only if:

- student s has preference $p \succ_s Y(s)$,
- there is a student $s' \in Y(p)$ such that p prefers $s \succ_p s'$,
- and matching $Y \setminus (Y_s \cup Y_{s'}) \cup \{(s,p)\}$ is feasible.[2]

A feasible matching (Y, μ) is fair if it has no envious pair. $\lrcorner$

[1] To handle the case where $p \notin \mu(R)$ and then $\mu^{-1}(p) = \emptyset$, we assume the standard convention that an empty sum equals zero.

[2] Since matching Y is made feasible by μ, matching $Y \setminus (Y_s \cup Y_{s'}) \cup \{(s,p)\}$ is also feasible by same allocation μ.

Definition 6 (Stability)**.** A feasible matching (Y, μ) is stable if it is nonwasteful and fair. That is, it admits no improving pair and no envious pair. ⌟

We assume that following concepts are common knowledge: decision problem, function problem, length function, complexity classes P, XP, NP, coNP, FP^{NP}, NP^{NP}, coNP^{NP}, complementation, reduction, hardness and completeness. An SPR instance has length function $\Theta(nm + mk)$.

Definition 7. We study the following sequence of problems.

- SPR/FA:
  Given an SPR (instance) and a matching, is it feasible?
- SPR/NW/VERIF:
  Given an SPR and a feasible matching, is it nonwasteful?
- SPR/NW/FIND:
  Given an SPR, find a nonwasteful matching.
- SPR/STABLE/VERIF:
  Given an SPR and a feasible matching, is it stable?
- SPR/STABLE/EXIST:
  Given an SPR, does a stable matching exist?

For this purpose, we create two new fundamental problems.

- PARETOPARTITION:
  Given positive integers multiset $W = \{w_1, \dots, w_n\}$, and m targets $\theta_1, \dots, \theta_m \in \mathbb{N}$, any partition of W into a list $V_1, \dots, V_m$ of m subsets is mapped to deficit vector $\boldsymbol{\delta} \in \mathbb{Z}^m$ that is defined for every $i \in [m]$ by:
  $$\delta_i = \min \{w(V_i) - \theta_i, 0\},$$
  where $w(V_i) = \sum_{w \in V_i} w$. (Subset V_i has negative value if it sums below θ_i, and value zero if it surpasses θ_i.)
  Find a partition of W into a list $V_1, \dots, V_m$ that is Pareto-efficient[3] with respect to the deficit vector.
- $\forall\exists$-4-PARTITION:
  Given positive integers multiset $W = \{w_1, \dots, w_{4m}\}$, target $\theta \in \mathbb{N}$ and list of couples $u_1 v_1, \dots, u_\ell v_\ell$ of W, for map $\sigma : [\ell] \to \{0, 1\}$, we say that a partition of W into m subsets $V_1, \dots, V_m$ is σ-satisfying if and only if:
  - for every $i \in [m]$, it holds that $|V_i| = 4$ and $w(V_i) = \theta$,
  - for every $j \in [\ell]$:
    if $\sigma(j) = 1$ then u_j and v_j are in the same subset,
    if $\sigma(j) = 0$ then u_j and v_j are in different subsets.

  Does, for every map $\sigma : [\ell] \to \{0, 1\}$, there exist a σ-satisfying partition of W into m subsets? ⌟

Preliminaries

Our main interest lies in computing a nonwasteful and fair matching. On one hand, it is well known that a nonwasteful matching can be obtained by mechanism Serial Dictatorship (SD) (Goto et al. 2017, Th. 1). The matching is constructed following a fixed priority on students: every student decides her most preferred project that is still feasible. Unfortunately, mechanism SD requires to verify feasibility – an NP-complete problem (Th. 1 below) – $O(nm)$ times. Hence SPR/NW/FIND is in class FP^{NP}[poly]. On the other hand, mechanism Artificial Caps Deferred Acceptance (ACDA) computes a fair matching (Goto et al. 2017, Th. 2) in polynomial-time. The idea is to fix (Pareto-efficient) artificial capacities on projects (e.g. by allocating every resource on projects that are top-preferred by some students) and use mechanism Deferre Acceptance (DA). Both SD and ACDA are strategy-proof. However, nonwastefulness and fairness are not compatible, since there exist SPR instances with no stable matching.

First, we settle the complexity of matching feasibility:

Theorem 1. SPR/FA *is NP-complete.*

Proof. An allocation μ is a yes-certificate that can be verified in polynomial-time, hence SPR/FA is in class NP.

To show NP-hardness, any instance of 4-PARTITION, defined by positive integers multiset $W = \{w_1, \dots, w_{4m}\}$ and target $\theta \in \mathbb{N}$ is reduced to an instance of SPR/FA, as follows. One can assume that $\sum_{w \in W} w = m\theta$. There are m projects $P = \{p_1, \dots, p_m\}$. In matching Y, θ students are matched to every project. Resources R are identified to multiset W: $q_R = (w_1, \dots, w_{4m})$ and $T_r \equiv P$. Crucially, since 4-PARTITION is NP-hard even if integers w_i and θ are polynomially bounded, there is a polynomial number of students.

(yes⇔yes) There is a straightforward correspondence between a partition of W into m sets that hit θ, and an allocation that provides capacity for θ students on m projects. □

In Th. 1, the *strong* NP-hardness of 4-PARTITION is necessary: a similar construct from PARTITION with two projects would require an exponential number of students, which is not polynomial. This technical detail pushes us to generalize 4-PARTITION into PARETOPARTITION and $\forall\exists$-4-PARTITION, also shown *strongly* hard for their classes.

The Complexity of Nonwastefulness

In this section, we first show that verifying nonwastefulness for a given matching is complete for class coNP. Hence, there is no natural verification procedure that makes SPR/NW/FIND lie in class NP. Indeed, we then show that computing a nonwasteful matching (which existence is guaranteed by SD) is FP^{NP}-complete; that is complete for a polynomial number of calls to (e.g.) SAT. The proof involves polynomial number encoding in a new PARETOPARTITION problem that we show strongly FP^{NP}-hard.

Theorem 2. SPR/NW/VERIF *is coNP-complete.*
(Even if each student only has one acceptable project.)

Proof. An improving pair (s, p) along with the assignment ν that makes it feasible are no-certificates that are efficiently verifiable. Hence, SPR/NW/VERIF is in coNP.

To show coNP-hardness, we reduce any instance $W = \{w_1, \dots, w_{4m}\}$ of 4-PARTITION with target θ and assumption $\sum_{w \in W} w = m\theta$ to the following co-instance, which yes-answers are for existent deviations. There are $m + 2$ projects: For $i \in [m]$, θ students want to attend p_i. There are $m\theta$ students who want to attend p_{m+1}, and $m\theta + m$ who want to attend p_{m+2}. In matching Y, all students are matched but one student s^* from p_{m+2}. Every project p_i, for

[3] Given two vectors $\delta, \delta' \in \mathbb{Z}^m$, vector δ Pareto-dominates δ' if and only if: $\forall i \in [m], \delta_i \geq \delta'_i$ and $\exists i \in [m], \delta_i > \delta'_i$. A vector is Pareto-efficient when no vector Pareto-dominates it.

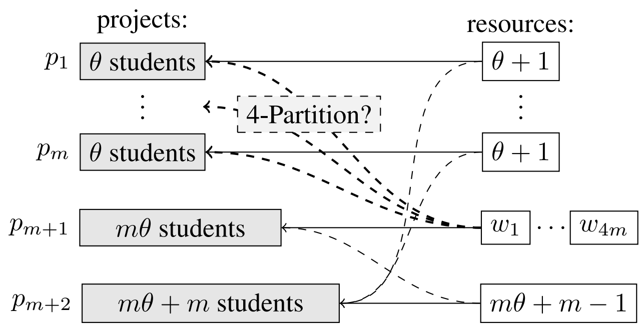

Figure 1: Reducing 4-PARTITION to SPR/NW/VERIF: One more student can be matched to p_{m+2} if and only if the dashed assignment is feasible (solution to 4-PARTITION).

$i \in [m]$ receives a resource r_{x_i} with capacity $q_{r_{x_i}} = \theta + 1$ and $T_{q_{r_{x_i}}} = \{p_i, p_{m+2}\}$. Project p_{m+1} receives $4m$ resources r_i identified with integer set $W = \{w_1, \ldots, w_{4m}\}$: every resource r_i for $i \in [m]$ has capacity $q_{r_i} = w_i$ and $T_{r_i} = \{p_i \mid i \in [m+1]\}$. Project p_{m+2} receives a resource r_z with capacity $q_{r_z} = m\theta + m - 1$ and $T_{r_z} = \{p_{m+1}, p_{m+2}\}$. Since integers w_i and θ are polynomially bounded, there is a polynomial number of students.

(yes⇒yes) If the 4-PARTITION instance admits a solution $V_1, \ldots, V_m$, then (s^*, p_{m+2}) is a feasible improving pair, as following allocation (dashed in Fig. 1) ν shows: $\nu^{-1}(p_i) \equiv V_i$ for $i \in [m]$, $\nu^{-1}(p_{m+1}) = \{r_z\}$ and $\nu^{-1}(p_{m+2}) = \{r_{x_i} \mid i \in [m]\}$. Indeed, allocation ν provides capacities $q_\nu(p_1) = \ldots = q_\nu(p_m) = \theta$, $q_\nu(p_{m+1}) = m\theta + m - 1$ and $q_\nu(p_{m+2}) = m\theta + m$.

(yes⇐yes) The only possible improving pair is (s^*, p_{m+2}). The only way matching $Y \cup \{(s^*, p_{m+2})\}$ is feasible, is when $\nu^{-1}(p_{m+2}) = \{r_{x_i} \mid i \in [m]\}$, $\nu^{-1}(p_{m+1}) = \{r_z\}$ and projects p_i for $i \in [m]$ use resources r_i for $i \in [4m]$ in a perfectly balanced manner $q_\nu(p_1) = \ldots = q_\nu(p_m) = \theta$. Hence, $V_i \equiv \nu^{-1}(p_i)$ for $i \in [m]$ is a solution for 4-PARTITION. □

Theorem 3. SPR/NW/FIND *is FP*^{NP}*-complete. (Even if each student only has one acceptable project.)*

Proof. Mechanism SD shows that SPR/NW/FIND belongs to FP$^{\text{NP}}$. Hardness follows from Lemma 1 and 2 below. □

Lemma 1. PARETOPARTITION *is strongly FP*^{NP}*-hard. (This result still holds when all targets are the same.)*

Proof. Let any instance of MAX3DM be defined by finite sets A, B, C with $|A| = |B| = |C|$ and triplets set $M \subseteq A \times B \times C$, $|M| = m$. Triplet $t = (a, b, c) \in M$ is mapped to payoff $v_t \in \mathbb{N}$. In a (partial) 3-dimensional matching (3DM), any element of $A \cup B \cup C$ occurs at most once in M'. The goal is to maximize $\sum_{t \in M'} v_t$ for $M' \subseteq M$ a (partial) 3-dimensional matching. This problem is FP$^{\text{NP}}$[poly]-complete (Gasarch, Krentel, and Rappoport 1995, Th. 3.5). For every $a_i \in A$ (resp. $b_j \in B$, $c_k \in C$), let $\#a_i$ (resp. $\#b_j$, $\#c_k$) denote the number of occurrences of a_i (resp. b_j, c_k) in M: the number of triplets that contain a_i (resp. b_j, c_k). Let integer v_M denote total payoff $\sum_{t \in M} v_t$. Elements are identified with integers $i, j, k \in [n]$ and $t \in [m]$.

We reduce this problem to the following instance of PARETOPARTITION for which finding a solution gives out the optimum (solution) for the given MAX3DM instance. Formally, it is a many-one metric reduction. Set W contains $6m$ different integers that must be partitioned into $m + 1$ different subsets of various cardinalities. Every subset bears an objective for Pareto-efficiency. The idea is that a Pareto-efficient deficit vector will always have deficit zero on the m first subsets, and will give the optimal value of MAX3DM by the deficit of the last subset. Given basis $\beta \in \mathbb{N}_{\geq 2}$ and integer sequence $(z_i)_{i \in \mathbb{N}}$, we define integer $\langle \ldots z_2\ z_1\ z_0 \rangle_\beta$ by $\sum_{i \geq 0} z_i \beta^i$, with $z_i = 0$ when it is not written. Let β be an integer large enough for such representation in basis β (as below) to never have remainders, even when one adds all the integers in W. Choosing $\beta = 32m|A| + 1$ will largely fit the purpose. The integers in set W are represented below. For every triplet $t = (a_i, b_j, c_k) \in M$, there is an integer $w(a_i, b_j, c_k)$. For every element $a_i \in A$, we introduce one *actual* integer $w(a_i)$ representing the actual element intended to go with the triplets in a 3-dimensional matching, and $\#a_i - 1$ *dummies* who will go with the triplets that are not in the 3-dimensional matching. Similarly, we introduce $\#b_j$ integers for every $b_j \in B$ and $\#c_k$ integers for every $c_k \in C$. For every triplet $t \in M$, we introduce two integers $w(v_t)$ and $w'(v_t)$ which roles we precise later.

$w(\overbrace{(a_i, b_j, c_k)}^{\text{triplet } t}) = \langle 1$	1	$-i$	$-j$	$-k$	$-t$	0	$0 \rangle_\beta$
actual $w(a_i) = \langle 1$	2	i	0	0	0	2	$0 \rangle_\beta$
$\#a_i - 1$ $w'(a_i) = \langle 1$	2	i	0	0	0	1	$0 \rangle_\beta$
actual $w(b_j) = \langle 1$	4	0	j	0	0	2	$0 \rangle_\beta$
$\#b_j - 1$ $w'(b_j) = \langle 1$	4	0	j	0	0	1	$0 \rangle_\beta$
actual $w(c_k) = \langle 1$	8	0	0	k	0	2	$0 \rangle_\beta$
$\#c_k - 1$ $w'(c_k) = \langle 1$	8	0	0	k	0	1	$0 \rangle_\beta$
actual $w(v_t) = \langle 1$	16	0	0	0	t	0	$0 \rangle_\beta$
goal $w'(v_t) = \langle 1$	16	0	0	0	t	3	$v_t \rangle_\beta$
targets $\theta_{1 \ldots m} = \langle 5$	31	0	0	0	0	6	$0 \rangle_\beta$
target $\theta_{m+1} = \langle m$	$16m$	0	0	0	0	0	$v_M \rangle_\beta$

The idea is that every subset V_i (which corresponds to an objective for Pareto-efficiency) has a preference on integers with respect to columns, from the heaviest weight β^7 to the lower one β^0, because in basis β, sums of integers in W never have remainders from a column to a heavier one. Firstly, for Pareto-efficiency, due to the heaviest digits, each subset $V_1, \ldots, V_m$ must contain five elements, and subset V_{m+1} must contain m elements, in order to induce a deficit of approximately zero (if we round the digits of lower weight $\beta^6 \ldots \beta^0$). Indeed, since the sum of the heaviest digits is $6m$, any other repartition would induce (approximate) deficits in multiples of $-\beta^7$ for some subset, hence would be Pareto-dominated. Secondly, for similar reasons, because of the second heaviest digits (the powers of 2), each subset $V_1, \ldots, V_m$ must contain one $w(a, b, c)$ integer, one $w(a)$ or $w'(a)$, one $w(b)$ or $w'(b)$, one $w(c)$

or $w'(c)$, and one $w(v)$ or $w'(v)$. Also, subset V_{m+1} must contain a number m of $w(v)$ or $w'(v)$ integers. Digits on $\beta^5, \ldots, \beta^2$ (that contain integers $\pm i, \pm j, \pm k, \pm t$) make every triple integer $w(a_i, b_j, c_k)$ be precisely with its own elements (actual or dummies): $w(a_i)$ or $w'(a_i)$, $w(b_j)$ or $w'(b_j)$, $w(c_k)$ or $w'(c_k)$, and own payoff integer: $w(v_t)$ or $w'(v_t)$. To sum up, rounding the two lower digits, any partition $V_1, \ldots, V_m, V_{m+1}$ that respects the constraints above has (approximate) deficits zero for every subset V_i, and any other partition would be Pareto-dominated by this (approximate) ideal point, hence not Pareto-efficient. Then, we already know that for every triple $t = (a_i, b_j, c_k)$, either payoff integer $w(v_t)$ is with triple integer $w(t)$, and $w'(v_t)$ in V_{m+1}, either $w'(v_t)$ is with $w(t)$, and $w(v_t)$ in V_{m+1}. While deficit on digit β^1 for V_{m+1} is always zero, and deficit on digits β^0 for $V_{1\ldots m}$ are also always zero, assuming Pareto-efficiency, payoff integer $w'(v_t)$ only goes in V_{m+1} when the (only) three *actual* elements integers are together, like in a (partial) 3-dimensional matching, in order to yield deficits zero on digit β^1 for $V_{1\ldots m}$. All in all, Pareto-efficiency, while requiring a partition which structure follows any (partial) 3-dimensional matching M', simply asks an optimal deficit $\delta_{m+1} = \sum_{t \in M'} v_t - v_M$ for subset V_{m+1}.

To conclude, there is a correspondence between optimal 3-dimensional matchings M' (resp. their values $\sum_{t \in M'} v_t$) and Pareto-efficient partitions (resp. deficit $\delta_{m+1} = \sum_{t \in M'} v_t - v_M$; recall that $\delta_1 = \ldots = \delta_m = 0$). This is a polynomial many-one metric reduction. Crucially, no integer is larger than polynomial β^8; hence ParetoPartition is *strongly* FP^{NP}-hard. Also, one can build the exact same reduction with $\theta_1 = \ldots = \theta_m = \theta_{m+1} = m\beta^7 + 16m\beta^6 + v_M$, by introducing m gap integers $(m - 5)\beta^7 + (16m - 31)\beta^6 + v_M$ in set W. □

Lemma 2. ParetoPartition $\leq_p$ SPR/Nw/Find

Proof. We reduce any instance $W = \{w_1, \ldots, w_n\}$ and $\theta_1, \ldots, \theta_m \in \mathbb{N}$ of ParetoPartition to the following (very simple) instance of SPR/Nw/Find. There are m projects $p_1, \ldots, p_m$; and for each project p_i there is a set of θ_i students who consider only p_i acceptable (and reciprocally), strictly above $\emptyset$. Crucially, since numbers in the ParetoPartition instance are polynomially bounded, there is only a polynomial number of students. Set of resources R is identified with integers set W: any resource is compatible with any project and $q_R = (w_1, \ldots, w_n)$.

Computing a nonwasteful matching (Y, μ) precisely outputs a partition $V_1, \ldots, V_m \equiv \mu^{-1}(p_1), \ldots, \mu^{-1}(p_m)$ with Pareto-efficient deficits: if there was a partition (allocation) which deficits (unmatched students) Pareto-dominated the deficits of $V_1, \ldots, V_m$, then an improving pair would exist. In other words, it is not possible to obtain one more capacity for an unmatched student without decreasing capacity on an other project p_i (with $q_\mu(p_i) \leq \theta_i$). □

The Complexity of Stability

A matching that is both nonwasteful and fair (i.e.: stable) may not exist. In this section, we settle the complexity of deciding whether one exists in a given SPR, as Σ_2^P-complete.

Theorem 4. SPR/Stable/Verif *is also coNP-complete. (Even if each student only has one acceptable project.)*

Proof. It is the same proof as for verifying nonwastefulness, since no envious pair is possible. Furthermore, verifying fairness is straightforward in P (see Def. 5, footnote). □

Theorem 5. SPR/Stable/Exist *is NP^{NP}-complete.*

Proof. A stable matching is a yes-certificate that can be verified in coNP time. Therefore, SPR/Stable/Exist belongs to NP^{NP}. Hardness follows from Lem. 3 and 4 below. □

Lemma 3. $\forall\exists$-4-Partition *is strongly coNP^{NP}-hard. (It holds even if couples are disjoint and couple heads u go in distinct subsets.)*

Proof. Let any instance of $\forall\exists$-3DM be defined by finite sets A, B, C with $|A| = |B| = |C|$ and two disjoint triplets set $M, N \subseteq A \times B \times C$, with $|M| = m$ and $|N| = n$. This decision problem asks whether:

$$\forall M' \subseteq M, \quad \exists N' \subseteq N, \quad M' \cup N' \text{ is a 3DM},$$

where $M' \cup N'$ a 3DM means that any element of $A \cup B \cup C$ occurs exactly once in $M' \cup N'$. It is a Π_2^p-complete problem (McLoughlin 1984). For every $a_i \in A$ (resp. $b_j \in B$, $c_k \in C$), let $\#a_i$ (resp. $\#b_j$, $\#c_k$) denote the number of occurrences of a_i (resp. b_j, c_k) in M: the number of triplets that contain a_i (resp. b_j, c_k). One can identify elements and triplets with integers $i, j, k \in [n]$ and $t \in [m]$.

We reduce it to the following $\forall\exists$-4-Partition instance. Set W contains the $4(m + n)$ integers depicted below in basis $\beta = 4(m + n)|A| + 1$ (def. in proof for Lem. 1). For every triplet $t = (a_i, b_j, c_k) \in M \cup N$, there is one "triplet" integer $w(a_i, b_j, c_k) \in \mathbb{N}$. For every element $a_i \in A$, we introduce one *actual* integer $w(a_i)$ representing the actual element intended to go with the triplets in the 3DM, and $\#a_i - 1$ *dummies* who will go with the triplets that are not in the 3-dimensional matching. Similarly, we introduce $\#b_j$ integers for each $b_j \in B$ and $\#c_k$ integers for each $c_k \in C$. Target $\theta = \beta^5 + 15\beta^4$ is given below.

$$\begin{array}{rl}
w(\overbrace{(a_i, b_j, c_k)}^{\text{triplet } t}) = & \langle 1 \quad 1 \quad -i \quad -j \quad -k \quad 0 \rangle_\beta \\
{\scriptstyle \text{one actual} \atop \#a_i - 1 \text{ dum.}} \; w(a_i) = & \langle 1 \quad 2 \quad i \quad 0 \quad 0 \quad {\scriptstyle -2 \text{ (actual)} \atop 0 \text{ (dummy)}} \rangle_\beta \\
{\scriptstyle \text{one actual} \atop \#b_j - 1 \text{ dum.}} \; w(b_j) = & \langle 1 \quad 4 \quad 0 \quad j \quad 0 \quad {\scriptstyle +1 \text{ (actual)} \atop 0 \text{ (dummy)}} \rangle_\beta \\
{\scriptstyle \text{one actual} \atop \#c_k - 1 \text{ dum.}} \; w(c_k) = & \langle 1 \quad 8 \quad 0 \quad 0 \quad k \quad {\scriptstyle +1 \text{ (actual)} \atop 0 \text{ (dummy)}} \rangle_\beta \\
\text{target } \theta = & \langle 4 \quad 15 \quad 0 \quad 0 \quad 0 \quad 0 \rangle_\beta
\end{array}$$

We define a list of couples of length $\ell = |M|$ in W: for every triple $t = (a_i, b_j, c_k) \in M$, we associate in a couple $u_t v_t$ "triplet" integer $u_t = w(a_i, b_j, c_k)$ with the "actual" integer $v_t = w(a_i)$. This instance asks whether:

$$\forall \sigma : [\ell] \to \{0, 1\}, \quad \exists \sigma\text{-satisfying partition of } W.$$

First let us observe that since β is large enough, additions in W never have remainders. Hence, subsets must hit the target on each column of this representation. Consequently, in any 4-partition of W, there are 4 elements (in case it wasn't required), one of each in: "triplet" integers, element-a integers, element-b integers and element-c integers. Moreover, "triplet" integer $w(a_i, b_j, c_k)$ is with "its" elements $w(a_i)$, $w(b_j)$ and $w(c_k)$. Also, *actual* elements must be in the same subset, and dummies in others. Therefore, there is a correspondence between any (full) 3-dimensional matching $M' \cup N'$ and a 4-partition, since in a 4-partition, the actual elements are regrouped according to triplets.

(yes⇒yes) Assume the 3DM instance is a yes, and let $\sigma : [\ell] \to \{0, 1\}$ be any couple enforcement/forbidding function. We construct a σ-satisfying 4-partition in correspondence with the following 3-dimensional matching $M' \cup N'$: for $t \in [\ell] \equiv M$, triplet t is in M' if and only if $\sigma(t) = 1$; then the assumption gives N' such that $M' \cup N'$ is a 3DM. We construct the corresponding 4-partition (see paragraph above), and it is σ-satisfying.

(yes⇐yes) Assume the partition instance is a yes, and let us show that for any $M' \subseteq M$, there is $N' \subseteq N$ such that $M' \cup N'$ is a 3DM. Let σ be defined as $\sigma(t) = 1$ if and only if $t \in M'$. A σ-satisfying 4-partition exists, and is in correspondence with some 3DM $M' \cup N'$ (see above).

It follows from this construction that the lemma holds even if couples are disjoint and no subset shall contain two heads, that is two u elements. Crucially, hardness holds even if numbers are polynomially bounded above by β^6. □

Lemma 4. ∀∃-4-PARTITION $\leq_p$ CO-SPR/STABLE/EXIST

Proof. We reduce any ∀∃-4-PARTITION instance to the CO-SPR/STABLE/EXIST instance depicted in Figure 2. The idea is that capacity requirements of projects $p_1, \ldots, p_m$ model the m targets of a 4-partition. Since integers $u_1, \ldots, u_\ell$ are in different subsets, we remove them from the targets of $p_1, \ldots, p_\ell$ (assuming they are already therein). We have a correspondance between enforcing u_t and v_t together in a 4-partition (by $\sigma(t) = 1$) and letting capacity requirement of project p_t be $\theta - u_t - v_t$ (that's u_t and v_t therein together), by matching $\overline{s_{v_t}}$ with p'_t and allocating r_{v_t} on p'_t (out of the remaining feasibility problem on $p_1, \ldots, p_m$). Conversely, there is a correspondance between forbidding u_t and v_t to be together in a 4-partition (by $\sigma(t) = 0$), and trying to match $\overline{s_{v_t}}$ with p_t, hence bringing its capacity requirement to $\theta - u_t$, while resource r_{v_t} can't be allocated to p_t. Remark that no student in any class $\overline{s_{v_t}}$ can be involved in an improving or envious pair (unless unmatched).

(yes⇒yes) For any $\sigma : [\ell] \to \{0, 1\}$, there exists a σ-satisfying 4-partition $V_1, \ldots, V_\ell, V_{\ell+1}, \ldots, V_m$ such that for any $t \in [\ell]$, firstly $u_t \in V_t$ and secondly $v_t \in V_t$ if and only if $\sigma(t) = 1$. For the sake of contradiction, let us assume a stable matching (Y, μ). By nonwastefulness, for every $t \in [\ell]$, either $\mu(r_{v_t}) = p'_t$ (think to $\sigma(t) = 1$) or $\mu(r_{v_t}) \in \{p_i \mid i \neq t\}$ (think to $\sigma(t) = 0$). Provided by the σ-satisfying 4-partition, there is an allocation of $v_1, \ldots, v_\ell$ and $W \setminus \mathcal{L}$ that makes feasible a full matching $Y(\overline{s_i}) = p_i$ for any $i \in [m]$. Hence it would be wasteful to use resource r_1 on projects $\{p_1, \ldots, p_m\}$ and it is allocated to p_a or p_b. The SPR defined by s_a, s_b, p_a, p_b, r_1 cannot be stable.

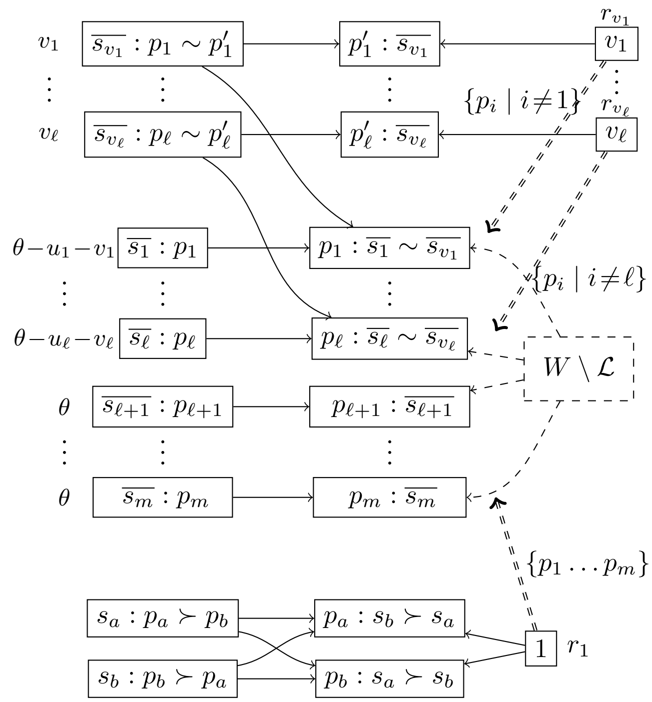

Figure 2: Given a ∀∃-4-PARTITION instance defined by $m \in \mathbb{N}$, positive integers multiset $W = \{w_1, \ldots, w_{4m}\}$, target $\theta \in \mathbb{N}$ and list of couples $\mathcal{L} = u_1v_1, \ldots, u_\ell v_\ell$ of W, we construct the CO-SPR/STABLE/EXIST instance depicted above, which contains $\ell + m + 2$ projects $p'_1, \ldots, p'_\ell, p_1, \ldots, p_m, p_a, p_b$ which preferences (strictly above $\emptyset$) are in the boxes. There are $m\theta - \sum_{t=1}^{\ell} u_t + 2$ students, distributed in $\ell + m$ different classes $\overline{s_{v_1}}, \ldots, \overline{s_{v_\ell}}, \overline{s_1}, \ldots, \overline{s_m}$ which sizes are on the left of classes, and two students s_a, s_b. Crucially, since ∀∃-4-PARTITION is strongly hard, there is a polynomial number of students. There are $|W| - \ell + 1$ resources with compatibilities depicted by right-left arrows, and capacities inside the boxes. For every $t \in [\ell]$, resource r_{v_t} is compatible with $\{p'_t\} \cup \{p_i \mid i \neq t\}$.

(no⇒no) There exists $\sigma : [\ell] \to \{0, 1\}$ such that no σ-satisfying 4-partition exists. Let us construct a stable matching (Y, μ). For any $t \in [\ell]$:

- if $\sigma(t) = 1$, then $Y(\overline{s_{v_t}}) = p'_t$ and $\mu(r_{v_t}) = p'_t$;
- if $\sigma(t) = 0$, then $\mu(r_{v_t}) \in \{p_i \mid i \neq t\}$.

More precisely, we allocate the resources in a way that minimizes the number of unmatched students in $\overline{s_1}, \ldots, \overline{s_m}$ and $\overline{s_{v_t}}$ for $\sigma(t) = 0$. However, since no σ-satisfying 4-partition exists, some projects in $p_1, \ldots, p_m$ have deficits of capacity. We allocate r_1 with one of those. Then, the SPR defined by s_a, s_b, p_a, p_b is stable in its emptyness. □

Related Work

When the capacity of every project is fixed, a matching that satisfies stability, fairness and efficiency can be found by using the celebrated Gale-Shapley mechanism (Gale and Shapley 1962) (also referred to as Deferred Acceptance (DA) mechanism), which is also strategyproof. The present work deals with constrained two-sided matching, which has been attracting attention from AI researchers (Aziz et al. 2017; Hamada et al. 2017). Many real-world matching markets are subject to a variety of distributional constraints, including regional maximum quotas, which restrict the total number of students assigned to a set of schools (Kamada and Kojima 2015), minimum quotas, which guarantee that a certain number of students are assigned to each school (Fragiadakis et al. 2016; Sönmez and Switzer 2013; Sönmez 2013), and diversity constraints, which enforce that a school satisfies a balance between different types (e.g., socioeconomic status) of students (Hafalir, Yenmez, and Yildirim 2013; Ehlers et al. 2014). Also, there exists a stream of works that examines the computational complexity for finding a matching that satisfies some desirable properties under distributional constraints, including (Biró et al. 2010; Fleiner and Kamiyama 2012).

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