A PROOF FOR BEAL'S CONJECTURE

Abstract. In the first part of this paper, we show how $a^x - b^y$ can be expressed as a new non-standard binomial formula (to an indeterminate power, n). In the second part, by fixing n to the value of z we compare this binomial formula to the standard binomial formula for c^z to prove the Beal Conjecture.

Introduction The Beal Conjecture (BC),[1] also commonly known as Tijdeman-Zagier Conjecture or sometimes the Mauldin Conjecture, states that if $A^x = B^y + C^z$, where A, B, C, x, y, z are positive integers and x, y and z are greater than 2, then A, B and C must have a common prime factor. Equivalently, no solutions to this equation exist in positive integers A, B, C, x, y, z with A, B, and C being pairwise co-prime and all of x, y, z being greater than 2. We will use this second formulation to prove BC. Of course, infinite invalid counterexamples exist outside the parameters required by BC.
For example, when A, B, and C are pairwise co-prime but x, y or z is equal to 1 or 2:

$$1^7 + 2^3 = 3^2,$$

$$2^5 + 7^2 = 3^4.$$

Or when A, B, C share a common prime factor but x, y, z are greater than 2:

$$3^9 + 54^3 = 3^{11},$$

$$19^4 + 38^3 = 57^3.$$

But in this paper, we rearrange the equation as $a^x - b^y = c^z$ without loss of integrity to demonstrate how $a^x - b^y$ can be expressed as a binomial formula, containing not only the standard factors found in a standard binomial formula for a single power but also an additional non-standard factor. We will use a proof by contradiction before showing what happens when $x, y, z = 1, 2$ and a, b, c are co-prime.

Definition 0.1. For $a^x - b^y = c^z$ we define $\gcd(a, b, c) = 1$; $a, b, c, x, y, z \in \mathbb{Z}^+$; and $x, y, z > 2$.

Lemma 0.2. *To demonstrate how $a^x - b^y$ can be expressed as a binomial formula.*

We first observe that by adding $[ab(x^{x-2} - b^{y-2}) - b^y]$ to a^x and b^y respectively, and then rearranging, it is possible to reconfigure the expression such that:

$$a^x - b^y = (a + b)(a^{x-1} - b^{y-1}) - ab(a^{x-2} - b^{y-2}). \tag{0.1}$$

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[1]See www.bealconjecture.com. Last accessed 14.12.17.

Now, since $a^x - b^y = (a+b)(a^{x-1} - b^{y-1}) - ab(a^{x-2} - b^{y-2})$, we can repeat the process in the same way by reconfiguring the components $(a^{x-1} - b^{y-1})$ and $(a^{x-2} - b^{y-2})$ respectively, and expanding it as follows:

(0.2)

$$(a+b)[(a+b)(a^{x-2}-b^{y-2})-ab(a^{x-3}-b^{y-3)})]-ab[(a+b)(a^{x-3}-b^{y-3})-ab(a^{x-4}-b^{y-4)})]$$

$$= (a+b)^2(a^{x-2} - b^{y-2}) - 2ab(a+b)(a^{x-3} - b^{y-3}) + (ab)^2(a^{x-4} - b^{y-4}). \tag{0.3}$$

Repeating the process for $(a^{x-2} - b^{y-2})$, $(a^{x-3} - b^{y-3})$ and $(a^{x-4} - b^{y-4})$, we get:

(0.4)

$$(a+b)^3(a^{x-3}-b^{y-3})-3ab(a+b)^2(a^{x-4}-b^{y-4})+3(ab)^2(a+b)(a^{x-5}-b^{y-5})-(ab)^3(a^{x-6}-b^{y-6}).$$

We can continue to expand $(a^x - b^y)$ *ad infinitum*, and using the binomial formula we can generalise it, for all $n \in \mathbb{Z}^+$, in a non-standard binomial formula:

$$a^x - b^y = \sum_{k=0}^{n} \binom{n}{k} (a+b)^{n-k}(-ab)^k(a^{x-n-k} - b^{y-n-k}). \tag{0.5}$$

By a similar process we also find that

$$a^x + b^y = \sum_{k=0}^{n} \binom{n}{k} (a+b)^{n-k}(-ab)^k(a^{x-n-k} + b^{y-n-k}).$$

However, using the form in (0.5) keeps the proof neater, but bearing both forms in mind we only need consider the value of z in Lemma 0.9 below. And just for comparison, the *standard form* of the binomial theorem (also in n) is:

$$(p-q)^n = \sum_{k=0}^{n} \binom{n}{k} p^{n-k}(-q)^k.$$

Remark 0.3. This new non-standard binomial formula in (0.5) is strange in two respects. First, the n-power is indeterminate. That is to say, its value is not determined by $a^x - b^y$, so that regardless of the value we give to n, the value of $(a^x - b^y)$ never changes. It is this property that allows us to compare the two sides of the title equation more easily. So to fix the value of n, we will let $n = z$ in our non-standard binomial formula.

Secondly, the first three factors of the sum in (0.5), i.e. $\binom{n}{k}$, $(a+b)^{n-k}$, and $(-ab)^k$ all obviously conform to the conventional forms of a standard binomial formula. But the last factor, $(a^{x-n-k} - b^{y-n-k})$, does not *obviously* conform, but may do. This is what we need to investigate.

Let us now turn to the main proof.

Theorem 0.4. *To prove that, for the equation $a^x - b^y = c^z$, integer solutions only exist for the values of x or y or $z = 1, 2$, but not for values of $x, y, z > 2$ when a, b, c are co-prime.*

Proof. Using proof by contradiction, we first assume that there exists a solution for the equation $a^x - b^y = c^z$ for values of $x, y, z > 2$. So if $c^z = a^x - b^y$ then from (0.5) where $n = z$, it follows that:

$$c^z = \sum_{k=0}^{z} \binom{z}{k} (a+b)^{z-k}(-ab)^k(a^{x-z-k} - b^{y-z-k}). \tag{0.6}$$

Now for all $s, t \in \mathbb{Z}$, every integer value of c can be represented in the equation:

$$c = [(a+b)|s| - ab|t|], \tag{0.7}$$

$$\Rightarrow c^z = [(a+b)|s| - ab|t|]^z. \tag{0.8}$$

Remark 0.5. We are using the absolute values of s and t in order to preserve a positive value for c and to preserve the minus sign in both $(-ab)^k$ factors in (0.10).

Using the standard binomial theorem, it follows from (0.8) that;

$$c^z = \sum_{k=0}^{z} \binom{z}{k} [(a+b)|s|]^{z-k} [(-ab)|t|]^k. \tag{0.9}$$

Rearranging this slightly, from (0.6) and (0.9) it follows that:

(0.10)
$$\sum_{k=0}^{z} \binom{z}{k} (a+b)^{z-k} (-ab)^k (a^{x-z-k} - b^{y-z-k}) = \sum_{k=0}^{z} \binom{z}{k} (a+b)^{z-k} (-ab)^k |s|^{z-k} |t|^k.$$

Remark 0.6. According to the normal rules of binomial expansion, the common factors on both sides of the equation (i.e. $\binom{z}{k}$, $(a+b)^{z-k}$ and $(-ab)^k$) will exactly correspond in each and every k^{th} term. So for this equation to have solutions it is also necessary that the remaining factors on both sides, $(a^{x-z-k} - b^{y-z-k})$ and $(|s|^{z-k}|t|^k)$, must also correspond exactly in each counterpart (k^{th}) term, for any given value of z. If it does, then the whole of the left hand side will be a power to z (as we know the right hand side is), and the main equation will have solutions. But if just one term of the corresponding binomials exists where $(a^{x-z-k} - b^{y-z-k})$ does not equal $(|s|^{z-k}|t|^k)$, then not only will the integrity of that particular k^{th} term be compromised as a valid standard binomial term, but also the whole expression as an expansion of a power to z. In the latter scenario, no solutions will exist.

What we will now proceed to show is that when $z > 2$ an inequality always arises in at least one of the corresponding k^{th} terms (for all k terms of the sum *simultaneously* from $k = 0$ to $k = z$), but that when $z = 1, 2$ every corresponding k^{th} term is equal, thus allowing solutions. Without testing for every value of z one by one *ad infinitum* we can test for all values of $z > 2$ in one go, using this equation:

$$\sum_{k=0}^{z} |s|^{z-k}|t|^k = \sum_{k=0}^{z} (a^{x-z-k} - b^{y-z-k}) \tag{0.11}$$

Remark 0.7. If our initial assumption that $c^z = a^x - b^y$ has solutions is correct, this equation in (0.11) should hold true for *all* values of z (*ad infinitum*). But in three steps we show that when $x, y, z > 2$ an inequality arises, thus creating a contradiction to our assumption. In the first step, from the first and last terms (where s and t occur on their own) we will establish the respective values of $|s|, |s|^z, |s|^{z-1}, |t|, |t|^z$, and $|t|^{z-1}$ (in terms of a and b). In the second step, we will use these results to evaluate what the *second* and *penultimate* terms are, and compare them with the second and penultimate terms directly derived from $(a^{x-z-k} - b^{y-z-k})$. In the third step, we will substitute like-terms to reveal the contradictions when they occur. [We will not need to look beyond the second and penultimate terms (even if z is very large) since this is where we find the contradiction in all cases of $z > 2$.]

STEP 1

Using the equation in (0.11), we can establish the respective values of $|s|$, $|s|^z$, $|s|^{z-1}$, $|t|$, $|t|^z$, and $|t|^{z-1}$ in terms of a and b, using first and last terms (i.e. $k = 0$ and $k = z$). So when $k = 0$, the first term in the binomial series is $|s|^z$, such that:

$$|s|^z = \pm(a^{x-z} - b^{y-z}). \tag{0.12}$$

From this equation it follows that:

$$|s| = \pm(a^{x-z} - b^{y-z})^{1/z}, \tag{0.13}$$

and

$$|s|^{z-1} = \pm(a^{x-z} - b^{y-z})^{(z-1)/z}. \tag{0.14}$$

Likewise, when $k = z$, the last term in the binomial series is $|t|^z$, such that:

$$|t|^z = \pm(a^{x-2z} - b^{y-2z}), \tag{0.15}$$

from which it follows that:

$$|t| = \pm(a^{x-2z} - b^{y-2z})^{1/z}, \tag{0.16}$$

and

$$|t|^{z-1} = \pm(a^{x-2z} - b^{y-2z})^{(z-1)/z}. \tag{0.17}$$

STEP 2

Using these different values of $|s|$ and $|t|$, we are now in a position to work out what the *second* and *penultimate* terms are (in terms of a and b). Thus, from (0.14) and (0.16), it follows that the *second* term, $|s|^{z-1}|t|$, is:

$$\pm(a^{x-z} - b^{y-z})^{(z-1)/z}(a^{x-2z} - b^{y-2z})^{1/z}. \tag{0.18}$$

But we also know, from the right hand side of the equation in (0.11), that the second term in the binomial expansion is $\pm(a^{x-z-1} - b^{y-z-1})$, i.e. when $k = 1$, from which this equation follows:

$$\pm(a^{x-z} - b^{y-z})^{(z-1)/z}(a^{x-2z} - b^{y-2z})^{1/z} = \pm(a^{x-z-1} - b^{y-z-1}). \tag{0.19}$$

Dividing both sides by $\pm(a^{x-z} - b^{y-z})^{(z-2)/z}$ we get:

$$\pm(a^{x-z} - b^{y-z})^{1/z}(a^{x-2z} - b^{y-2z})^{1/z} = \frac{\pm(a^{x-z-1} - b^{y-z-1})}{\pm(a^{x-z} - b^{y-z})^{(z-2)/z}}. \tag{0.20}$$

What about the *penultimate* term? It follows from (0.13) and (0.17) that the penultimate term, $|s||t|^{z-1}$, is:

$$\pm(a^{x-z} - b^{y-z})^{1/z}(a^{x-2z} - b^{y-2z})^{(z-1)/z}. \tag{0.21}$$

But we also know, from the right hand side of the equation in (0.11), that the penultimate term in the binomial expansion is $\pm(a^{x-2z+1} - b^{y-2z+1})$, i.e. when $k = z - 1$, from which this equation follows:

$$\pm(a^{x-z} - b^{y-z})^{1/z}(a^{x-2z} - b^{y-2z})^{(z-1)/z} = \pm(a^{x-2z+1} - b^{y-2z+1}). \tag{0.22}$$

This time we divide both sides by $(a^{x-2z} - b^{y-2z})^{(z-2)/z}$ to get:

$$\pm(a^{x-z} - b^{y-z})^{1/z}(a^{x-2z} - b^{y-2z})^{1/z} = \frac{\pm(a^{x-2z+1} - b^{y-2z+1})}{\pm(a^{x-2z} - b^{y-2z})^{(z-2)/z}}. \tag{0.23}$$

STEP 3

Thirdly, we are in a position to substitute like-terms. For the left hand sides of the equations in (0.20) and (0.23) are exactly the same. Therefore by substituting like-terms we get:

$$\frac{\pm(a^{x-z-1} - b^{y-z-1})}{\pm(a^{x-z} - b^{y-z})^{(z-2)/z}} = \frac{\pm(a^{x-2z+1} - b^{y-2z+1})}{\pm(a^{x-2z} - b^{y-2z})^{(z-2)/z}} \tag{0.24}$$

We raise both sides by the power of z and rearrange to get:

$$\pm\left(\frac{a^{x-z-1} - b^{y-z-1}}{a^{x-2z+1} - b^{y-2z+1}}\right)^{z} = \pm\left(\frac{a^{x-z} - b^{y-z}}{a^{x-2z} - b^{y-2z}}\right)^{(z-2)} \tag{0.25}$$

We will return shortly to the case of $z = 1, 2$, but at this point we can ignore the $\pm$. The reason for this is that while we accept that different signs might create inequality under certain circumstances, we are only trying to prove a contradiction when there *is* equality. Thus we can simply assume equal polarity and remove the $\pm$ sign. Thereafter, solutions will exist either a) if the large bracketed factors on each side of the equation in (0.25) have a value of 1 (since the main outer exponents are not equal), or b) if both numerators (with respective outer exponents) are equal and simultaneously if both denominators (with respective outer exponents) are equal. Taking these two options in turn:

a) since $(a^{x-z-1} - b^{y-z-1}) \neq (a^{x-2z+1} - b^{y-2z+1})$, and $(a^{x-2z} - b^{y-2z}) \neq (a^{x-z} - b^{y-z})$, neither side in (0.25) has a value of 1, eliminating this option;

b) even without exponents outside the brackets, the base $(a^{x-2z+1} - b^{y-2z+1})$ is greater than $(a^{x-2z} - b^{y-2z})$; but when the exponent, z, is greater (i.e. than $z-2$), then the inequality is even greater. So it follows that $(a^{x-2z+1} - b^{y-2z+1})^{z} \neq (a^{x-2z} - b^{y-2z})^{(z-2)}$.

Having now eliminated both options it follows that, for all values of $x, y, z > 2$:

$$\sum_{k=0}^{z} |s|^{z-k}|t|^{k} \neq \sum_{k=0}^{z}(a^{x-z-k} - b^{y-z-k}). \tag{0.26}$$

However, this contradicts our equation in (0.11). Under these circumstances there is no equality. And so our initial assumption that for any value of $x, y, z > 2$ solutions exist for the equation $c^{z} = a^{x} - b^{y}$ is false. Thus, BC is true. □

Remark 0.8. We have now formally proved BC, but the question remains about what happens when a) $z = 1, 2$ and a, b, c remain co-prime.

Lemma 0.9. *Prove that solutions exist when $z = 1, 2$ and a, b, c remain co-prime.*

What happens when $z = 1, 2$? Well, these cases resolve smoothly even when $x, y > 2$. From (0.25), when $z = 1$ (and again when there is equal polarity of pre-bracket signs) it follows that:

$$\left(\frac{a^{x-2} - b^{y-2}}{a^{x-1} - b^{y-1}}\right)^{1} = \left(\frac{a^{x-1} - b^{y-1}}{a^{x-2} - b^{y-2}}\right)^{-1}, \tag{0.27}$$

$$\Rightarrow \left(\frac{a^{x-2} - b^{y-2}}{a^{x-1} - b^{y-1}}\right) = \left(\frac{a^{x-2} - b^{y-2}}{a^{x-1} - b^{y-1}}\right), \tag{0.28}$$

Thus, when the signs are equal on both sides, there is no contradiction. And again from (0.25), when $z = 2$ (and again when there is equal polarity of pre-bracket signs), it follows that:

$$\left(\frac{a^{x-3}-b^{y-3}}{a^{x-3}-b^{y-3}}\right)^2 = \left(\frac{a^{x-2}-b^{y-2}}{a^{x-4}-b^{y-4}}\right)^0, \tag{0.29}$$

$$\Rightarrow 1 = 1. \tag{0.30}$$

Again, no contradiction. So in both cases, when $z = 1$ and when $z = 2$, the standard rules of binomial expansion can be applied to our non-standard binomial expression without contradiction such that $(a^{x-z-k} - b^{y-z-k})$ is equal to $|s|^{z-k}|t|^k$, and therefore that in these cases solutions to the original equation exist.

Remark 0.10. Finally, it is worth mentioning the obvious point that we can apply the same method to Fermat's Last Theorem with the same result.

References

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