A joint multifractal analysis of finitely many non Gibbs-Ahlfors type measures

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Abstract

In the present paper, new multifractal analysis of vector valued Ahlfors type measures is developed. Mutual multifractal generalizations f fractal measures such as Hausdorff and packing have been introduced with associated dimensions. Essential properties of these measures have been shown using convexity arguments.

Key words: Hausdorff and packing measures, Hausdorff and packing dimensions, Multifractal formalism, Mixed cases, Hölderian Measures.
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1 Introduction

In the present work we are concerned with the whole topic of multifractal analysis of measures and the validity of multifractal formalisms. We aim to consider some cases of simultaneous behaviors of measures instead of a single measure as in the classic or original multifractal analysis of measures. We call such a study mixed multifractal analysis. Such a mixed analysis has been generating a great attention recently and thus proved to be powerful in describing

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the local behavior of measures especially fractal ones. (See [3], [4], [21], [22], [24], [25], [26], [35], [38], [39], [40], [43], [44], [45], [46]).

In this paper, multi purposes will be done. Firstly we review the classical multifractal analysis of measures and recall all basics about fractal measures as well as fractal dimensions. We review Hausdorff measures, Packing measures, Hausdorff dimensions, Packing dimensions as well as Renyi dimensions and we recall the eventual relations linking these notions. A second aim is to develop a type of multifractal analysis, multifractal spectra, multifractal formalism which permit to study simultaneously a higher number of measures. As it is noticed from the literature on multifractal analysis of measures, this latter always considered a single measure and studies its scaling behavior as well as the multifractal formalism associated. Recently, many works have been focused on the study of simultaneous behaviors of finitely many measures. In [21], a mixed multifractal analysis is developed dealing with a generalization of Rényi dimensions for finitely many self similar measures. This was one of the motivations leading to our present paper. Secondly, we intend to combine the generalized Hausdorff and packing measures and dimensions recalled after with Olsen's results in [26] to define and develop a more general multifractal analysis for finitely many measures by studying their simultaneous regularity, spectrum and to define a mixed multifractal formalism which may describe better the geometry of the singularities's sets of these measures. We apply the techniques of L. Olsen especially in [21] and [26] with the necessary modifications to give a detailed study of computing general mixed multifractal dimensions of simultaneously many finite number of measures one of them at least is characterized by a quasi-Ahlfors property and try to project our results for the case of a single measure to show the generecity of our's.

The assumption of being Ahlfors for one of the measures is essential contrarily to some existing works that have forgotten such assumption and developed some questionable version of multifractal densities, eventhough published ([1] and [7]). Indeed, in such references the authors referred to similar techniques as in [6] to show the existence of a real valued dimension without assuming. However, the authors did not pay attention to the fact that general probability measures (eventhough being doubling) may not lead to multifractal dimensions. Indeed, it is already mentioned in [6] (but nowhere in [1] and [7]) that

- for a Borel probability measure, the infimum for the μ-Hausdorff measure (and thus the supremum for μ-packing measures) extends over μ-ρ-coverings (packings). A μ-ρ-covering being a covering by cylinders C with $\mu(C) < \rho$.
- The measure μ is nonatomic, since otherwise there may be no μ-ρ-covering at all.

It is therefore questionable for both [1] and [7] the existence of multifractal dimensions in a general framework not taking into account some control of the measure of balls by means of their diameters. Comparing with the first multifractal generalisations due to Olsen ([21]), the cases developed in [1] and [7] are different, as in [21], the measure ν is replaced by an equivalent of Lebesgue's measure, the diameter of the ball. To overcome this problem, we proposed in the present work to assume some weak hypothesis on the measures applied. It consists of a weak form of the so-called Alhfors measures. For more details on such measures, we may refer to citeEdgar, [18], [30].

Definition 1.1 *A borel probability measure ν on $\mathbb{R}^d$ is said to be quasi-Ahlfors with index (regularity) $\alpha > 0$ if there*

$$\limsup_{|U|\longrightarrow 0} \frac{\mu(U)}{|U|^\alpha} < +\infty.$$

We denote $\mathcal{QAHP}(\mathbb{R}^n)$ the set of quasi-Ahlfors probability measures on $\mathbb{R}^d$. A borel probability measure ν on $\mathbb{R}^d$ is said to be Ahlfors with index (regularity) $\alpha > 0$ if there

$$0 < \liminf_{|U|\longrightarrow 0} \frac{\mu(U)}{|U|^\alpha} \leq \limsup_{|U|\longrightarrow 0} \frac{\mu(U)}{|U|^\alpha} < +\infty.$$

We denote $\mathcal{AHP}(\mathbb{R}^n)$ the set of quasi-Ahlfors probability measures on $\mathbb{R}^d$.

Using this assumption, the multifractal generalizations of Hausdorff and packing measures introduced in [1] and [7] induce in a usual way multifractal generalizations of Hausdorff and packing dimensions introduced in [6], [7], [8], [19], [21]. Otherwise, the task remains questionable and thus the set of coverings applied there may be empty!!! For backgrounds and details on multifractal dimensions, readers may refer to [21], [22], [35], [38], [39], [40], [41], [41], [43], [44], [45], [46]).

Resuming, mixed multifractal analysis is a natural extension of multifractal analysis of single objects such as measures, fuctions, statistical data, distributions... It is developed quite recently (since 2014) in the pure mathematical point of view. In physics and statistics, it was appearing on different forms but not really and strongly linked to the mathematical theory. See for example [13], [16]. In many applications such as clustering topics, each attribute in a data sample may be described by more than one type of measure. This leads researchers to apply measures well adopted for mixed-type data. See for example [13].

The next section is devoted to some preliminaries. Section 3 is devoted to the main results. In section 4, proofs of main results are developed.

2 Preliminaries and results

Denote $\mathcal{P}(\mathbb{R}^d)$ the family of probability measures on $\mathbb{R}^d$. For a single or vector valued measure μ denote S_μ its topological support.

Let $k \in \mathbb{N}$ fixed and consider a vector valued measure $\mu = (\mu_1, \mu_2, ..., \mu_k) \in \mathcal{P}(\mathbb{R}^d)^k$. For $q = (q_1, q_2, ..., q_k) \in \mathbb{R}^k$, $x \in \mathbb{R}^d$ and $r > 0$, denote

$$[\mu(B(x,r)]^q = [\mu_1(B(x,r)]^{q_1} \times ... \times [\mu_k(B(x,r)]^{q_k},$$

where $B(x,r)$ is the ball of center x and radius r. Next, given $E \subseteq \mathbb{R}^d$ and $\varepsilon > 0$, we call an ε-covering of E any countable set $(U_i)_i$ of non-empty subsets $U_i \subseteq \mathbb{R}^d$ satisfying

$$E \subseteq \bigcup_i U_i \text{ and } |U_i| < \varepsilon, \tag{1}$$

where $|.|$ is the diameter.

The last assumption is already assumed in [6], [3], [4], [19], [21], [22], [23], [24], [25], [27] but unfortunately not in [1] and [7]. When assuming that the measure is quasi-Ahlfors, this assumption is not necessary and may be replaced by the original one in [6] on $\mu - \varepsilon$-coverings.

We now proceed in introducing the multifractal generalisations of Hausdorff and packing measures and the associated dimensions. We will see after that bening quasi-Ahlfors is necessary for at least one measure.

For $(\mu, \nu) = (\mu_1, \mu_2, ..., \mu_k, \nu) \in \mathcal{P}(\mathbb{R}^d)^k \times \mathcal{QAHP}(\mathbb{R}^d)$, $(q,t) = (q_1, q_2, ..., q_k, t) \in \mathbb{R}^{k+1}$, $E \subset \mathbb{R}^d$ and $\varepsilon > 0$, let

$$\overline{\mathcal{H}}^{q,t}_{\mu,\nu,\varepsilon}(E) = \inf\{\sum_{(} \mu(B(x_i,r_i)))^q(\nu(B(x_i,r_i)))^t\}$$

and

$$\overline{\mathcal{H}}^{q,t}_{\mu,\nu}(E) = \lim_{\varepsilon\downarrow 0} \overline{\mathcal{H}}^{q,t}_{\mu,\nu,\varepsilon}(E),$$

where the infimum above is taken over the set of all centred ε-coverings of E. Similarly, let

$$\overline{\mathcal{P}}^{q,t}_{\mu,\nu,\varepsilon}(E) = \sup\{\sum_i \mu(B(x_i,r_i)))^q(\nu(B(x_i,r_i)))^t\}$$

and

$$\overline{\mathcal{P}}^{q,t}_{\mu,\nu}(E) = \lim_{\varepsilon\downarrow 0} \overline{\mathcal{P}}^{q,t}_{\mu,\nu,\varepsilon}(E),$$

where the supremum above is taken over the set of all centred ε-packings of E.

Definition 2.1 *The mixed generalized Hausdorff measure relatively to* (μ, ν) *is defined by*

$$\mathcal{H}^{q,t}_{\mu,\nu}(E) = \sup_{F \subseteq E} \overline{\mathcal{H}}^{q,t}_{\mu,\nu}(F).$$

The mixed generalized packing measure relatively to (μ, ν) *is defined by*

$$\mathcal{P}^{q,t}_{\mu,\nu}(E) = \inf_{E \subseteq \bigcup\limits_i E_i} \sum_i \overline{\mathcal{P}}^{q,t}_{\mu,\nu}(E_i).$$

It is straightforward that $\mathcal{H}^{q,t}_{\mu,\nu}$ and $\mathcal{P}^{q,t}_{\mu,\nu}$ are outer metric and regular measures on $\mathbb{R}^d$. Borel sets are thus measurable relatively to them. Furthermore, we may prove using the well known Besicovitch covering theorem that

$$\mathcal{H}^{q,t}_{\mu,\nu}(E) \leq \xi \mathcal{P}^{q,t}_{\mu,\nu} \leq \xi \overline{\mathcal{P}}^{q,t}_{\mu,\nu}(E), \ \forall (q,t) \in \mathbb{R}^{k+1}, \ \forall E \subseteq \mathbb{R}^d. \tag{2}$$

ξ is the number related to the Besicovitch covering theorem.

We now introduce the associated mixed generalisations of Hausdorff ad packing dimensions to $\mathcal{H}^{q,t}_{\mu,\nu}$ and $\mathcal{P}^{q,t}_{\mu,\nu}$. We will notice the necessity of the quasi-Ahlfors assumption. We have the following result.

Proposition 2.1 *Let* $(\mu, \nu) \in \mathcal{P}(\mathbb{R}^d)^k \times \mathcal{QAHP}(\mathbb{R}^d)$ *and* $E \subseteq \mathbb{R}^d$. $\forall q \in \mathbb{R}^k$, *the set* $\Gamma_q = \left\{t \ ; \mathcal{H}^{q,t}_{\mu,\nu}(E) < +\infty\right\}$ *is nonempty.*

Proof. Let $\alpha, M \in \mathbb{R}_+$ be such that

$$\overline{\lim_{|U| \to 0}} \frac{\nu(U)}{|U|^\alpha} < M.$$

There exists $\delta > 0$ such that $\forall \ r$, $0 < r < \delta$,

$$\nu(U) \leq M|U|^\alpha; \ \forall U; \ |U| < r.$$

Let next $(B(x_i, r_i))_i$ be an ε-covering of E and consider the ξ-families defined by the Besicovitch covering theorem. We get

$$\sum_i \mu(B(x_i, r_i))^q \nu(B(x_i, r_i))^t \leq \sum_{i=1}^{\xi} \sum_j \mu(B(x_{ij}, r_{ij}))^q \nu(B(x_{ij}, r_{ij}))^t.$$

Whenever $q \geq 0$, the right hand term is bounded by

$$\sum_{i=1}^{\xi} \sum_j \nu(B(x_{ij}, r_{ij}))^t.$$

For $t = 1$, this becomes

$$\sum_{i=1}^{\xi} \sum_j \nu(B(x_{ij}, r_{ij})).$$

As the $(B(x_{ij}, r_{ij}))_j$ are disjoint, the last quantity will be bounded by

$$\sum_{i=1}^{\xi} \nu\left(\underset{j}{\cup} B(x_{ij}, r_{ij})\right) \leq \xi \nu(\mathbb{R}^d) = \xi.$$

Consequently

$$\mathcal{H}_{\mu,\nu}^{q,1}(E) < +\infty.$$

Assume now that there exist i, $1 \leq i \leq k$ such that $q_i \leq 0$. For $t > 0$, we get

$$\nu(B(x_i, r_i))^t \leq M^t r_i^{\alpha t}, \forall i.$$

Consequently

$$\sum_i \mu(B(x_i, r_i))^q \nu(B(x_i, r_i))^t \leq 2^{-\alpha} M^t \sum_i \mu(B(x_i, r_i))^q (2r_i)^{\alpha t}.$$

Let next $t > \frac{1}{\alpha}\left[\max\left(1, dim_{\mu}^{q}(E)\right)\right]$. We obtain

$$\mathcal{H}_{\mu,\nu}^{q,t}(E) \leq 2^{-\alpha} M^t \mathcal{H}_{\mu,\nu}^{q,\alpha t}(E) < +\infty.$$

As a consequence of Proposition 2.1, we get the following result, which is a first step to introduce the associated mixed multifractal dimensions.

Proposition 2.2 *i)* $\mathcal{H}_{\mu,\nu}^{q,t}(E) < +\infty \Rightarrow \mathcal{H}_{\mu,\nu}^{q,s}(E) = 0, \ \forall \ s > t.$
ii) $\mathcal{H}_{\mu,\nu}^{q,t}(E) > 0 \Rightarrow \mathcal{H}_{\mu,\nu}^{q,s}(E) = +\infty, \ \forall \ s < t.$

Proof.
i) Let $(B(x_i, r_i))_i$ a ε-covering of E. It follows from Definition 1.1 that

$$\begin{aligned}\sum_i \mu(B(x_i, r_i))^q \nu(B(x_i, r_i))^s &= \sum_i \mu(B(x_i, r_i))^q \nu(B(x_i, r_i))^t \nu(B(x_i, r_i))^{s-t} \\ &\leq M^{s-t} \delta^{s-t} \sum_i \mu(B(x_i, r_i))^q \nu(B(x_i, r_i))^t.\end{aligned}$$

Consequently

$$\overline{\mathcal{H}}_{\mu,\nu,\delta}^{q,s}(E) \leq M^{s-t} \delta^{s-t} \overline{\mathcal{H}}_{\mu,\nu,\delta}^{q,t}(E).$$

Hence

$$\mathcal{H}_{\mu,\nu}^{q,s}(E) = 0.$$

ii) Using the same arguments, we get

$$\overline{\mathcal{H}}_{\mu,\nu,\delta}^{q,s}(E) \geq M^{s-t} \delta^{s-t} \overline{\mathcal{H}}_{\mu,\nu,\delta}^{q,t}(E) \quad (\text{as } s - t < 0).$$

Consequently,

$$\mathcal{H}_{\mu,\nu}^{q,s}(E) = +\infty.$$

We are now able to introduce the generalised mixed multifractal Hausdorff and packing dimensions.

Proposition 2.3 *1. There exists a unique number* $dim^q_{\mu,\nu}(E) \in [-\infty, +\infty]$ *such that*

$$\mathcal{H}^{q,t}_{\mu,\nu}(E) = \begin{cases} \infty \ \text{if } t < dim^q_{\mu,\nu}(E), \\ 0 \ \text{if } t > dim^q_{\mu,\nu}(E). \end{cases}$$

2. There exists a unique number $\Delta^q_{\mu,\nu}(E) \in [-\infty, +\infty]$ *such that*

$$\overline{\mathcal{P}}^{q,t}_{\mu,\nu}(E) = \begin{cases} \infty \ \text{if } t < \Delta^q_{\mu,\nu}(E), \\ 0 \ \text{if } t > \Delta^q_{\mu,\nu}(E). \end{cases}$$

3. There exists a unique number $Dim^q_{\mu,\nu}(E) \in [-\infty, +\infty]$ *such that*

$$\mathcal{P}^{q,t}_{\mu,\nu}(E) = \begin{cases} \infty \ \text{if } t < Dim^q_{\mu,\nu}(E), \\ 0 \ \text{if } t > Dim^q_{\mu,\nu}(E). \end{cases}$$

The **proof** follows immediately from Propositions 2.1 and 2.2 by setting

$$dim^q_{\mu,\nu}(E) = \inf\{\, t \in \mathbb{R}\,;\ \mathcal{H}^{q,t}_{\mu,\nu}(E) = 0\},$$

$$\Delta^q_{\mu,\nu}(E) = \inf\{\, t \in \mathbb{R}\,;\ \overline{\mathcal{P}}^{q,t}_{\mu,\nu}(E) = 0\}$$

and

$$Dim^q_{\mu,\nu}(E) = \inf\{\, t \in \mathbb{R}\,;\ \mathcal{P}^{q,t}_{\mu,\nu}(E) = 0\},$$

Definition 2.2 *The quantities* $dim^q_{\mu,\nu}(E)$, $Dim^q_{\mu,\nu}(E)$ *and* $\Delta^q_{\mu,\nu}(E)$ *are called mixed multifractal generalizations of the Hausdorff dimension, the packing dimension and the logarithmic index of* E *respectively.*

Remark that for $k = 1$, we come back to the classical definitions of the Hausdorff and packing measures and dimensions in their original form (by taking $q = 0$) and their generalized multifractal variants for q being arbitrary. The mixed case studied here may be also applied for a single measure and thus the results and characterizations outpointed in the present work remains valid for a single measure. Indeed, denote $Q_i = (0, 0, ..., q_i, 0, ..., 0)$ the vector with zerp coordinates except the ith one which equals q_i, we obtain the multifractal generalizations of the Hausdorff measure and dimension relatively to μ and ν, the packing ν-dimension and the logarthmic ν-index of the set E for the single measure μ_i,

$$dim^{Q_i}_{\mu,\nu}(E) = dim^{q_i}_{\mu_i,\nu}(E),\ Dim^{Q_i}_{\mu,\nu}(E) = Dim^{q_i}_{\mu_i,\nu}(E),\ \Delta^{Q_i}_{\mu,\nu}(E) = \Delta^{q_i}_{\mu_i,\nu}(E).$$

Similarly, for the null vector of $\mathbb{R}^d$, we obtain

$$dim^0_{\mu,\nu}(E) = dim_\nu(E),\ Dim^0_{\mu,\nu}(E) = Dim_\nu(E),\ \Delta^0_{\mu,\nu}(E) = \Delta_\nu(E).$$

In the rest of the paper, we adopt the following notations. For $E \subseteq \mathbb{R}^d$,

$q=(q_1,q_2,...,q_n)\in\mathbb{R}^k$, $t\in\mathbb{R}$, $\mu\in\mathcal{P}(\mathbb{R}^d)$ and $\nu\in\mathcal{QAHP}(\mathbb{R}^d)$, we denote

$$b_{\mu,\nu}(E,q)=dim^q_{\mu,\nu}(E),\ B_{\mu,\nu}(E,q)=Dim^q_{\mu,\nu}(E),\ \Delta_{\mu,\nu}(E)=\Lambda^q_{\mu,\nu}(E).$$

When $E=S_{(\mu,\nu)}$ we denote

$$b_{\mu,\nu}(q)=dim^q_{\mu,\nu}(S_{(\mu,\nu)}),\ B_{\mu,\nu}(q)=Dim^q_{\mu,\nu}(S_{(\mu,\nu)}),\ \Delta_{\mu,\nu}(q)=\Lambda^q_{\mu,\nu}(S_{(\mu,\nu)}).$$

For $x=(x_1,x_2,\ldots,x_k)$ and $q=(q_1,q_2,\ldots,q_k)$ in $\mathbb{R}^k$ we denote

$$|x|=x_1+x_2+\cdots+x_k \ \text{ and } \ x^q=x_1^{q_1}x_2^{q_2}\ldots x_k^{q_k}.$$

Proposition 2.4 *The following assertions hold.*
***a**. $b_{\mu,\nu}(.,q)$ and $B_{\mu,\nu}(.,q)$ and $\Delta_{\mu,\nu}(.,q)$ are non decreasing with respect to the inclusion proprety in $\mathbb{R}^d$.*
***b**. $b_{\mu,\nu}(.,q)$ and $B_{\mu,\nu}(.,q)$ are σ-stable.*

Proof.
a. follows from the non decreasing property of $\mathcal{H}^{q,t}_{\mu,\nu}$, $\mathcal{P}^{q,t}_{\mu,\nu}$ and $\overline{\mathcal{P}}^{q,t}_{\mu,\nu}$ with respect to the inclusion in $\mathbb{R}^d$.
b. follows from the sub-additivity property of $\mathcal{H}^{q,t}_{\mu,\nu}$ and $\mathcal{P}^{q,t}_{\mu,\nu}$ in $\mathbb{R}^d$.

Proposition 2.5 *The following assertions are true.*
***a.** The functions $q\mapsto B_{\mu,\nu}(q)$ and $q\mapsto\Lambda_{\mu,\nu}(q)$ are convex.*
***b.** For $i=1,2,...,k$ and $\overset{\wedge}{q_i}=(q_1,...,q_{i-1},q_{i+1},...,q_k)$ fixed, the functions $q_i\mapsto b_{\mu,\nu}(q)$, $q_i\mapsto B_{\mu,\nu}(q)$ and $q_i\mapsto\Lambda_{\mu,\nu}(q)$ are non increasing.*

Proof. a. We start by proving that $\Lambda_{\mu,\nu}(E,.)$ is convex. Let $p,q\mathbb{R}^k$, $\alpha\in]0,1[$ and s,t such that

$$s>\Lambda_{\mu,\nu}(E,p) \text{ and } t>\Lambda_{\mu,\nu}(E,q).$$

For $\varepsilon>0$ and $(B_i=B(x_i,r_i))_i$ a centered ε-packing of E, we have

$$\begin{aligned}&\sum_i(\mu(B_i))^{\alpha q+(1-\alpha)p}(\nu(B_i))^{\alpha t+(1-\alpha)s}\\ &\le\left[\sum_i(\mu(B_i))^q(\nu(B_i))^t\right]^\alpha\left[\sum_i(\mu(B_i))^p(\nu(B_i))^s\right]^{1-\alpha}.\end{aligned}$$

Hence,

$$\overline{\mathcal{P}}^{\alpha q+(1-\alpha)p,\alpha t+(1-\alpha)s}_{\mu,\nu,\varepsilon}(E)\le(\overline{\mathcal{P}}^{q,t}_{\mu,\nu,\varepsilon}(E))^\alpha(\overline{\mathcal{P}}^{p,s}_{\mu,\nu,\varepsilon}(E))^{1-\alpha}.$$

The limit on $\varepsilon\downarrow 0$ gives

$$\overline{\mathcal{P}}^{\alpha q+(1-\alpha)p,\alpha t+(1-\alpha)s}_{\mu,\nu}(E)\le(\overline{\mathcal{P}}^{q,t}_{\mu,\nu}(E))^\alpha(\overline{\mathcal{P}}^{p,s}_{\mu,\nu}(E))^{1-\alpha}.$$

Consequently,

$$\overline{\mathcal{P}}_{\mu,\nu}^{\alpha q+(1-\alpha)p,\alpha t+(1-\alpha)s}(E)=0,\ \forall\ s>\Lambda_{\mu,\nu}(E,p) \text{ and } t>\Lambda_{\mu,\nu}(E,q).$$

It results that

$$\Lambda_{\mu,\nu}(\alpha q+(1-\alpha)p,\ E)\leq\alpha\ \Lambda_{\mu,\nu}(E,q)+(1-\alpha)\Lambda_{\mu,\nu}(E,p).$$

We now prove the convexity of $B_{\mu,\nu}(E,.)$. We set in this case

$$t=B_{\mu,\nu}(E,q) \text{ and } s=B_{\mu,\nu}(E,p).$$

We have

$$\mathcal{P}_{\mu,\nu}^{q,t+\varepsilon}(E)=\mathcal{P}_{\mu,\nu}^{q,s+\varepsilon}(E)=0.$$

Therefore, there exists $(H_i)_i$ and $(K_i)_i$ coverings of the set E for which

$$\sum_i\overline{\mathcal{P}}_{\mu,\nu}^{q,t+\varepsilon}(H_i)\leq C<+\infty \text{ and } \sum_i\overline{\mathcal{P}}_{\mu,\nu}^{p,s+\varepsilon}(K_i)\leq C<+\infty.$$

C being a positive constant. Then, the sequence $\Big(E_n=\underset{1\leq i,j\leq n}{\cup}(H_i\cap K_i)\Big)_{n\in\mathbb{N}}$ is a covering of E. So that,

$$\begin{aligned}
&\mathcal{P}_{\mu,\nu}^{\alpha q+(1-\alpha)p,\alpha t+(1-\alpha)s}(E_n)\\
\leq&\sum_{i,j=1}\mathcal{P}_{\mu,\nu}^{\alpha q+(1-\alpha)p,\alpha t+(1-\alpha)s}(H_i\cap K_j)\\
\leq&\sum_{i,j=1}\overline{\mathcal{P}}_{\mu,\nu}^{\alpha q+(1-\alpha)p,\alpha t+(1-\alpha)s}(H_i\cap K_j)\\
\leq&\ (\sum_{i,j=1}\overline{\mathcal{P}}_{\mu,\nu}^{q,t+\varepsilon}(H_i\cap K_j))^{\alpha}(\sum_{i,j=1}\overline{\mathcal{P}}_{\mu,\nu}^{p,s+\varepsilon}(H_i\cap K_j))^{1-\alpha}\\
\leq&\ (nC)^{\alpha}(nC)^{1-\alpha}=nC<\infty.
\end{aligned}$$

Consequently,

$$B_{\mu,\nu}(E_n,\alpha q+(1-\alpha)p)\leq\alpha t+(1-\alpha)s+\varepsilon,\ \forall\,\varepsilon>0.$$

Hence,

$$B_{\mu,\nu}(E,\alpha q+(1-\alpha)p)\leq\alpha B_{\mu,\nu}(E,q)+(1-\alpha)B_{\mu,\nu}(E,p).$$

b. For $i=1,2,...,n$ and $\overset{\wedge}{q_i}=(q_1,...,q_{i-1},q_{i+1},...,q_k)$ fixed and $p_i\leq q_i$ denote

$$q=(q_1,...,q_{i-1},q_i,q_{i+1},...,q_k) \text{ and } p=(q_1,...,q_{i-1},p_i,q_{i+1},...,q_k).$$

For any $A\subseteq E$ and $(B(x_i,r_i))_i$ a centered ε-covering of A we have

$$(\mu(B(x_i,r_i)))^q(\nu(B(x_i,r_i)))^t\leq(\mu(B(x_i,r_i)))^p(\nu(B(x_i,r_i)))^t,\ \forall\,t\in\mathbb{R}.$$

Hence,
$$\overline{\mathcal{H}}^{q,t}_{\mu,\nu,\varepsilon}(A) \le \overline{\mathcal{H}}^{p,t}_{\mu,\nu,\varepsilon}(A),\ \forall\, A \subseteq E.$$
When $\varepsilon \downarrow 0$, we obtain
$$\overline{\mathcal{H}}^{q,t}_{\mu,\nu}(A) \le \overline{\mathcal{H}}^{p,t}_{\mu,\nu}(A),\ \forall\, A \subseteq E.$$
Therefore,
$$\mathcal{H}^{q,t}_{\mu,\nu}(E) \le \mathcal{H}^{p,t}_{\mu,\nu}(E).$$
As a result,
$$\mathcal{H}^{q,t}_{\mu,\nu}(E) = 0,\ \forall t > b_{\mu,\nu}(E,p).$$
Consequently
$$b_{\mu,\nu}(E,q) < t,\ \forall t > b_{\mu,\nu}(E,p).$$
Which means that
$$b_{\mu,\nu}(E,q) \le b_{\mu,\nu}(E,p).$$
The proof of the monotonicity of $B_{\mu,\nu}(E,.)$ and $\Lambda_{\mu,\nu}(E,.)$ is similar.

Proposition 2.6 *Let $\mu = (\mu_1, \mu_2, ..., \mu_k) \in \mathcal{P}(\mathbb{R}^d)^k$ and $\nu \in \mathcal{QAHP}(\mathbb{R}^d)$. We have*
1. $0 \le b_{\mu,\nu}(q) \le B_{\mu,\nu}(q) \le \Lambda_{\mu,\nu}(q)$, *whenever $q_i < 1$ for all $i = 1, 2, ..., k$.*
2. $b_{\mu,\nu}(e_i) = B_{\mu,\nu}(e_i) = \Lambda_{\mu,\nu}(e_i) = 0$ *with $e_i = (0, 0, 0..., 1, 0, 0..., 0)$.*
3. $b_{\mu,\nu}(q) \le B_{\mu,\nu}(q) \le \Lambda_{\mu,\nu}(q) \le 0$, *whenever $q_i > 1$ for all $i = 1, 2, ..., k$.*

Proof. Using (2) we get
$$b_{\mu,\nu}(E,q) \le B_{\mu,\nu}(E,q) \le \Lambda_{\mu,\nu}(E,q),\ \forall q \in \mathbb{R}^k.$$

We are going to prove now that $b_{\mu,\nu}(e_i) \ge 0$ and $\Lambda_{\mu,\nu}(e_i) \le 0$ with $e_i = (0, 0, ..., 0, 1, 0, ..., 0)$. Indeed if $t < 0$, $0 < \varepsilon < \frac{1}{2}$ and $(B(x_i, r_i))_i$ is a centered ε-covering of E, then
$$\sum_i \mu^{e_i}(B(x_i, r_i))\nu^t(B(x_i, r_i)) \ge 1 \ \Rightarrow\ \overline{\mathcal{H}}^{e_i,t}_{\mu,\nu,\varepsilon}(E) \ge 1,\ \forall t > 0.$$

Therefore
$$t \le b_{\mu,\nu}(e_i), \forall t < 0.$$
Consequently
$$b_{\mu,\nu}(e_i) \ge 0.$$
Consider now $t > 0$, $0 < \delta < \frac{1}{2}$ and $(B(x_i, r_i))_i$ is a centered ε-packing of E, then
$$\overline{\mathcal{P}}^{e_i,t}_{\mu,\nu,\varepsilon}(E) \le \sum_i \mu^{e_i}(B(x_i, r_i))\nu^t(B(x_i, r_i)) \le 1,$$
consequently
$$\overline{\mathcal{P}}^{e_i,t}_{\mu,\nu,\varepsilon}(E) \le 1,\ \forall\ t > 0.$$
which implies
$$\Lambda_{\mu,\nu}(e_i) \le t,\ \forall t > 0.$$

Finally

$$\Lambda_{\mu,\nu}(e_i) \leq 0.$$

Conclusion:
1) If $q_i > 1 \; \forall \; i = 1, 2, ..., n$, we have $\Lambda^q_{\mu,\nu}(q) < \Lambda_{\mu,\nu}(e_i) \leq 0$, then

$$b_{\mu,\nu}(q) \leq B_{\mu,\nu}(q) \leq \Lambda^q_{\mu,\nu}(q) \leq \Lambda_{\mu,\nu}(e_i) \leq 0.$$

2) If $q_i < 1 \; \forall \; i = 1, 2, ..., n$, we have $b_{\mu,\nu}(q) > b_{\mu,\nu}(e_i) \geq 0$, then

$$0 \leq b_{\mu,\nu}(e_i) \leq b_{\mu,\nu}(q) \leq B_{\mu,\nu}(q) \leq \Lambda_{\mu,\nu}(q).$$

we also have

$$\forall q \in \mathbb{R}^k, \; b_{\mu,\nu}(q) \leq B_{\mu,\nu}(q) \leq \Lambda^q_{\mu,\nu}(q),$$

Then

$$0 \leq b_{\mu,\nu}(e_i) \leq B_{\mu,\nu}(e_i) \leq \Lambda_{\mu,\nu}(e_i) \leq 0.$$

Which implies that

$$b_{\mu,\nu}(e_i) = B_{\mu,\nu}(e_i) = \Lambda_{\mu,\nu}(e_i) = 0.$$

Next we need to introduce the following quantities which will be useful later. Let $\mu = (\mu_1, \mu_2, ..., \mu_k)$ be a vector valued measure composed of probability measures on $\mathbb{R}^d$. For $j = 1, 2, ..., k$, $a > 1$ and $E \subseteq S_\mu$ we denote

$$T^j_a(\mu) = \limsup_{r \downarrow 0} (\sup_{x \in S_\mu} \frac{\mu_j(B(x, ar))}{\mu_j(B(x, r))})$$

and for $x \in S_\mu$, $T^j_a(x) = T^j_a(\{x\})$. We define also $P_D(\mathbb{R}^n)$ the family of doubling probability measures on $\mathbb{R}^n$ by

$$P_D(\mathbb{R}^d) = \left\{ \mu \in P(\mathbb{R}^d); \; T^j_a(\mu) < \infty \text{ for some } a, \; \forall \; j \right\}.$$

We denote also

$$\mathcal{QAHP}_D(\mathbb{R}^d) = \mathcal{QAHP}(\mathbb{R}^d) \cap P_D(\mathbb{R}^d).$$

Obviously, these sets are independent of a.

Proposition 2.7 *Let $\mu = (\mu_1, \mu_2, ..., \mu_k) \in \mathcal{P}(\mathbb{R}^d)^k$ and $\nu \in \mathcal{QAHP}_D(\mathbb{R}^d)$, $E \subset \mathbb{R}^d$, $p, q \in \mathbb{R}^k$ and $\alpha \in [0, 1]$. Then we have*

$$b_{\mu,\nu}(E, \alpha p + (1 - \alpha)q) \leq \alpha B_{\mu,\nu}(E, p) + (1 - \alpha) b_{\mu,\nu}(E, q).$$

Proof.
Let $t = B_{\mu,\nu}(E, p)$ and $s = b_{\mu,\nu}(E, q)$. We will prove that

$$b_{\mu,\nu}(E, \alpha p + (1 - \alpha)q) \leq \alpha t + (1 - \alpha)s, \; \forall \; \varepsilon > 0.$$

Let $\varepsilon > 0$, $m \in \mathbb{N}^*$ and denote

$$E_m = \{x \in E;\ \frac{\mu_j(B(x,5r))}{\mu_j(B(x,r))} < m,\ \forall j,\ \frac{\nu(B(x,5r))}{\nu(B(x,r))} < m,\ 0 < r < \frac{1}{m}\}.$$

As $E = \underset{m}{\cup} E_m$, we shall prove that

$$\mathcal{H}_{\mu,\nu}^{\alpha p+(1-\alpha)q,\alpha t+(1-\alpha)s+\varepsilon}(E_m) < \infty\ ,\ \forall\ m \in \mathbb{N}^*.$$

So, let $F \subset E_m$ and consider a covering $(F_i)_i$ of F and $\delta > 0$. Let next $\varepsilon > 0$, $i \in \mathbb{N}$ and choose $0 < \delta_i$ such that

$$\overline{\mathcal{P}}_{\mu,\nu,\delta_i}^{p+\varepsilon,t}(F_i) \leq \overline{\mathcal{P}}_{\mu,\nu}^{p+\varepsilon,t}(F_i) + \frac{1}{2^i}.$$

Since $F_i \cap F \subset F \subset E$ then,

$$b_{\mu,\nu}(F_i \cap F, q) \leq b_{\mu,\nu}(E, q) = s < s + \varepsilon.$$

Consequently

$$b_{\mu,\nu}(F_i \cap F, q) < s + \varepsilon.$$

Which yields that

$$\overline{\mathcal{H}}_{\mu,\nu}^{q,s+\varepsilon}(F_i \cap F) = 0.$$

There exists consequently a centered $(\frac{\delta}{5} \wedge \frac{1}{m} \wedge \delta_i)$-covering $(B(x_{ij}, r_{ij}))_{j\in I_i}$ of $F_i \cap F$ such that

$$\sum_{j\in I_i} \mu^q(B(x_{ij}, r_{ij}))\nu^{s+\varepsilon}(B(x_{ij}, r_{ij})) \leq \frac{1}{2^i}.$$

Let now $J_i \subset I_i$ composed of disjoint balls such that

$$\underset{j\in I_i}{\cup} B(x_{ij}, r_{ij}) \subset \underset{j\in J_i}{\cup} B(x_{ij}, 5r_{ij}).$$

Since $(B(x_{ij}, 5r_{ij}))_{j\in J_i}$ is a centered δ-covering of $F_i \cap F$ and $(B(x_{ij}, r_{ij}))_{j\in J_i}$ is a centered δ_i-packing of F_i, we obtain

$$\begin{aligned} \overline{\mathcal{H}}_{\mu,\nu,\delta}^{\alpha(p,q),\alpha_\varepsilon(t,s)}(F) &\leq \overline{\mathcal{H}}_{\mu,\nu,\delta}^{\alpha(p,q),\alpha_\varepsilon(t,s)}\Big(\underset{i}{\cup}\underset{j\in J_i}{\cup} B(x_{ij}, 5r_{ij})\Big) \\ &\leq \sum_i \sum_{j\in J_i} [\mu(B(x_{ij}, 5r_{ij}))]^{\alpha(p,q)} [\nu(B(x_{ij}, 5r_{ij}))]^{\alpha_\varepsilon(t,s)} \end{aligned} \tag{3}$$

where $\alpha(p,q) = \alpha p + (1-\alpha)q$ and $\alpha_\varepsilon(t,s) = \alpha t + (1-\alpha)s + \varepsilon$. Consequently, whenever $\alpha(p,q) \in (0,+\infty)^k 0$ and $\alpha_\varepsilon(t,s) \in (0,+\infty)$, we get

$$[\mu(B(x_{ij}, 5r_{ij}))]^{\alpha(p,q)} \leq\ m^{|\alpha(p,q)|}\Big[\mu(B(x_{ij}, r_{ij}))\Big]^{\alpha(p,q)}$$

and

$$[\nu(B(x_{ij}, 5r_{ij}))]^{\alpha_\varepsilon(t,s)} \leq\ m^{\alpha_\varepsilon(t,s)}\Big[\nu(B(x_{ij}, r_{ij}))\Big]^{\alpha_\varepsilon(t,s)}.$$

Consequently, using (3) we get

$$\overline{\mathcal{H}}^{\alpha(p,q),\alpha_\varepsilon(t,s)}_{\mu,\nu,\delta}(F) \leq m^{|\alpha(p,q)|+\alpha_\varepsilon(t,s)}(\sum_i(\overline{\mathcal{P}}^{p,t+\varepsilon}_{\mu,\nu}(F_i) + \frac{1}{2^i}))^\alpha. \tag{4}$$

Which yields that

$$\overline{\mathcal{H}}^{\alpha(p,q),\alpha_\varepsilon(t,s)}_{\mu,\nu}(F) \leq m^{|\alpha(p,q)|+\alpha_\varepsilon(t,s)}(\sum_i\overline{\mathcal{P}}^{p,t+\varepsilon}_{\mu,\nu}(F_i) + 1)^\alpha. \tag{5}$$

Hence,

$$\overline{\mathcal{H}}^{\alpha(p,q),\alpha_\varepsilon(t,s)}_{\mu,\nu}(F) \leq m^{|\alpha(p,q)|+\alpha_\varepsilon(t,s)}(\mathcal{P}^{p,t+\varepsilon}_{\mu,\nu}(F) + 1)^\alpha,\ \forall\ F \subseteq E_m.$$

Which in turn implies that

$$\overline{\mathcal{H}}^{\alpha(p,q),\alpha_\varepsilon(t,s)}_{\mu,\nu}(E_m) \leq m^{|\alpha(p,q)|+\alpha_\varepsilon(t,s)}(\mathcal{P}^{p,t+\varepsilon}_{\mu,\nu}(E_m) + 1)^\alpha.$$

Consequently,

$$\overline{\mathcal{H}}^{\alpha(p,q),\alpha_\varepsilon(t,s)}_{\mu,\nu}(E_m) < \infty, \forall m.$$

Therefore,

$$b_{\mu,\nu}(E_m, \alpha(p,q)) \leq \alpha_\varepsilon(t,s), \forall \varepsilon > 0,\ \forall m.$$

Which yields finally that

$$b_{\mu,\nu}(E, \alpha p + (1-\alpha)q) \leq \alpha t + (1-\alpha)s = \alpha B_{\mu,\nu}(E,p) + (1-\alpha)b_{\mu,\nu}(E,q).$$

Corollary 2.1 *Let $\mu = (\mu_1, \mu_2, ..., \mu_k) \in \mathcal{P}(\mathbb{R}^d)^k$ and $\nu \in \mathcal{QAHP}_D(\mathbb{R}^d)$, $q \in \mathbb{R}^k$ and $E \subseteq S_\mu \cap S_\nu$. The following assertions hold.*
1. Whenever $q_i \leq 0$, $\forall k$, we have

$$b^q_{\mu,\nu}(E) \geq \dim_\nu(E)\left(1 - \frac{|q|}{k}\right). \tag{6}$$

2. Whenever $0 \leq q_i \leq 1$, $\forall k$, we have

$$b^q_{\mu,\nu}(E) \leq \dim_\nu(E)\left(1 - \frac{|q|}{k}\right) \leq \frac{n}{k}(k - |q|). \tag{7}$$

3. Whenever $q_i \geq 1$, $\forall k$, we have

$$b^q_{\mu,\nu}(E) \geq \frac{\beta}{\beta-1}\dim_\nu(E) \geq \frac{\beta}{\beta-1}n, \quad \textit{with } \beta = \max_i(1 - \frac{1}{q_i}). \tag{8}$$

Proof. 1. For $q=(q_1,...,q_k)\in\mathbb{R}^k$ take in Proposition 2.7, $p=e_i$, $\widetilde{q}_i=q_ie_i$ and $\alpha=\frac{-q_i}{1-q_i}$. As $q_i\leq 0$, $\forall i$, we get in one hand

$$b_{\mu,\nu}(0)\leq\alpha B_{\mu,\nu}(e_i)+(1+\frac{q_i}{1-q_i})b_{\mu,\nu}(\widetilde{q}_i).$$

Recall now that $B_{\mu,\nu}(e_i)=0$. Therefore,

$$(1-q_i)b_{\mu,\nu}(0)\leq b_{\mu,\nu}(\widetilde{q}_i)\leq b_{\mu,\nu}(q).$$

Which implies that

$$(1-q_i)\dim_\nu(E)\leq b^q_{\mu,\nu}(E).$$

The summution on $i=1,2,...,k$ gives

$$b^q_{\mu,\nu}(E)\geq\dim_\nu(E)\big(1-\frac{|q|}{k}\big).$$

2. For $q=(q_1,...,q_k)\in\mathbb{R}^k$ take in Proposition 2.7, $p=e_i$, $\widetilde{q}_i=q_ie_i$ and $\alpha=q_i$ and follow similar techniques as in assertion **1.**
3. For $q=(q_1,...,q_k)\in\mathbb{R}^k$ take in Proposition 2.7, $p=0$ and $\alpha=\beta$ and follow as usual similar techniques as previously.

Corollary 2.2 *Let $\mu=(\mu_1,\mu_2,...,\mu_k)\in\mathcal{P}(\mathbb{R}^d)^k$ and $\nu\in\mathcal{QAHP}_D(\mathbb{R}^d)$, $q\in\mathbb{R}^k$ and $E\subseteq S_\mu\cap S_\nu$. The following assertions hold.*
1. Whenever $q_i\leq 0$, $\forall k$, we have

$$B^q_{\mu,\nu}(E)\geq\ Dim_\nu(E)\left(1-\frac{|q|}{k}\right).\tag{9}$$

2. Whenever $0\leq q_i\leq 1$, $\forall k$, we have

$$B^q_{\mu,\nu}(E)\leq\ Dim_\nu(E)\left(1-\frac{|q|}{k}\right)\leq\frac{n}{k}(k-|q|).\tag{10}$$

3. Whenever $q_i\geq 1$, $\forall k$, we have

$$B^q_{\mu,\nu}(E)\geq\frac{\beta}{\beta-1}Dim_\nu(E)\geq\frac{\beta}{\beta-1}n,\quad with\ \beta=\ \max_i(1-\frac{1}{q_i}).\tag{11}$$

The proof follows similar techniques as for **Corollary 2.1**.

Proposition 2.8 *Let $\mu=(\mu_1,\mu_2,...,\mu_k)\in\mathcal{P}(\mathbb{R}^d)^k$ and $\nu\in\mathcal{AHP}_D(\mathbb{R}^d)$, $q\in\mathbb{R}^k$ be compactly supported Radon measures on $\mathbb{R}^d$ with $S_\mu\subseteq S_\nu$. Suppose further that the μ_i's are absolutely continuous with respect to Lebesgue measure on S_{μ_i}. Then for all Borel set $E\subset S_\mu$ such that $\mu_i(E)>0$, $\forall i$ and $|q|\in[0,1]$, we have*

$$\alpha b_{\mu,\nu}(E,q)\geq(1-|q|),$$

with $|q| = q_1 + q_2 + ... + q_k$ *and* α *is the Ahlfor's regularity index of* ν*.*

Proof. Let $q \in [0,1]^k$. Since μ_i is absolutely continuous with respect to Lebesgue measure on S_{μ_i} then forall i, there exists a function $g_i \geq 0$ such that $\mu_i = g_i \lambda^n_{|S_{\mu_i}}$. Therefore, as $\mu_i(E) > 0$, there exists a Borel set $B_i \subset E$ such that $\lambda^n(B_i) > 0$ and a constant $\gamma_i > 0$ satsifying

$$g_i(x) \geq \gamma_i, \ \forall x \in B_i. \tag{12}$$

Let next $\varepsilon > 0$ and $(B(x_i, r_i))_i$ be a centred ε-covering of E. For $t > 0$, we have

$$\sum_i (\mu(B(x_i,r_i)))^q (\nu(B(x_i,r_i)))^t \geq C_\nu \sum_i (\mu(B(x_i,r_i)))^q (\lambda^n(B(x_i,r_i)))^{\alpha t},$$

where C_ν is a constant due to Ahlfor's regularity of ν. Denote next $B = \underset{i}{\cup} B_i \subset E$. It holds from (12) that

$$(\mu(B(x_i,r_i)))^q \geq \gamma^q \Big(\lambda^n(B(x_i,r_i) \cap B \Big)^q.$$

As a result,

$$\sum_i (\mu(B(x_i,r_i)))^q (\nu(B(x_i,r_i)))^t \geq \ C_\nu \gamma^q \sum_i \Big(\lambda^n(B(x_i,r_i) \cap B) \Big)^{|q|+\alpha t}.$$

For $\alpha t < 1 - |q|$, it byields that

$$\sum_i (\mu(B(x_i,r_i)))^q (\nu(B(x_i,r_i)))^t \geq \ C_\nu \gamma^q \lambda^n(B) > 0.$$

Consequently, $\forall\, t$ such that $0 < \alpha t < 1 - |q|$, we get

$$\mathcal{H}^{q,t}_{\mu,\nu}(E) > 0.$$

Which implies that

$$b_{\mu,\nu}(E,q) \geq t.$$

By letting $t \to \dfrac{1-|q|}{\alpha}$, we obtain

$$b^q_{\mu,\nu}(E) \geq \frac{1-|q|}{\alpha}.$$

Proposition 2.9 *Let* $p > 1$*,* $(\mu,\nu) = (\mu_1, \mu_2, ..., \mu_k, \nu) \in \mathcal{P}(\mathbb{R}^d)^k \times \mathcal{AHP}$ *be a vector valued measure composed of compactly supported Radon measures on*

$\mathbb{R}^d$ *with* $S_\mu \subset S_\nu$. *Suppose further that* $\mu_i \in L^p(\mathbb{R}^d)$. *Then for* $q_i \geq 1$, *we have*

$$\alpha B_{\mu,\nu}(q) \leq \max\left\{k - |q|, \frac{-|q|(p-1)}{p}\right\}.$$

Proof. Let for $i = 1, 2, ..., k$, $g_i \in L^p(\mathbb{R}^d)$ be such $d\mu_i = g_i d\lambda^n$ on S_{μ_i}. Of course, the g_i's are compactly supported functions. Assume for instence that $q_i \geq p > 1$, $\forall i$ and let $t > \frac{-|q|(p-1)}{p}$, $\delta > 0$ and $(B(x_i, r_i))_i$ a centred δ-packing of S_μ. Let finally, $g = \max_i g_i$. It holds as in the proof of **Proposition 2.8** that

$$\sum_i (\mu(B(x_i, r_i)))^q (\nu(B(x_i, r_i)))^t \leq C_\nu \sum_i \left(\lambda^n(B(x_i, r_i))\right)^{\alpha t + \frac{|q|(p-1)}{p}} \left(\int_{B(x_i, r_i)} g^p \, d\lambda^n\right)^{\frac{|q|}{p}}.$$

As $\alpha t > \frac{-|q|(p-1)}{p}$, we get

$$\sum_i (\mu(B(x_i, r_i)))^q (\nu(B(x_i, r_i)))^t \leq C_\mu \sum_i \left(\int_{B(x_i, r_i)} g^p \, d\lambda^n\right).$$

Which in turns yields that

$$\sum_i (\mu(B(x_i, r_i)))^q (\nu(B(x_i, r_i)))^t \leq C_\mu (\int g^p d\lambda^n)^{\frac{|q|}{p}} < \infty.$$

Hence,

$$\overline{\mathcal{P}}^{q,t}_{\mu,\nu}(S_\mu) < \infty$$

and consequently,

$$\mathcal{P}^{q,t}_{\mu,\nu}(S_\mu) < \infty.$$

Therefore,

$$B_{\mu,\nu}(E, q) \leq t, \ \forall t > \frac{-|q|(p-1)}{\alpha p}.$$

As a result

$$\alpha B_{\mu,\nu}(E, q) \leq \frac{-|q|(p-1)}{p}.$$

Now, assume that $1 \leq q_i < p$, $\forall i$. Let $t > n - |q|$, $\delta > 0$ and $(B_i = B(x_i, r_i))_i$ a centred δ-packing of S_μ. We get

$$\sum_i (\mu(B_i))^q (\nu(B_i))^t \leq C_\mu \sum_i (\mu(B_i))^q ((\lambda^n(B_i \cap S_\mu))^{\alpha t}).$$

Therefore,

$$\sum_i (\mu(B_i))^q (\nu(B_i))^t \leq C_\mu \sum_i \prod_l \left(\int_{B_i \cap S_\mu} g_l \, d\lambda^n\right)^{q_l} (\lambda^n(B_i \cap S_\mu))^{\alpha t}.$$

Next, using Hölder's inequality, we obtain

$$\sum_i (\mu(B_i))^q (\nu(B_i))^t \leq C_\mu \sum_i \prod_l \left(\int_{B_i \cap S_\mu} g_l^{q_l} d\lambda^n \right) [\lambda^n(B_i \cap S_\mu)]^{q_l - 1} \lambda^n(B_i \cap S_\mu))^{\alpha t}.$$

Which yields that

$$\sum_i (\mu(B_i))^q (\nu(B_i))^t \leq C_\mu \sum_i (\lambda^n(B_i \cap S_\mu))^{\alpha t + |q| - k} \prod_l \left(\int_{B_i \cap S_\mu} g^{q_l} d\lambda^n \right) < C < \infty.$$

As a consequence we get

$$\overline{\mathcal{P}}^{q,t}_{\mu,\nu}(S_\mu) < \infty,$$

which means that

$$B_{\mu,\nu}(q) \leq t, \ \forall t > \frac{k - |q|}{\alpha}.$$

Which in turns yields

$$B_{\mu,\nu}(q) \leq \frac{k - |q|}{\alpha}.$$

Remark 2.1 *For $\alpha = 1$, the measure ν is equivalent to the Lebesgue's one. If further $k = 1$, Propositions 2.8 and 2.9 are the classical cases raised by Olsen et al.*

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