A Generalized Solution to a Specific Type of Power Series and its Trigonometric and Hyperbolic Extensions

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1 Abstract

In this paper, the power series is looked at from a different perspective.The Summation of $n^m a^{-bn}$ is evaluated using a new method.An assumption is made that the power $-bn$ is multiplied by x where $x = 1$;then the series is integrated m times in order to cancel the term n^m out leaving the term a^{-bn}.By simply taking the derivative of the result m-times, an expression to evaluate the series arises ,which include only a constant term and the mth order derivative of the summation of a simple geometric series. Further applications of the method are used to evaluate the series with the cosine, sine, hyperbolic sine, and hyperbolic cosine. Finally, the method is used to further simplify some of the hardest forms of series to deal with: the series $n^m \sinh^v(ny)$, and $n^m \cosh^v(ny)$ and $n^m \sin^v(ny)$.

2 Introduction

In Boltzman-Gibbs distribution of energy, the probability as a function of temperature is given by the relation $\frac{\sum_k x_k e^{-\varepsilon_k/T}}{\sum_k e^{-\varepsilon_k/T}}$ Yet, this formula and many others (like the quantum system of Oscillators [2]) are evaluated using computers and software programs due to the difficulty in evaluating the series $\sum_k x_k e^{-\varepsilon_k/T}$ using a manual technique, especially for numerous number of particles. Additionally, no general formula was introduced to reduce this type of series when x_k is raised to the power of m, where m can be any integer number. The purpose of this paper is to introduce a method to evaluate the general form of this type of power series. This is achieved by integrating and then differentiating the series m-times with respect to an implemented variable, which is set =1. Finally, the method is applied in more complex forms that contain hyperbolic and trigonometric functions to the power of v.

3 Derivation of the method

It is too difficult to deal with sums in general, especially when they contain power terms. Therefore, a shortcut was formulated to be used as a method of expanding a certain form of power series.

Theorem(Th1): Suppose there is a finite sum $\sum_{n=0}^{k} n^m a^{-bn}$ where ($m \in N, n$ and $a \in R, b \neq 0$)

Then, this sum can be expressed as : ‘

$$(\frac{-1}{b \ln a})^m \frac{d^m}{dx^m}(\frac{1 - a^{-b(k+1)x}}{1 - a^{-bx}}) \tag{1}$$

(where: $m \in N, n$ and $a \in R, b \neq 0, x = 1$.)

Proof: The power of a can be multiply by x where $x = 1$. Then the sum becomes:

$$S = \sum_{n=0}^{k} n^m a^{-bnx} \tag{2}$$

Integrating S with respect to x,using geometric series law,

$$\int S dx = \frac{-1}{b \ln a} \sum_{n=0}^{k} n^{m-1} a^{-bnx} \tag{3}$$

Then, by integrating S (m times), the mth integral of S becomes:

$$\int\int .....\int S(dx)^m = (\frac{-1}{b \ln a})^m \sum_{n=0}^{k} a^{-bnx} \tag{4}$$

Now, by solving $\sum_{n=0}^{k} a^{-bnx}$using geometric series law,

$$\sum_{n=0}^{k} a^{-bnx} = (\frac{1 - a^{-b(k+1)x}}{1 - a^{-bx}}) \tag{5}$$

From equations (4), (5), it is concluded that the mth integral of

$$S = (\frac{-1}{b \ln a})^m (\frac{1 - a^{-b(k+1)x}}{1 - a^{-bx}}) \tag{6}$$

Now, by applying the fundamental theorem of calculus, it is found that:

$$S = \sum_{n=0}^{k} n^m a^{-bn} = \frac{d^m}{dx^m}(\frac{-1}{b \ln a})^m (\frac{1 - a^{-b(k+1)x}}{1 - a^{-bx}}) = (\frac{-1}{b \ln a})^m \frac{d^m}{dx^m}(\frac{1 - a^{-b(k+1)x}}{1 - a^{-bx}}) \tag{7}$$

Similarly, an expression for the positive power of a can be derived :

$$S = (\frac{1}{b \ln a})^m \frac{d^m}{dx^m}(\frac{1 - a^{b(k+1)x}}{1 - a^{bx}}) \tag{8}$$

Further simplification can be achieved in the case that the series is evaluated between zero and infinity, where m is a natural number bigger than zero, provided that it converges.To determine when S converges, the series $S = \sum_{n=0}^{k} n^m a^{-bn}$is evaluated between zero and infinity.By testing convergence of the function by D'Alembert test, $\lim_{n\to\infty}(\frac{S(n+1)}{S(n)})$ has to be found. By solving the limit,

$$\lim_{n\to\infty}(\frac{(n+1)^m.a^{-b(n+1)}}{n^m.a^{-bn}}) = \lim_{n\to\infty}((\frac{n+1}{n})^m.\frac{a^{-b(n+1)}}{a^{-bn}}) = \lim_{n\to\infty}(\frac{(1+\frac{1}{n})^m}{a^b}) = a^{-b} \tag{9}$$

And thus, three cases are to be considered.The first case is when $a^{-b} = 1$, then $b = 0$, and the sum turns into $S = \sum_{n=0}^{k} n^m$.Now, applying the integral test of convergence,

$$\lim_{c\to\infty}(\int_1^c n^m dn) = \lim_{c\to\infty}(\frac{c^{m+1}}{m+1}) - \frac{1}{m+1} = \infty \tag{10}$$

Thus, the series diverges if $a^{-b} = 1$.The second case is when $a^{-b} > 1$; the series automatically fails D'Alembert test, and consequently, it diverges.The third case is when $a^{-b} < 1$; it passes D'Alembert test because $\lim_{n\to\infty}(\frac{S(n+1)}{S(n)}) < 1$, and so, the series converges.

From the previous examination, it is concluded that the series converges if and only if $a^{-b} < 1$ and $m > 0$.

Applying the theorem(Th1) and (7) then taking the limit at infinity, the series can be written as:

$$\lim_{k\to\infty}(\frac{-1}{b\ln a})^m \frac{d^m}{dx^m}(\frac{1-a^{-b(k+1)x}}{1-a^{-bx}}) = (\frac{-1}{b\ln a})^m \frac{d^m}{dx^m}(\frac{1}{1-a^{-bx}}) \tag{11}$$

4 The Applications of the method

It is possible to apply the method in evaluating sums further complicated functions : sinh and cosh functions [4.1], sin and cos functions [4.2], hyperbolic sine and cosine Raised to the power of v [4.3], and finally sine and cosine functions raised to the power of v [4.4].

4.1 Application of the method in series that include hyperbolic Sine and Cosine

First, let us examine the sinh function. Substituting $\sinh(ny) = \frac{e^{ny}-e^{-ny}}{2}$ in the general form of the series,(where $n \in N, y \in R$), it becomes:

$$\sum_{n=0}^{k} n^m.\sinh(ny) = \frac{1}{2}(\sum_{n=0}^{k} n^m.e^{ny} - \sum_{n=0}^{k} n^m.e^{-ny}) = \frac{1}{2}(\sum_{n=0}^{k} n^m.e^{nyx} - \sum_{n=0}^{k} n^m.e^{-nyx})(\textbf{where}x = 1, y \neq 0) \tag{12}$$

By applying (Th1) and equation (8), putting $a = e$, the following expression is obtained :

$$\sum_{n=0}^{k} n^m.\sinh(ny) = \frac{1}{2}((\frac{1}{y})^m \frac{d^m}{dx^m}(\frac{1-e^{y(k+1)x}}{1-e^{yx}}) - (\frac{-1}{y})^m \frac{d^m}{dx^m}(\frac{1-e^{-y(k+1)x}}{1-e^{-yx}})) \tag{13}$$

Simplifying the expression leads to

$$\sum_{n=0}^{k} n^m.\sinh(ny) = \frac{1}{2}(\frac{1}{y})^m(\frac{d^m}{dx^m}(\frac{1-e^{y(k+1)x}}{1-e^{yx}}) - (-1)^m \frac{d^m}{dx^m}(\frac{1-e^{-y(k+1)x}}{1-e^{-yx}})) \tag{14}$$

when putting $y=0$, the series vanishes: $\sum_{n=0}^{k} n^m.\sinh(ny) = \sum_{n=0}^{k} n^m.0 = 0$

Second, cosh function is examined. Putting cosh function as $\frac{e^{ny}+e^{-ny}}{2}$ in the general form of the series,(where $n \in N, y \in R$), it turns into

$$\sum_{n=0}^{k} n^m.\cosh(ny) = \frac{1}{2}(\sum_{n=0}^{k} n^m.e^{ny} + \sum_{n=0}^{k} n^m.e^{-ny}) = \frac{1}{2}(\sum_{n=0}^{k} n^m.e^{nyx} + \sum_{n=0}^{k} n^m.e^{-nyx})(\textbf{where} x=1, y \neq 0) \tag{15}$$

Now, method(Th1) and equation (8) are applied, putting $a = e$:

$$\sum_{n=0}^{k} n^m.\cosh(ny) = \frac{1}{2}((\frac{1}{y})^m \frac{d^m}{dx^m}(\frac{1-e^{y(k+1)x}}{1-e^{yx}}) + (\frac{-1}{y})^m \frac{d^m}{dx^m}(\frac{1-e^{-y(k+1)x}}{1-e^{-yx}})) \tag{16}$$

Simplifying the expression leads to

$$\sum_{n=0}^{k} n^m.\cosh(ny) = \frac{1}{2}(\frac{1}{y})^m(\frac{d^m}{dx^m}(\frac{1-e^{y(k+1)x}}{1-e^{yx}}) + (-1)^m \frac{d^m}{dx^m}(\frac{1-e^{-y(k+1)x}}{1-e^{-yx}})) \tag{17}$$

when putting $y=0$, the series turns into: $\sum_{n=0}^{k} n^m.\cosh(ny) = \sum_{n=0}^{k} n^m$. $\sum_{n=0}^{k} n^m$ is evaluated using **formula for Bernoulli numbers by using only the recurrence relation.** For reference , the relation is $\sum_{k=1}^{n} = \frac{1}{m+1}\sum_{k=1}^{m}\binom{m+1}{k}B_k n^{m+1-k}$, **where** $B_n = \sum_{k=0}^{n}\frac{1}{k+1}\sum_{r=0}^{k}(-1)^r\binom{k}{r}$[5].

4.2 Application of the method in series with trigonometric Sine and Cosine using complex analysis

The definition $\cos(x) = \mathrm{Re}(e^{ix})$ and $\sin(x) = \mathrm{Im}(e^{ix})$ is used to convert the series containing Sine and Cosine into a series containing $(n^m e^{iny})$ analogously to what was done in [3.2].

First, let us study the cosine function $\cos(ny)$ can be written as the real part of e^{iny} and equally as the real part of the whole series, (where $n \in N, y \in R$).

$$\sum_{n=0}^{k} n^m.\cos(ny) = \mathrm{Re}(\sum_{n=0}^{k} n^m.e^{iny}) = \mathrm{Re}(\sum_{n=0}^{k} n^m.e^{inyx})(\textbf{where}, x=1, y \neq 0) \tag{18}$$

By applying equation (8),putting $a = e$,the series is written as

$$\sum_{n=0}^{k} n^m.\cos(ny) = \mathrm{Re}((\frac{1}{iy})^m \frac{d^m}{dx^m}(\frac{1-e^{iy(k+1)x}}{1-e^{iyx}})) \tag{19}$$

Also, if $y = 0, \cos(0) = 1$, and thus, the series turns into$\sum_{n=0}^{k} n^m$, which is evaluated using **formula for Bernoulli numbers by using only the recurrence relation.**

Second, going through the Sine function, $\sin(ny)$ is,by definition, written as the imaginary part of e^{iny}. Since the imaginary part of $\sin(ny)$ is the same as taking the imaginary part of the whole series,(where $n \in N, y \in R$); the series can be written as

$$\sum_{n=0}^{k} n^m. \sin(ny) = \mathrm{Im}(\sum_{n=0}^{k} n^m.e^{iny}) = \mathrm{Im}(\sum_{n=0}^{k} n^m.e^{inyx})(\textbf{where}, x = 1, y \neq 0) \tag{20}$$

Now, equation (8) is applied putting $a = e$:

$$\sum_{n=0}^{k} n^m. \sin(ny) = \mathrm{Im}((\frac{1}{iy})^m \frac{d^m}{dx^m}(\frac{1 - e^{iy(k+1)x}}{1 - e^{iyx}})) \tag{21}$$

Also, it is obvious that if $y = 0, \sin(0) = 0$, and thus, the series vanishes.

4.3 Application of the method in Hyperbolic Sine and Cosine functions raised to the power of v

In this section, the same series is evaluated, but this time sinh and cosh functions are raised to the power of v (where $v \in N$)

First, let us consider the sinh function.Putting $\sinh^v(y) = (\frac{e^y - e^{-y}}{2})^v$, a binomial expansion can be performed.

$$\sinh^v(y) = (\frac{e^y - e^{-y}}{2})^v = (\frac{1}{2})^v \sum_{u=0}^{v} (-1)^u.({}^vC_u).(e^y)^{v-u}.(e^{-y})^u = (\frac{1}{2})^v \sum_{u=0}^{v} (-1)^u({}^vC_u)(e^y)^{v-2u} \tag{22}$$

Then, two cases arise : when v is odd and when v is even
If v is odd, then there is an even number of terms after expansion, and so there is no single middle term.

$$\begin{aligned} \sinh^v(y) = (\frac{1}{2})^v(({}^vC_0)(-1)^0(e^y)^{v-2(0)} + ({}^vC_1)(-1)^1(e^y)^{v-2(1)} + ... \\ +({}^vC_{v-1})(-1)^{v-1}(e^y)^{v-2(v-1)} + ({}^vC_v)(-1)^v(e^y)^{v-2(v)}) \end{aligned} \tag{23}$$

Since ${}^vC_u ={}^v C_{v-u}$ whenever $v > u$, the 1^{st} term can be combined with the last term, the second term can be combined with the $(v-1)^{th}$ term, and so on.

$$\sinh^v(y) = (\frac{1}{2})^{v-1}(\frac{({}^vC_0)(e^{vy}-e^{-vy})}{2} - \frac{({}^vC_1)(e^{(v-2)y}-e^{-(v-2)y})}{2} + ... + (-1)^{\frac{v-1}{2}}\frac{({}^vC_{\frac{v-1}{2}})(e^{y}-e^{-y})}{2})$$
$$= (\frac{1}{2})^{v-1}({}^vC_0\sinh(vy) -{}^v C_1\sinh((v-2)y) + ... + (-1)^{\frac{v-1}{2}}({}^vC_{\frac{v-1}{2}})\sinh(y))$$
$$= (\frac{1}{2})^{v-1}\sum_{u=0}^{\frac{v-1}{2}}(-1)^u({}^vC_u)\sinh((v-2u)y) \quad (24)$$

If v is even, there is an odd number of terms after the expansion; therefore, there is a single middle term which can no be combined with a similar term. From equation (22)

$$\sinh^v(y) = (\frac{1}{2})^v[({}^vC_0)(-1)^0(e^y)^{v-2(0)} + ({}^vC_1)(-1)^1(e^y)^{v-2} + ... + ({}^vC_{\frac{v}{2}-1})(-1)^{(\frac{v}{2}-1)}(e^y)^{v-2(\frac{v}{2}-1)}$$
$$+({}^vC_{\frac{v}{2}})(-1)^{\frac{v}{2}}(e^y)^{v-2(\frac{v}{2})} + ({}^vC_{(\frac{v}{2}+1)})(-1)^{(\frac{v}{2}+1)}(e^y)^{v-2(\frac{v}{2}+1)} + ... + ({}^vC_v)(-1)^v(e^y)^{v-2(v)}]. \quad (25)$$

Since ${}^vC_u ={}^v C_{v-u}$ whenever $v > u$, the 1^{st} term can be combined with the last term, the second term can be combined with the $(v-1)^{th}$ term and so on, leaving the middle term.

$$\sinh^v(y) = (\tfrac{1}{2})^{v-1}[\tfrac{{}^vC_0(e^{vy}+e^{-vy})}{2} - \tfrac{{}^vC_1(e^{(v-2)y}+e^{-(v-2)y})}{2} + ... + (-1)^{(\frac{v}{2}-1)}\tfrac{{}^vC_{(\frac{v}{2}-1)}(e^{2y}+e^{-2y})}{2} + (-1)^{\frac{v}{2}}\tfrac{{}^vC_{(\frac{v}{2})}}{2}] = (\tfrac{1}{2})^{v-1}[(-1)^0({}^vC_0)\cosh(vy) + (-1)^1({}^vC_1)\cosh((v-2)y) + ... + (-1)^{(\frac{v}{2}-1)}({}^vC_{(\frac{v}{2}-1)})\cosh(2y) + (-1)^{\frac{v}{2}}\tfrac{{}^vC_{\frac{v}{2}}}{2}] = (\tfrac{1}{2})^{v-1}[(\sum_{u=0}^{\frac{v}{2}-1}({}^vC_u)(-1)^u\cosh((v-2u)y)) + (-1)^{\frac{v}{2}}\tfrac{{}^vC_{(\frac{v}{2})}}{2}]$$
(26)

To summarize :

$$\sinh^v(y) = \begin{cases} (\frac{1}{2})^{v-1}[(\sum_{u=0}^{\frac{v}{2}-1}({}^vC_u)(-1)^u\cosh((v-2u)y)) + (-1)^{\frac{v}{2}}\frac{{}^vC_{\frac{v}{2}}}{2}], \text{if } v \text{ is even} \\ (\frac{1}{2})^{v-1}\sum_{u=0}^{\frac{v}{2}-1}(-1)^u({}^vC_u)\sinh((v-2u)y), \text{if } v \text{ is odd} \end{cases}$$
(27)

Now, the series $\sum_{n=0}^{k} n^m\sinh^v(ny)$ is evaluated using the previous results. Since $\sum_{n=0}^{k} n^m\sinh^v(ny) = (\frac{1}{2})^v\sum_{n=0}^{k} n^m.[\sum_{u=0}^{v}(-1)^{u}{}^vC_u(e^{ny})^{v-2u}]$, the series can be expanded as following(where $x = 1$) :

$$\sum_{n=0}^{k} n^m \sinh^v(ny) = (\frac{1}{2})^v (\sum_{n=0}^{k} n^m ({}^vC_0)(-1)^0 (e^{nyx})^{v-2(0)}) + (\frac{1}{2})^v (\sum_{n=0}^{k} n^m ({}^vC_1)(-1)^1 (e^{nyx})^{v-2})$$
$$+.... + (\frac{1}{2})^v (\sum_{n=0}^{k} n^m ({}^vC_{v-1})(-1)^{v-1} (e^{nyx})^{v-2(v-1)}) + (\frac{1}{2})^v (\sum_{n=0}^{k} n^m ({}^vC_v)(-1)^v (e^{nyx})^{v-2(v)}) \tag{28}$$

First, if v is even, the method is applied on equation (26). Noticeably, the term in which $u = \frac{v}{2}$ violates the condition $b \neq 0$, and so the method is not applied on it. Equation (8) is applied on all the terms, leaving the term where $u = \frac{v}{2}$ as it is.

$$\sum_{n=0}^{k} n^m \sinh^v(ny) = (\frac{1}{2})^v ({}^vC_0)(-1)^0 (\frac{1}{vy})^m \frac{d^m}{dx^m} (\frac{1 - e^{yv(k+1)x}}{1 - e^{yvx}}) + (\frac{1}{2})^v ({}^vC_1)(-1)^1 (\frac{1}{(v-2)y})^m$$
$$\frac{d^m}{dx^m} (\frac{1 - e^{y(v-2)(k+1)x}}{1 - e^{y(v-2)x}}) + ... + (\frac{1}{2})^v ({}^vC_{\frac{v}{2}-1})(-1)^{\frac{v}{2}-1} (\frac{1}{y(v - 2(\frac{v}{2}-1))})^m \frac{d^m}{dx^m} (\frac{1 - e^{y(v-2(\frac{v}{2}-1))(k+1)x}}{1 - e^{y(v-2(\frac{v}{2}-1))x}})$$
$$+(\frac{1}{2})^v ({}^vC_{\frac{v}{2}})(-1)^{\frac{v}{2}} \sum_{n=0}^{k} n^m + (\frac{1}{2})^v ({}^vC_{\frac{v}{2}+1})(-1)^{\frac{v}{2}+1} (\frac{1}{y(v - 2(\frac{v}{2}+1))})^m \frac{d^m}{dx^m} (\frac{1 - e^{y(v-2(\frac{v}{2}+1))(k+1)x}}{1 - e^{y(v-2(\frac{v}{2}+1))x}})$$
$$+.... + (\frac{1}{2})^v ({}^vC_v)(-1)^v (\frac{1}{y(v-2v)})^m \frac{d^m}{dx^m} (\frac{1 - e^{y(v-2v)(k+1)x}}{1 - e^{y(v-2v)x}}) \tag{29}$$

Simplifying the expression leads to

$$\sum_{n=0}^{k} n^m \sinh^v(ny) = \frac{d^m}{dx^m} [\sum_{u=0}^{\frac{v}{2}-1} (\frac{1}{2})^v ({}^vC_u)(-1)^u (\frac{1}{y(v-2u)})^m (\frac{1 - e^{y(v-2u)(k+1)x}}{1 - e^{y(v-2u)x}})$$
$$+ \sum_{u=\frac{v}{2}+1}^{v} (\frac{1}{2})^v ({}^vC_u)(-1)^u (\frac{1}{y(v-2u)})^m (\frac{1 - e^{y(v-2u)(k+1)x}}{1 - e^{y(v-2u)x}})]$$
$$+(\frac{1}{2})^v ({}^vC_{\frac{v}{2}})(-1)^{\frac{v}{2}} \sum_{n=0}^{k} n^m \tag{30}$$

The term containing $\sum_{n=0}^{k} n^m$ can be evaluated using the **Explicit formula for Bernoulli numbers by using only the recurrence relation** [5].

If v is odd, the number of terms is even; therefore, there is no middle term to violate the condition $b \neq 0$

$$\sum_{n=0}^{k} n^m \sinh^v(ny) = (\frac{1}{2})^v ({}^vC_0)(-1)^0 (\frac{1}{vy})^m \frac{d^m}{dx^m}(\frac{1-e^{yv(k+1)x}}{1-e^{yvx}}) + (\frac{1}{2})^v ({}^vC_1)(-1)^1 (\frac{1}{(v-2)y})^m$$
$$\frac{d^m}{dx^m}(\frac{1-e^{y(v-2)(k+1)x}}{1-e^{y(v-2)x}}) + ... + (\frac{1}{2})^v ({}^vC_v)(-1)^v (\frac{1}{y(v-2v)})^m \frac{d^m}{dx^m}(\frac{1-e^{y(v-2v)(k+1)x}}{1-e^{y(v-2v)x}})$$
$$= \frac{d^m}{dx^m}[\sum_{u=0}^{v}(\frac{1}{2})^v ({}^vC_u)(-1)^u (\frac{1}{y(v-2u)})^m (\frac{1-e^{y(v-2u)(k+1)x}}{1-e^{y(v-2u)x}})] \tag{31}$$

putting $v = 1$, the expression reduces to equation (14)

$$\sum_{n=0}^{k} n^m \sinh(ny) = \frac{d^m}{dx^m}[(\frac{1}{2})^1 ({}^1C_0)(-1)^0 (\frac{1}{y(1-2(0))})^m (\frac{1-e^{y(1-2(0))(k+1)x}}{1-e^{y(1-2(0))x}})]$$
$$+\frac{d^m}{dx^m}[(\frac{1}{2})^1 ({}^1C_1)(-1)^1 (\frac{1}{y(1-2)})^m (\frac{1-e^{y(1-2)(k+1)x}}{1-e^{y(1-2)x}})]$$
$$= \frac{1}{2}(\frac{1}{y})^m (\frac{d^m}{dx^m}(\frac{1-e^{y(k+1)x}}{1-e^{yx}}) - (-1)^m \frac{d^m}{dx^m}(\frac{1-e^{-y(k+1)x}}{1-e^{-yx}})) \tag{32}$$

Then, moving to $\cosh^v(y)$,similar steps are done on the cosh function, but there is no changing negative signs like the $(-1)^u$ and the $(-1)^{\frac{v}{2}}$ as in the sinh function.

$$\cosh^v(y) = \begin{cases} (\frac{1}{2})^{v-1}[(\sum_{u=0}^{\frac{v}{2}-1}({}^vC_u)\cosh((v-2u)y)) + \frac{{}^vC_{\frac{v}{2}}}{2}], \text{if } v \text{ is even} \\ (\frac{1}{2})^{v-1}\sum_{u=0}^{\frac{v}{2}-1}({}^vC_u)\cosh((v-2u)y), \text{if } v \text{ is odd} \end{cases} \tag{33}$$

Then, substituting $\cosh^v(ny)$ as $(\frac{1}{2})^v \sum_{u=0}^{v}({}^vC_u)e^{nyx(v-2u)}$ gives two conditions: odd v and even v.

If v is even:

$$\sum_{n=0}^{k} n^m \cosh^v(ny) = \frac{d^m}{dx^m}[\sum_{u=0}^{\frac{v}{2}-1}(\frac{1}{2})^v ({}^vC_u)(\frac{1}{y(v-2u)})^m (\frac{1-e^{y(v-2u)(k+1)x}}{1-e^{y(v-2u)x}})$$
$$+\sum_{u=\frac{v}{2}+1}^{v}(\frac{1}{2})^v ({}^vC_u)(\frac{1}{y(v-2u)})^m (\frac{1-e^{y(v-2u)(k+1)x}}{1-e^{y(v-2u)x}})]$$
$$+(\frac{1}{2})^v ({}^vC_{\frac{v}{2}})\sum_{n=0}^{k} n^m \tag{34}$$

If v is odd:

$$\sum_{n=0}^{k} n^m \cosh^v(ny) = \frac{d^m}{dx^m}[\sum_{u=0}^{v}(\frac{1}{2})^v({}^vC_u)(\frac{1}{y(v-2u)})^m(\frac{1-e^{y(v-2u)(k+1)x}}{1-e^{y(v-2u)x}})] \tag{35}$$

4.4 Application of the method in series with Trigonometric Sine and Cosine raised to the power of v.

By using the substitution $\cos(y) = \cosh(iy) = \frac{e^{iy}+e^{-iy}}{2}, \sin(y) = \frac{\sinh(iy)}{i} = \frac{e^{iy}-e^{-iy}}{2i}$ and substituting in the results of (35), (34) and (33) , a reduction formula for $(\cos^v(y)$ and $(\sin^v(y))$ can be reached.
If v is an even number:

$$\sum_{n=0}^{k} n^m \cos^v(ny) = (\frac{1}{2})^{v-1}[\sum_{u=0}^{\frac{v}{2}-1}({}^vC_u)(\frac{1}{y(v-2u)})^m \frac{d^m}{dx^m}(\frac{1-e^{iy(v-2u)(k+1)x}}{1-e^{iy(v-2u)x}}) +^v C_{\frac{v}{2}} \sum_{n=0}^{k} n^m] \tag{36}$$

If v is an odd number:

$$\sum_{n=0}^{k} n^m \cos^v(ny) = (\frac{1}{2})^{v-1}[\sum_{u=0}^{\frac{v}{2}-1}({}^vC_u)(\frac{1}{y(v-2u)})^m \frac{d^m}{dx^m}(\frac{1-e^{iy(v-2u)(k+1)x}}{1-e^{iy(v-2u)x}})] \tag{37}$$

The same pattern can be applied to$\sum_{n=0}^{k} n^m \sin^v(ny)$ to obtain the formulas for even and odd v

For an even v:

$$\sum_{n=0}^{k} n^m \sin^v(ny) = (\frac{1}{2})^{v-1}(-1)^{\frac{v}{2}}[\sum_{u=0}^{\frac{v}{2}-1}({}^vC_u)(\frac{1}{iy(v-2u)})^m \frac{d^m}{dx^m}(\frac{1-e^{iy(v-2u)(k+1)x}}{1-e^{iy(v-2u)x}}) +^v C_{\frac{v}{2}} \sum_{n=0}^{k} n^m] \tag{38}$$

For an odd v:

$$\sum_{n=0}^{k} n^m \sin^v(ny) = (\frac{1}{2})^{v-1}(-1)^{\frac{v}{2}}[\sum_{u=0}^{\frac{v}{2}-1}({}^vC_u)(-1)^u(\frac{1}{iy(v-2u)})^m \frac{d^m}{dx^m}(\frac{1-e^{iy(v-2u)(k+1)x}}{1-e^{iy(v-2u)x}})] \tag{39}$$

5 Conclusion

A reduction method is introduced for a certain type of power series in this paper. Additionally, it was used to obtain a general formula for this sequence with basic hyperbolic and trigonometric functions. The method is seen to be exceedingly useful when m is much smaller than k because taking the mth derivative will be much easier than summing up so many terms. Moreover, it helps transforming the series of hyperbolic and trigonometric functions raised to the power of v from $(k+1)$ terms to $(v+1)$ terms. This transformation is very beneficial when k is a large number compared to v. In the future, it is planned to design a software in which the method is used. This software is expected to reduce the execution time of the processors to compute the results of the sequence as it will transform the summation process from summing numerous complex terms ,like the hyperbolic and the trigonometric functions in the series into shorter operations for the processors to perform -taking the derivative of a function. For all these reasons, this method will be revolutionary in branches like statistical physics.

6 References

[1] V.M.Yakovenko, J.B.Rosser,"Colloquium: Statistical mechanics of money, wealth, and income.", PP.14, 2009.

[2] B. C. Berndt, "How to Write Mathematical Papers", http://www.math.uiuc.edu/ berndt/ writingmath.pdf

[3] K.F.RILEY,M.P.HOBSON,S.J.BENCE "Mathematical Methods for Physics and Engineering: Third Edition.", PP.115-144.

[4] G.B.Thomas,Jr, "Thomas Calculus: 12th Edition", PP.532-605.

[5] T.Agoh,K.Dilcher,"Shortened recurrence relations for Bernoulli numbers ",Discrete Mathematics Vol.309, no.4, PP.887-898, 2009.