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Research Article

On Pseudohyperbolical Smarandache Curves in Minkowski 3-Space

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We define pseudohyperbolical Smarandache curves according to the Sabban frame in Minkowski 3-space. We obtain the geodesic curvatures and the expression for the Sabban frame vectors of special pseudohyperbolic Smarandache curves. Finally, we give some examples of such curves.

1. Introduction

In the theory of curves in the Euclidean and Minkowski spaces, one of the interesting problems is the problem of characterization of a regular curve. In the solution of the problem, the curvature functions κ and τ of a regular curve have an effective role. It is known that we can determine the shape and size of a regular curve by using its curvatures κ and τ. Another approach to the solution of the problem is to consider the relationship between the corresponding Frenet vectors of two curves. For instance, Bertrand curves and Mannheim curves arise from this relationship. Another example is the Smarandache curves. They are the objects of Smarandache geometry, that is, a geometry which has at least one Smarandachely denied axiom [1]. An axiom is said to be Smarandachely denied if it behaves in at least two different ways within the same space. Smarandache geometries are connected with the theory of relativity and the parallel universes.

If the position vector of a regular curve α is composed by the Frenet frame vectors of another regular curve β, then the curve α is called a Smarandache curve [2]. Special Smarandache curves in Euclidean and Minkowski spaces are studied by some authors [3–8]. The curves lying on a pseudohyperbolic space H_0^2 in Minkowski 3-space $\mathbb{E}_1^3$ are characterized in [9].

In this paper, we define pseudohyperbolical Smarandache curves according to the Sabban frame $\{\alpha, T, \xi\}$ in Minkowski 3-space. We obtain the geodesic curvatures and the expressions for the Sabban frame's vectors of special pseudohyperbolical Smarandache curves. In particular, we prove that special $T\xi$-pseudohyperbolical Smarandache curves do not exist. Besides, we give some examples of special pseudohyperbolical Smarandache curves in Minkowski 3-space.

2. Basic Concepts

The Minkowski 3-space $\mathbb{R}_1^3$ is the Euclidean 3-space $\mathbb{R}^3$ provided with the standard flat metric given by

$$\langle \cdot, \cdot \rangle = -dx_1^2 + dx_2^2 + dx_3^2, \tag{1}$$

where (x_1, x_2, x_3) is a rectangular Cartesian coordinate system of $\mathbb{R}_1^3$. Since g is an indefinite metric, recall that a nonzero vector $\vec{x} \in \mathbb{R}_1^3$ can have one of the three Lorentzian causal characters: it can be spacelike if $\langle \vec{x}, \vec{x} \rangle > 0$, timelike if $\langle \vec{x}, \vec{x} \rangle < 0$, and null (lightlike) if $\langle \vec{x}, \vec{x} \rangle = 0$. In particular, the norm (length) of a vector $\vec{x} \in \mathbb{R}_1^3$ is given by $\|\vec{x}\| = \sqrt{|\langle \vec{x}, \vec{x} \rangle|}$ and two vectors $\vec{x}$ and $\vec{y}$ are said to be orthogonal if $\langle \vec{x}, \vec{y} \rangle = 0$. Next, recall that an arbitrary curve $\alpha = \alpha(s)$ in $\mathbb{E}_1^3$ can locally be *spacelike, timelike,* or *null (lightlike)* if all of its velocity vectors $\alpha'(s)$ are, respectively, *spacelike, timelike,* or *null (lightlike)* for every $s \in I$ [10]. If $\|\alpha'(s)\| \neq 0$ for every $s \in I$, then α is a *regular curve* in $\mathbb{R}_1^3$. A spacelike (timelike) regular curve α is parameterized by pseudo-arclength parameter s which is given by $\alpha : I \subset \mathbb{R} \to \mathbb{R}_1^3$, and then the tangent vector $\alpha'(s)$ along α has unit length, that is, $\langle \alpha'(s), \alpha'(s) \rangle = 1(\langle \alpha'(s), \alpha'(s) \rangle = -1)$ for all $s \in I$, respectively.

For any $\vec{x} = (x_1, x_2, x_3)$ and $\vec{y} = (y_1, y_2, y_3)$ in the space $\mathbb{R}_1^3$, the *pseudovector product* of $\vec{x}$ and $\vec{y}$ is defined by

$$\vec{x} \times \vec{y} = \left(-x_2 y_3 + x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1\right). \tag{2}$$

Remark 1. Let $\vec{x} = (x_1, x_2, x_3)$, $\vec{y} = (y_1, y_2, y_3)$, and $\vec{z} = (z_1, z_2, z_3)$ be vectors in $\mathbb{R}_1^3$. Then,

(i) $\langle \vec{x} \times \vec{y}, \vec{z} \rangle = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$,

(ii) $\vec{x} \times (\vec{y} \times \vec{z}) = -\langle \vec{x}, \vec{z} \rangle \vec{y} + \langle \vec{x}, \vec{y} \rangle \vec{z}$,

(iii) $\langle \vec{x} \times \vec{y}, \vec{x} \times \vec{y} \rangle = -\langle \vec{x}, \vec{x} \rangle \langle \vec{y}, \vec{y} \rangle + \langle \vec{x}, \vec{y} \rangle^2$,

where $\times$ is the pseudovector product in the space $\mathbb{R}_1^3$.

Lemma 2. *In the Minkowski 3-space $\mathbb{R}_1^3$, the following properties are satisfied [10]:*

(i) *two timelike vectors are never orthogonal;*

(ii) *two null vectors are orthogonal if and only if they are linearly dependent;*

(iii) *timelike vector is never orthogonal to a null vector.*

Pseudohyperbolic space in the Minkowski 3-space $\mathbb{R}_1^3$ is a quadric defined by

$$H_0^2 = \left\{ \vec{x} \in \mathbb{R}_1^3 \mid -x_1^2 + x_2^2 + x_3^2 = -1 \right\}. \tag{3}$$

Let $\alpha : I \subset \mathbb{R} \to H_0^2$ be a regular unit speed curve lying fully in H_0^2 in $\mathbb{R}_1^3$. Then its position vector α is timelike vector, which implies that tangent vector $\vec{T} = \alpha'$ is the unit spacelike vector for all $s \in I$. Hence we have orthonormal Sabban frame $\{\alpha(s), T(s), \xi(s)\}$ along the curve α, where $\xi(s) = \alpha(s) \times T(s)$ is the unit spacelike vector. The corresponding Frenet formulae of α, according to the Sabban frame, read

$$\begin{bmatrix} \alpha' \\ T' \\ \xi' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & k_g(s) \\ 0 & -k_g(s) & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ T \\ \xi \end{bmatrix}, \tag{4}$$

where $k_g(s) = \det(\alpha(s), T(s), T'(s))$ is the geodesic curvature of α on H_0^2 in $\mathbb{R}_1^3$ and s is arc length parameter of *alpha*. In particular, the following relations hold:

$$\alpha \times T = \xi, \qquad T \times \xi = -\alpha, \qquad \xi \times \alpha = T. \tag{5}$$

3. Pseudohyperbolical Smarandache Curves in Minkowski 3-Space

In this section, we define and investigate pseudohyperbolical Smarandache curves in Minkowski 3-space according to the Sabban frame.

Let $\alpha = \alpha(s)$ and $\beta = \beta(s^*)$ be two regular unit speed curves lying fully in pseudohyperbolic space H_0^2 in $\mathbb{R}_1^3$ and let $\{\alpha, T, \xi\}$ and $\{\beta, T_\beta, \xi_\beta\}$ be the moving Sabban frames of these curves, respectively. Then we have the following definitions of pseudohyperbolical Smarandache curves.

Definition 3. Let $\alpha : I \subset \mathbb{R} \mapsto H_0^2$ be a regular unit speed curve lying fully in H_0^2. Then $\alpha\xi$-pseudohyperbolical Smarandache curve $\beta : I \subset \mathbb{R} \mapsto H_0^2$ of α is defined by

$$\beta\left(s^*(s)\right) = \frac{1}{\sqrt{2}}\left(a\alpha(s) + b\xi(s)\right), \tag{6}$$

where $a, b \in \mathbb{R}_0$ and $a^2 - b^2 = 2$.

Definition 4. Let $\alpha : I \subset \mathbb{R} \mapsto H_0^2$ be a regular unit speed curve lying fully in H_0^2.Then αT-pseudohyperbolical Smarandache curve $\beta : I \subset \mathbb{R} \mapsto H_0^2$ of α is defined by

$$\beta\left(s^*(s)\right) = \frac{1}{\sqrt{2}}\left(a\alpha(s) + bT(s)\right), \tag{7}$$

where $a, b \in R_0$ and $a^2 - b^2 = 2$.

Definition 5. Let $\alpha : I \subset R \mapsto H_0^2$ be a regular unit speed curve lying fully in H_0^2. Then $\alpha T\xi$ -pseudohyperbolical Smarandache curve $\beta : I \subset R \mapsto H_0^2$ of α is defined by

$$\beta\left(s^*(s)\right) = \frac{1}{\sqrt{3}}\left(a\alpha(s) + bT(s) + c\xi(s)\right), \tag{8}$$

where $a, b, c \in \mathbb{R}_0$ and $a^2 - b^2 - c^2 = 3$.

Theorem 6. *Let $\alpha : I \subset \mathbb{R} \mapsto H_0^2$ be a regular unit speed curve lying fully in H_0^2. Then $T\xi$-pseudohyperbolical Smarandache curve $\beta : I \subset \mathbb{R} \mapsto H_0^2$ of α does not exist.*

Proof. Assume that there exists $T\xi$-pseudohyperbolical Smarandache curve of α. Then it can be written as

$$\beta\left(s^*(s)\right) = \frac{1}{\sqrt{2}}\left(aT(s) + b\xi(s)\right), \tag{9}$$

where $a^2 + b^2 = -2$, which is a contradiction. □

In the theorems which follow, we obtain Sabban frame $\{\beta, T_\beta, \xi_\beta\}$ and geodesic curvature κ_g^β of pseudohyperbolical Smarandache curve β.

Theorem 7. *Let $\alpha : I \subset \mathbb{R} \mapsto H_0^2$ be a regular unit speed curve lying fully in H_0^2 with the Sabban frame $\{\alpha, T, \xi\}$ and* *the geodesic curvature k_g. If $\beta : I \subset \mathbb{R} \mapsto H_0^2$ is $\alpha\xi$-pseudohyperbolical Smarandache curve of α, then its frame $\{\beta, T_\beta, \xi_\beta\}$ is given by*

$$\begin{bmatrix} \beta \\ T_\beta \\ \xi_\beta \end{bmatrix} = \begin{bmatrix} \dfrac{a}{\sqrt{2}} & 0 & \dfrac{b}{\sqrt{2}} \\ 0 & \epsilon & 0 \\ \epsilon\dfrac{b}{\sqrt{2}} & 0 & \epsilon\dfrac{a}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \alpha \\ T \\ \xi \end{bmatrix}, \tag{10}$$

and the corresponding geodesic curvature k_g^β reads

$$k_g^\beta = \frac{ak_g - b}{\left|a - bk_g\right|}, \quad a, b \in \mathbb{R}_0, \tag{11}$$

where $a^2 - b^2 = 2$ and $\epsilon = \pm 1$.

Proof. Differentiating (6) with respect to s and using (4), we obtain

$$\begin{gathered} \beta'(s) = \frac{d\beta}{ds^*}\frac{ds^*}{ds} = \frac{a - bk_g}{\sqrt{2}}T, \\ T_\beta \frac{ds^*}{ds} = \frac{a - bk_g}{\sqrt{2}}T, \end{gathered} \tag{12}$$

where

$$\frac{ds^*}{ds} = \frac{\left|a - bk_g\right|}{\sqrt{2}}. \tag{13}$$

Therefore, the unit spacelike tangent vector of the curve β is given by

$$T_\beta = \epsilon T, \tag{14}$$

where $\epsilon = +1$ if $a - bk_g > 0$ for all s and $\epsilon = -1$ if $a - bk_g < 0$ for all s.

Differentiating (14) with respect to s, we find

$$\frac{dT_\beta}{ds^*}\frac{ds^*}{ds} = \epsilon\left(\alpha + k_g\xi\right), \tag{15}$$

and from (13) and (15) we get

$$T'_\beta = \frac{\sqrt{2}\epsilon}{\left|a - bk_g\right|}\left(\alpha + k_g\xi\right). \tag{16}$$

On the other hand, from (6) and (14) it can be easily seen that

$$\begin{aligned} \xi_\beta &= \beta \times T_\beta \\ &= \epsilon\frac{b}{\sqrt{2}}\alpha + \epsilon\frac{a}{\sqrt{2}}\xi \end{aligned} \tag{17}$$

is a unit spacelike vector.

Consequently, the geodesic curvature k_g^β of the curve $\beta = \beta(s^*)$ is given by

$$\begin{aligned} k_g^\beta &= \det\left(\beta, T_\beta, T'_\beta\right) \\ &= \frac{ak_g - b}{\left|a - bk_g\right|}. \end{aligned} \tag{18}$$

□

Theorem 8. *Let $\alpha : I \subset \mathbb{R} \mapsto H_0^2$ be a regular unit speed curve lying fully in H_0^2 with the Sabban frame $\{\alpha, T, \xi\}$ and the geodesic curvature k_g. If $\beta : I \subset \mathbb{R} \mapsto H_0^2$ is αT-pseudohyperbolical Smarandache curve of α, then its frame $\{\beta, T_\beta, \xi_\beta\}$ is given by*

$$\begin{aligned} \begin{bmatrix} \beta \\ T_\beta \\ \xi_\beta \end{bmatrix} &= \begin{bmatrix} \dfrac{a}{\sqrt{2}} & \dfrac{b}{\sqrt{2}} & 0 \\ \dfrac{b}{\sqrt{2 + \left(bk_g\right)^2}} & \dfrac{a}{\sqrt{2 + \left(bk_g\right)^2}} & \dfrac{bk_g}{\sqrt{2 + \left(bk_g\right)^2}} \\ -\dfrac{b^2k_g}{\sqrt{4 + 2\left(bk_g\right)^2}} & -\dfrac{abk_g}{\sqrt{4 + 2\left(bk_g\right)^2}} & \dfrac{\sqrt{2}}{\sqrt{2 + \left(bk_g\right)^2}} \end{bmatrix} \\ &\times \begin{bmatrix} \alpha \\ T \\ \xi \end{bmatrix}, \end{aligned} \tag{19}$$

and the corresponding geodesic curvature k_g^β reads

$$k_g^\beta = \frac{b^2k_g\varepsilon_1 - abk_g\varepsilon_2 + 2\varepsilon_3}{\left(2 + b^2k_g^2\right)^{5/2}}, \quad a, b \in \mathbb{R}_0, \tag{20}$$

where

$$\begin{gathered} \varepsilon_1 = -b^3k_gk'_g + a\left(2 + \left(bk_g\right)^2\right), \\ \varepsilon_2 = -ab^2k_gk'_g + \left(b - bk_g^2\right)\left(2 + \left(bk_g\right)^2\right), \\ \varepsilon_3 = -b^3k_g^2k'_g + \left(ak_g + bk'_g\right)\left(2 + \left(bk_g\right)^2\right), \end{gathered} \tag{21}$$

and $a^2 - b^2 = 2$.

Proof. Differentiating (7) with respect to s and using (4), we obtain

$$\begin{gathered} \beta'(s) = \frac{d\beta}{ds^*}\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\left(b\alpha + aT + bk_g\xi\right), \\ T_\beta\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\left(b\alpha + aT + bk_g\xi\right), \end{gathered} \tag{22}$$

where

$$\frac{ds^*}{ds} = \sqrt{\frac{2 + \left(bk_g\right)^2}{2}}. \tag{23}$$

Therefore, the unit spacelike tangent vector of the curve β is given by

$$T_\beta = \frac{1}{\sqrt{2 + \left(bk_g\right)^2}}\left(b\alpha + aT + bk_g\xi\right). \tag{24}$$

Differentiating (24) with respect to s, it follows that

$$\frac{dT_\beta}{ds^*}\frac{ds^*}{ds} = \frac{1}{\left(2+\left(bk_g\right)^2\right)^{3/2}}\left(\varepsilon_1\alpha+\varepsilon_2 T+\varepsilon_3\xi\right). \tag{25}$$

From (4) and (23), we get

$$T'_\beta = \frac{\sqrt{2}}{\left(2+\left(bk_g\right)^2\right)^2}\left(\varepsilon_1\alpha+\varepsilon_2 T+\varepsilon_3\xi\right), \tag{26}$$

where

$$\begin{aligned}\varepsilon_1 &= -b^3k_gk'_g + a\left(2+\left(bk_g\right)^2\right),\\ \varepsilon_2 &= -ab^2k_gk'_g + \left(b-bk_g^2\right)\left(2+\left(bk_g\right)^2\right),\\ \varepsilon_3 &= -b^3k_g^2k'_g + \left(ak_g+bk'_g\right)\left(2+\left(bk_g\right)^2\right).\end{aligned} \tag{27}$$

On the other hand, from (7) and (24), it can be easily seen that

$$\begin{aligned}\xi_\beta &= \beta\times T_\beta\\ &= -\frac{b^2k_g}{\sqrt{4+2\left(bk_g\right)^2}}\alpha - \frac{abk_g}{\sqrt{4+2\left(bk_g\right)^2}}T + \frac{\sqrt{2}}{\sqrt{2+\left(bk_g\right)^2}}\xi.\end{aligned} \tag{28}$$

Hence ξ_β is a unit spacelike vector.

Therefore, the geodesic curvature k_g^β of the curve $\beta = \beta(s^*)$ is given by

$$\begin{aligned}k_g^\beta &= \det\left(\beta, T_\beta, T'_\beta\right)\\ &= \frac{b^2k_g\varepsilon_1 - abk_g\varepsilon_2 + 2\varepsilon_3}{\left(2+b^2k_g^2\right)^{5/2}}.\end{aligned} \tag{29}$$

□

Theorem 9. *Let $\alpha : I \subset \mathbb{R} \mapsto H_0^2$ be a regular unit speed curve lying fully in H_0^2 with the Sabban frame $\{\alpha, T, \xi\}$ and the geodesic curvature k_g. If $\beta : I \subset \mathbb{R} \mapsto H_0^2$ is $\alpha T\xi$-pseudohyperbolical Smarandache curve of α, then its frame $\{\beta, T_\beta, \xi_\beta\}$ is given by*

$$\begin{bmatrix}\beta\\ T_\beta\\ \xi_\beta\end{bmatrix} = \begin{bmatrix}\dfrac{a}{\sqrt{3}} & \dfrac{b}{\sqrt{3}} & \dfrac{c}{\sqrt{3}}\\ \dfrac{b}{\sqrt{\left|\left(ak_g-c\right)^2-3k_g^2+3\right|}} & \dfrac{a-ck_g}{\sqrt{\left|\left(ak_g-c\right)^2-3k_g^2+3\right|}} & \dfrac{bk_g}{\sqrt{\left|\left(ak_g-c\right)^2-3k_g^2+3\right|}}\\ \dfrac{-\left(a^2-3\right)k_g+ac}{\sqrt{\left|3\left(ak_g-c\right)^2-9k_g^2+9\right|}} & \dfrac{bc-abk_g}{\sqrt{\left|3\left(ak_g-c\right)^2-9k_g^2+9\right|}} & \dfrac{3+c^2-ack_g}{\sqrt{\left|3\left(ak_g-c\right)^2-9k_g^2+9\right|}}\end{bmatrix}\begin{bmatrix}\alpha\\ T\\ \xi\end{bmatrix} \tag{30}$$

and the corresponding geodesic curvature k_g^β reads

$$\begin{aligned}k_g^\beta &= \left(-\left(ac-\left(a^2-3\right)k_g\right)\varepsilon_1\right.\\ &\quad \left.+\left(bc-abk_g\right)\varepsilon_2+\left(3+c^2-ack_g\right)\varepsilon_3\right)\\ &\quad \times\left(\left(\left(ak_g-c\right)^2-3k_g^2+3\right)^{5/2}\right)^{-1},\\ &\qquad a,b,c\in\mathbb{R}_0,\end{aligned} \tag{31}$$

where

$$\begin{aligned}\varepsilon_1 &= -\left(\left(ak_g-c\right)ak'_g-3k'_gk_g\right)b\\ &\quad +\left(\left(ak_g-c\right)^2-3k_g^2+3\right)\left(a-ck_g\right),\\ \varepsilon_2 &= -\left(\left(ak_g-c\right)ak'_g-3k'_gk_g\right)\left(a-ck_g\right)\\ &\quad +\left(\left(ak_g-c\right)^2-3k_g^2+3\right)\left(b-ck'_g-bk_g^2\right),\\ \varepsilon_3 &= -\left(\left(ak_g-c\right)ak'_g-3k'_gk_g\right)bk_g\\ &\quad +\left(\left(ak_g-c\right)^2-3k_g^2+3\right)\left(\left(a-ck_g\right)k_g+bk'_g\right)\end{aligned} \tag{32}$$

and $a^2-b^2-c^2=3$.

Proof. Differentiating (8) with respect to s and by using (4), we find

$$\beta'(s) = \frac{d\beta}{ds^*}\frac{ds^*}{ds} = \frac{1}{\sqrt{3}}\left(b\alpha+\left(a-ck_g\right)T+bk_g\xi\right), \tag{33}$$

and thus

$$T_\beta \frac{ds^*}{ds} = \frac{1}{\sqrt{3}} \left(b\alpha + \left(a - ck_g \right) T + bk_g \xi \right), \tag{34}$$

where

$$\frac{ds^*}{ds} = \sqrt{\frac{\left| \left(ak_g - c \right)^2 - 3k_g^2 + 3 \right|}{3}}. \tag{35}$$

The geodesic curvature $k_g \neq (ac \pm \sqrt{a^2c^2 - (a^2-3)(c^2+3)})/(a^2-3)$ for all s, since $\beta = \beta(s^*)$ is a unit speed regular curve in $\mathbb{R}_1^3$.

Therefore, the unit spacelike tangent vector of the curve β is given by

$$T_\beta = \frac{1}{\sqrt{\left| \left(ak_g - c \right)^2 - 3k_g^2 + 3 \right|}} \left(b\alpha + \left(a - ck_g \right) T + bk_g \xi \right). \tag{36}$$

Differentiating (36) with respect to s and from (4) and (35), it follows that

$$T'_\beta = \frac{\sqrt{3}}{\left(\left(ak_g - c \right)^2 - 3k_g^2 + 3 \right)^2} \left(\varepsilon_1 \alpha + \varepsilon_2 T + \varepsilon_3 \xi \right), \tag{37}$$

where

$$\begin{aligned} \varepsilon_1 &= -\left(\left(ak_g - c \right) ak'_g - 3k'_g k_g \right) b \\ &\quad + \left(\left(ak_g - c \right)^2 - 3k_g^2 + 3 \right) \left(a - ck_g \right), \\ \varepsilon_2 &= -\left(\left(ak_g - c \right) ak'_g - 3k'_g k_g \right) \left(a - ck_g \right) \\ &\quad + \left(\left(ak_g - c \right)^2 - 3k_g^2 + 3 \right) \left(b - ck'_g - bk_g^2 \right), \\ \varepsilon_3 &= -\left(\left(ak_g - c \right) ak'_g - 3k'_g k_g \right) bk_g \\ &\quad + \left(\left(ak_g - c \right)^2 - 3k_g^2 + 3 \right) \left(\left(a - ck_g \right) k_g + bk'_g \right). \end{aligned} \tag{38}$$

On the other hand, from (8) and (36) it can be easily seen that

$$\begin{aligned} \xi_\beta &= \beta \times T_\beta \\ &= \frac{-\left(a^2 - 3 \right) k_g + ac}{\sqrt{\left| 3\left(ak_g - c \right)^2 - 9k_g^2 + 9 \right|}} \alpha \\ &\quad + \frac{bc - abk_g}{\sqrt{\left| 3\left(ak_g - c \right)^2 - 9k_g^2 + 9 \right|}} T \\ &\quad + \frac{3 + c^2 - ack_g}{\sqrt{\left| 3\left(ak_g - c \right)^2 - 9k_g^2 + 9 \right|}} \xi. \end{aligned} \tag{39}$$

Hence ξ_β is a unit spacelike vector.

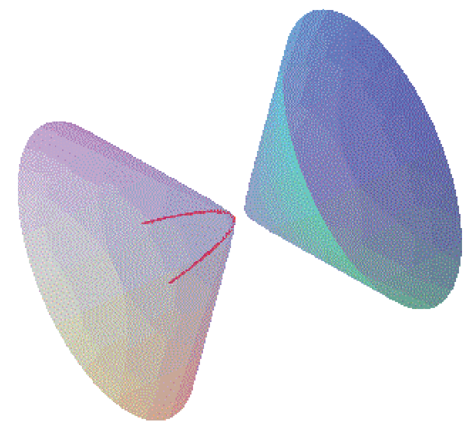

Figure 1: The curve α on $H_0^2(1)$.

Therefore, the geodesic curvature k_g^β of the curve $\beta = \beta(s^*)$ is given by

$$\begin{aligned} k_g^\beta &= \det\left(\beta, T_\beta, T'_\beta \right) \\ &= \left(-\left(ac - \left(a^2 - 3 \right) k_g \right) \varepsilon_1 + \left(bc - abk_g \right) \varepsilon_2 \right. \\ &\quad \left. + \left(3 + c^2 - ack_g \right) \varepsilon_3 \right) \\ &\quad \times \left(\left(\left(ak_g - c \right)^2 - 3k_g^2 + 3 \right)^{5/2} \right)^{-1}. \end{aligned} \tag{40}$$

□

Corollary 10. *If $\alpha : I \subset \mathbb{R} \mapsto H_0^2$ is a geodesic curve on H_0^2 in Minkowski 3-space $\mathbb{E}_1^3$, then*

(1) *αT-pseudohyperbolic Smarandache curve is also geodesic on H_0^2;*

(2) *$\alpha\xi$-pseudohyperbolic and $\alpha T\xi$-pseudohyperbolic Smarandache curves have constant geodesic curvatures on H_0^2.*

4. Examples

Example 1. Let α be a unit speed curve lying in pseudohyperbolic space $H_0^2(1)$ in the Minkowski 3-space $\mathbb{E}_1^3$ with parameter equation (see Figure 1)

$$\alpha(s) = \left(\frac{s^2}{2} + 1, \frac{s^2}{2}, s \right). \tag{41}$$

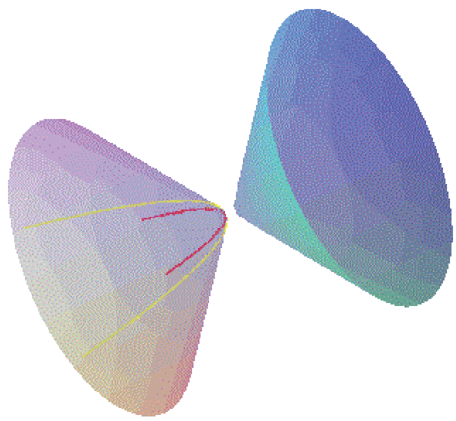

Figure 2: The $\alpha\xi$-pseudohyperbolical Smarandache curve β and the curve α on $H_0^2(1)$.

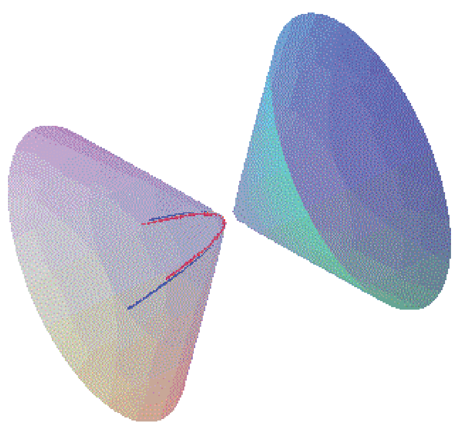

Figure 3: The αT-pseudohyperbolical Smarandache curve β and the curve α on $H_0^2(1)$.

The orthonormal Sabban frame $\{\alpha(s), T(s), \xi(s)\}$ along the curve α is given by

$$\alpha(s) = \left(\frac{s^2}{2}+1, \frac{s^2}{2}, s\right),$$
$$T(s) = \alpha'(s) = (s, s, 1), \qquad (42)$$
$$\xi(s) = \alpha(s) \times T(s) = \left(\frac{s^2}{2}, \frac{s^2}{2}-1, s\right).$$

In particular, the geodesic curvature k_g of the curve α has the form

$$k_g(s) = -1. \qquad (43)$$

Case 1. If we take $a = 2$, $b = \sqrt{2}$, then from (6) the $\alpha\xi$-pseudohyperbolical Smarandache curve β is given by (see Figure 2)

$$\beta(s^*(s)) = \frac{1}{2}\left(\left(\sqrt{2}+1\right)s^2 + 2\sqrt{2}, \left(\sqrt{2}+1\right)s^2 - 2, \left(2\sqrt{2}+2\right)s\right). \qquad (44)$$

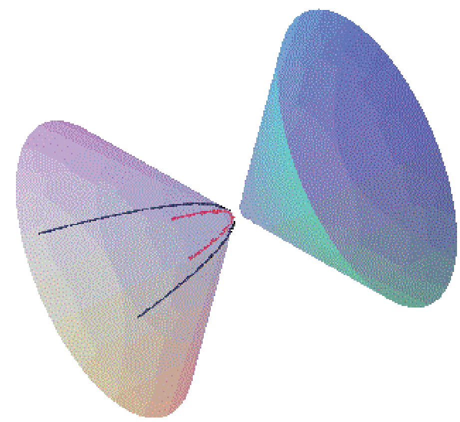

Figure 4: The $\alpha T\xi$-pseudohyperbolical Smarandache curve β and the curve α on $H_0^2(1)$.

From Theorem 7, its frame $\{\beta, T_\beta, \xi_\beta\}$ is given by

$$\begin{bmatrix} \beta \\ T_\beta \\ \xi_\beta \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} \alpha \\ T \\ \xi \end{bmatrix}, \qquad (45)$$

and the corresponding geodesic curvature k_g^β reads

$$k_g^\beta = -1. \qquad (46)$$

Case 2. If we take $a = \sqrt{3}$, $b = -1$, then from (7) the αT-pseudohyperbolical Smarandache curve is given by (see Figure 3)

$$\beta(s^*(s)) = \frac{\sqrt{2}}{4}\left(\sqrt{3}s^2 - 2s + 2\sqrt{3}, \sqrt{3}s^2 - 2s, 2\sqrt{3}s - 2\right). \qquad (47)$$

According to Theorem 8, its frame $\{\beta, T_\beta, \xi_\beta\}$ is given by

$$\begin{bmatrix} \beta \\ T_\beta \\ \xi_\beta \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{6}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{3}}{3} & 1 & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{6}}{6} & -\frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{3} \end{bmatrix} \begin{bmatrix} \alpha \\ T \\ \xi \end{bmatrix}, \qquad (48)$$

and the corresponding geodesic curvature k_g^β reads

$$k_g^\beta = -1. \qquad (49)$$

Case 3. If $a = 3$, $b = \sqrt{3}$, and $c = \sqrt{3}$, then from (7) the $\alpha T\xi$-pseudohyperbolical Smarandache curve β is given by (see Figure 4)

$$\beta(s^*(s)) = \frac{1}{2}\left(\left(\sqrt{3}+1\right)s^2 + 2s + 2\sqrt{3}, \left(\sqrt{3}+1\right)s^2 + 2s - 2, \left(2\sqrt{3}+2\right)s + 2\right). \qquad (50)$$

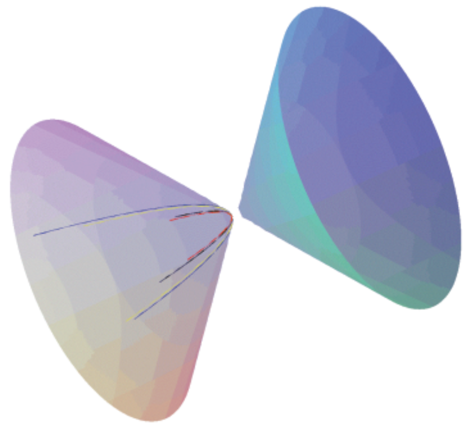

FIGURE 5: Pseudohyperbolical Smarandache curves of α and the curve α on $H_0^2(1)$.

By using Theorem 9, it follows that the frame $\{\beta, T_\beta, \xi_\beta\}$ is given by (see Figure 4)

$$\begin{bmatrix} \beta \\ T_\beta \\ \xi_\beta \end{bmatrix} = \begin{bmatrix} \sqrt{3} & 1 & 1 \\ \dfrac{\sqrt{3}-1}{2} & 1 & \dfrac{1-\sqrt{3}}{2} \\ \dfrac{\sqrt{3}+1}{2} & 1 & \dfrac{\sqrt{3}+1}{2} \end{bmatrix} \begin{bmatrix} \alpha \\ T \\ \xi \end{bmatrix}, \tag{51}$$

and the corresponding geodesic curvature k_g^β reads

$$k_g^\beta = -6\left(2+\sqrt{3}\right)\left(3+\sqrt{3}\right)^{1/2}. \tag{52}$$

Also, Pseudohyperbolical Smarandache curves of α and the curveα on H_0^2 (1) with Figure 5 are shown.

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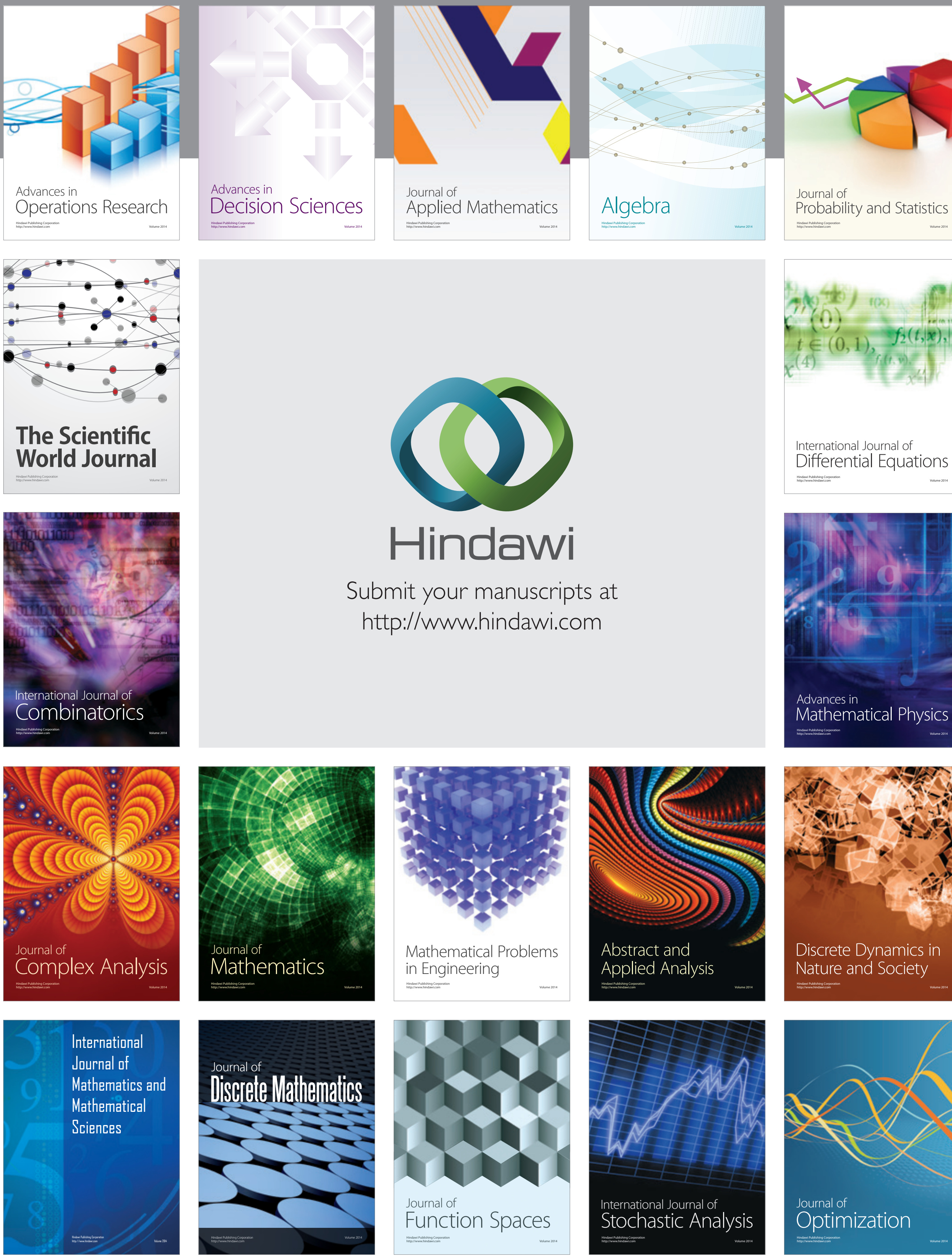
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