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ORIGINAL PAPER

Some studies on algebraic integers in $\mathbb{Q}(i, \sqrt{3})$ by using coset diagram

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Abstract In this paper, we studied the action of Picard modular group $PSL(2, \mathbb{Z}[i])$ denoted by Γ on the biquadratic field $\mathbb{Q}\left(i, \sqrt{3}\right)$. We found patern of algebraic integers formed by this action. To prove results we used coset diagrams for the action of Γ on $\mathbb{Q}\left(i, \sqrt{3}\right)$, proponded by Graham Higman.

Keywords Algebraic integers · Coset diagram · Biquadratic field

Mathematics Subject Classification Primary 05C38 · 15A15; Secondary 05A15 · 15A18

Picard modular group denoted by Γ is $PSL(2, \mathbb{Z}[i])$ or $PSL(2, O_1)$, where O_1 is the ring of Gaussian integers. Specifically, it is the group of linear fractional transforma-

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tions $T(z) = \dfrac{az+b}{cz+d}$ with $ad - bc = 1$ and $a, b, c, d \in \mathbb{Z}[i]$, for details see Anis and Hanif (2011), Mushtaq and Anis (2016), Özgür (2003).

A biquadratic field $\mathbb{Q}\left(i, \sqrt{3}\right)$ has i and $\sqrt{3}$ as roots of an irreducible polynomial $(t^2 - 3)(t^2 + 1)$ over $\mathbb{Q}$.

If Γ acts on projective line over$\mathbb{Q}\left(i, \sqrt{3}\right)$, that is, $PL\left(\mathbb{Q}\left(i, \sqrt{3}\right)\right)$ then the stabilizer Γ_α of α is the subgroup of Γ defined by $\Gamma_\alpha = \{g \in \Gamma : g(\alpha) = \alpha\}$ and the orbit $\Gamma\alpha$ of α is the subset of $\mathbb{Q}\left(i, \sqrt{3}\right)$ defined by $\Gamma\alpha = \{g(\alpha) : g \in \Gamma\}$. Any element of $\mathbb{Q}\left(i, \sqrt{3}\right)$ which satisfies a monic equation of degree greater than or equal to 1 with rational integral coefficients is called an algebraic integer of $\mathbb{Q}\left(i, \sqrt{3}\right)$.

In this paper, we studied the action of Γ on $\mathbb{Q}\left(i, \sqrt{3}\right)$ because it contains stablizers of $\dfrac{i \pm \sqrt{3}}{2}, \pm 1, \dfrac{-1 \pm \sqrt{3}i}{2}$ and $\pm i$. There is a natural one-to-one correspondence between the set Γ / Γ_α of cosets and the orbit $\Gamma\alpha$. We studied patern of algebraic integers formed by this action. To prove results we used coset diagrams for the action of Γ on $\mathbb{Q}\left(i, \sqrt{3}\right)$ (Mushtaq and Anis 2016).

It is shown in Fine (1989) that Γ has finite presentation

$$\left\langle A, B, C, D : A^3 = B^2 = C^3 = D^2 = (AC)^2 = (AD)^2 = (BC)^2 = (BD)^2 = 1 \right\rangle$$

where A, B, C and D are linear fractional transformations defined by $A(z) = \dfrac{1}{z-i}$, $B(z) = \dfrac{1}{z}$, $C(z) = \dfrac{1+z}{-z}$ and $D(z) = \dfrac{-1}{z}$.

Fixed points of generators of Γ, namely, A, B, C and D are $\dfrac{i \pm \sqrt{3}}{2}, \pm 1, \dfrac{-1 \pm \sqrt{3}i}{2}$ and $\pm i$ respectively. They all lie in $\mathbb{Q}\left(i, \sqrt{3}\right)$. The elements of $\mathbb{Q}\left(i, \sqrt{3}\right)$ are of the form $u + v\sqrt{3}$ where $u, v \in \mathbb{Q}(i)$. They can be written as $\dfrac{a + bi + c\sqrt{3} + d\sqrt{3}i}{e}$, where $a, b, c, d, e \in \mathbb{Z}$. Any element of $\mathbb{Q}\left(i, \sqrt{3}\right)$ which satisfies a monic equation of degree greater than or equal to 1 with rational integral coefficients is called an algebraic integer of $\mathbb{Q}\left(i, \sqrt{3}\right)$.

1 Coset diagram for Picard group

A diagramatic argument, called coset diagrams for the action of Picard group Γ on $\mathbb{Q}\left(i, \sqrt{3}\right)$, is used to prove results in this paper. Higman and Mushtaq have defined coset diagrams for modular group in Higman and Mushtaq (1983). Mushtaq and Anis defined coset diagrams for the Picard group in Mushtaq and Anis (2016). The coset diagrams are extensively used by many authors to solve identification problems of

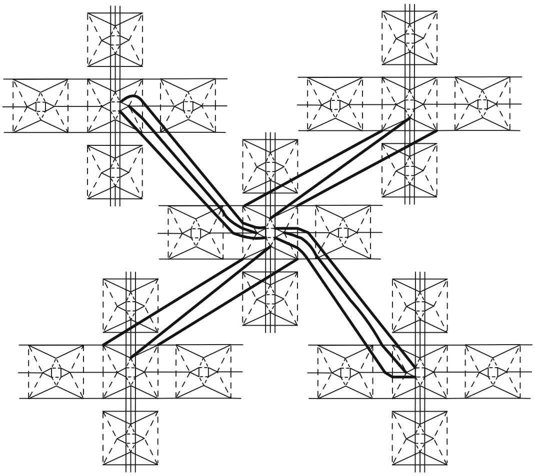


Fig. 1 Coset diagram for Picard group

groups Anis and Mushtaq (2008), Ashiq and Imran (2015), Everitt (1997), Torstensson (2010) as well as in signal processing [9].

The coset diagram for the action of Picard group Γ on $\mathbb{Q}\left(i, \sqrt{3}\right)$ is defined in detail in Mushtaq and Anis (2016). The group Γ consists of four generators, two of order 3 and two of order 2 so it is possible to avoid using colours. The generators A and C both have orders 3, so the 3-cycles of A and C are represented by triangles. But to distinguish generator A from generator C we have denoted the 3-cycles of the generator C by three unbroken edges of a triangle permuted anti-clockwise by C. The 3-cycles of the generator A are denoted by three broken edges of a triangle permuted anti-clockwise by A.

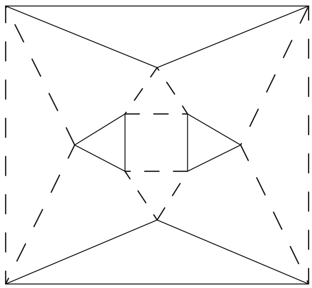
As generators B and D are involutions so we have represented them by edges without orientation. To distinguish generator B from generator D, the 2-cycles of generator B is represented as a bold edge and two vertices which are interchanged by D are joined by an hairline edge. Fixed points of A, B, C and D, if they exist, are denoted by heavy dots.

This fragment of coset diagram explains beautifully and clearly the amalgam structure of Γ that is $\Gamma = (A_4 \underset{\mathbb{Z}_3}{*} S_3) \underset{M}{*} (S_3 \underset{\mathbb{Z}_2}{*} D_2)$, here M is modular group whose finite presentation is $\left\langle D, C : D^2 = C^3 = 1 \right\rangle$. The coset diagram of Γ will be as follows Mushtaq and Anis (2016) (Fig. 1).

This diagram is 3-dimensional. The generators A and C together form a diagram of alternating group $A_4 = \left\langle A, C : A^3 = C^3 = (AC)^2 = 1 \right\rangle$ as shown in the Fig. 2 Mushtaq and Anis (2016).

If m and n are two distinct square-free rational integers, then the field formed by adjoining $\sqrt{m}$ and $\sqrt{n}$ to $\mathbb{Q}$ is denoted by $\mathbb{Q}\left(\sqrt{m}, \sqrt{n}\right)$ and is called a biquadratic field over $\mathbb{Q}$, where $\sqrt{m}$ and $\sqrt{n}$ are zeros of an irreducible quartic polynomial over $\mathbb{Q}$. The elements of $\mathbb{Q}\left(\sqrt{m}, \sqrt{n}\right)$ are of the form $a_0 + a_1\sqrt{m} + a_2\sqrt{n} + a_3\sqrt{mn}$, where $a_0, a_1, a_2, a_3 \in \mathbb{Q}$. Any element of $\mathbb{Q}\left(\sqrt{m}, \sqrt{n}\right)$ which satisfies a monic equation of degree $\geq$ 1 with rational integral coefficients is called an algebraic integer of

Fig. 2 Diagram of A_4



$\mathbb{Q}\left(\sqrt{m}, \sqrt{n}\right)$. In Williams (1970), the explicit form of the algebraic integers, an integral basis and the discriminant of $\mathbb{Q}\left(\sqrt{m}, \sqrt{n}\right)$ is given. These fields have degree 4 over $\mathbb{Q}$. Some of the subfields of $\mathbb{Q}\left(\sqrt{m}, \sqrt{n}\right)$ are $\mathbb{Q}\left(\sqrt{m}\right)$, $\mathbb{Q}\left(\sqrt{n}\right)$ and $\mathbb{Q}\left(\sqrt{mn}\right)$. The author supposed that l is the greatest common divisor of m and n, that is, $l = (m, n)$, so that $m = lm_1$, $n = ln_1$ and $(m_1, n_1) = 1$.

Theorem 1 *[Williams (1970), Theorem 1] Letting a, b, c, d denote rational integers, the algebraic integers of $\mathbb{Q}\left(\sqrt{m}, \sqrt{n}\right)$ are given as follows:*

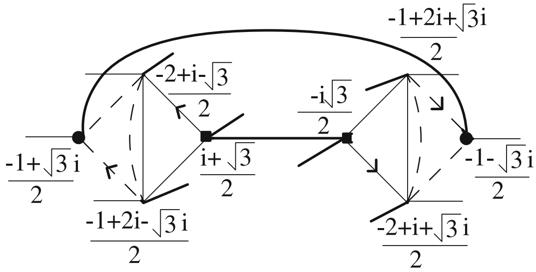
(i) if $(m, n) \equiv (m_1, n_1) \equiv (1, 1) \pmod 4$, then the algebraic integers are $\frac{1}{4}\left(a + b\sqrt{m} + c\sqrt{n} + d\sqrt{m_1 n_1}\right)$, where $a \equiv b \equiv c \equiv d \pmod 2$, $a - b + c - d \equiv 0 \pmod 4$;
(ii) if $(m, n) \equiv (1, 1)$, $(m_1, n_1) \equiv (3, 3) \pmod 4$, then the algebraic integers are $\frac{1}{4}\left(a + b\sqrt{m} + c\sqrt{n} + d\sqrt{m_1 n_1}\right)$, where $a \equiv b \equiv c \equiv d \pmod 2$, $a - b - c - d \equiv 0 \pmod 4$;
(iii) if $(m, n) \equiv (1, 2) \pmod 4$, then the algebraic integers are $\frac{1}{2}\left(a + b\sqrt{m} + c\sqrt{n} + d\sqrt{m_1 n_1}\right)$, where $a \equiv b$, $c \equiv d \pmod 2$;
(iv) if $(m, n) \equiv (2, 3) \pmod 4$, then the algebraic integers are $\frac{1}{2}\left(a + b\sqrt{m} + c\sqrt{n} + d\sqrt{m_1 n_1}\right)$, where $a \equiv c \equiv 0$, $b \equiv d \pmod 2$;
(v) if $(m, n) \equiv (3, 3) \pmod 4$, then the algebraic integers are $\frac{1}{2}\left(a + b\sqrt{m} + c\sqrt{n} + d\sqrt{m_1 n_1}\right)$, where $a \equiv d$, $b \equiv c \pmod 2$.

Remark 1 The algebraic integers of $\mathbb{Q}\left(i, \sqrt{3}\right)$ are of the form $\frac{1}{2}\left(a + b\sqrt{m} + c\sqrt{n} + d\sqrt{m_1 n_1}\right)$, where $a \equiv d$, $b \equiv c \pmod 2$.

Proposition 1 *The fixed points of a linear fractional transformation $T(z) = \frac{az + b}{cz + d}$ where $a, b, c, d \in \mathbb{Z}[i]$, are algebraic integers when $\frac{d - a}{c}$ and $\frac{b}{c} \in \mathbb{Z}$.*

Proof Let $k \in \mathbb{C}$ be a fixed point of a linear fractional transformation $T \in \Gamma$, that is, $T(k) = k$, where $T(z) = \frac{az + b}{cz + d}$, $a, b, c, d \in \mathbb{Z}[i]$. This implies that $ck^2 + (d -$

Fig. 3 Diagram having fixed points of generators A and C



$a)k - b = 0$, or $k^2 + (\frac{d-a}{c})k - \frac{b}{c} = 0$. If $\frac{d-a}{c}$ and $\frac{b}{c} \in \mathbb{Z}$, then the roots are algebraic integers. □

The fixed points of the generators A, B, C and D of Γ are algebraic integers. If α and $\bar{\alpha}$ are conjugates, then $T(\alpha)$ and $T(\bar{\alpha})$ are also conjugates, where T is a linear fractional transformation. This means the diagram formed by applying elements of Γ on α is same as the diagram formed by applying the same elements of Γ on $\bar{\alpha}$. We denote the latter diagram as "conjugate diagram". If an edge joins two vertices of a triangle, then we denote this edge by a "cap".

Proposition 2 *The fragment of the coset diagram containing the fixed points of generators A and C have four vertices and all of them are algebraic integers.*

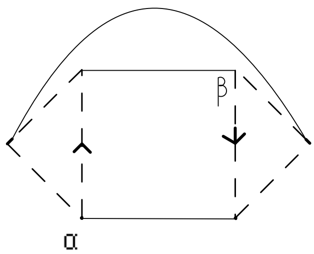
Proof The fixed points of generators A and C are $\frac{i \pm \sqrt{3}}{2}$ and $\frac{-1 \pm \sqrt{3}i}{2}$ respectively, which are of course the algebraic integers of $\mathbb{Q}(i, \sqrt{3})$ (Fig. 3).

First, consider a fragment of the coset diagram which contains $\frac{i+\sqrt{3}}{2}$. By applying generator C on $\frac{i+\sqrt{3}}{2}$, we get $\frac{-2+i-\sqrt{3}}{2}$ and $C^2\left(\frac{i+\sqrt{3}}{2}\right) = \frac{-1+2i-\sqrt{3}i}{2}$. Also, by applying generator A on above values, that is,

$$A\left(\frac{-2+i-\sqrt{3}}{2}\right) = \frac{-1+2i-\sqrt{3}i}{2} \text{ and } A^2\left(\frac{-2+i-\sqrt{3}}{2}\right) = \frac{-1+\sqrt{3}i}{2},$$

which is the fixed point of C. Further applications of A and C on these values give the same elements. This means that we get only four elements of $\mathbb{Q}(i, \sqrt{3})$ in the diagram containing the fixed points of A and C, namely, $\frac{i+\sqrt{3}}{2}$ and $\frac{-1+\sqrt{3}i}{2}$. So the Cayley's diagram of A_4 is reduced to a diagram having four vertices because of the fixed points of A and C. As $D\left(\frac{i+\sqrt{3}}{2}\right) = \frac{i-\sqrt{3}}{2}$ is conjugate of $\frac{i+\sqrt{3}}{2}$, so the diagram formed by applying generators A and C on $\frac{i-\sqrt{3}}{2}$ is conjugate diagram. It is same to the diagram formed by applying generators A and C on $\frac{i+\sqrt{3}}{2}$ because if α and $\bar{\alpha}$ are conjugates, then $T(\alpha)$ and $T(\bar{\alpha})$ are also conjugates, where T is a

Fig. 4 Diagram of S_3



linear fractional transformation. Similarly, it can be proved that the conjugate diagram containing $\frac{i-\sqrt{3}}{2}$ and $\frac{-1-\sqrt{3}i}{2}$ also contains four elements as shown in Fig. 4. □

Proposition 3 *The triangles in diagram of S_3, generated by A and D, in the coset diagram for the action of Γ on $\mathbb{Q}(i, \sqrt{3})$, have the same number of algebraic integers.*

Proof Let α be an algebraic integer of $\mathbb{Q}(i, \sqrt{3})$. By Theorem 1, $\alpha = \frac{1}{2}((a+bi) + (c+di)\sqrt{3})$ where $a \equiv d \pmod 2$, $b \equiv c \pmod 2$. By applying transformation DA on α, we get $DA(\alpha) = -\alpha + i = \frac{1}{2}\{-a-(b-2)i-(c+di)\sqrt{3}\}$, where $b-2 \equiv c \pmod 2$ since $b \equiv c \pmod 2$. This means that $DA(\alpha)$ is again an algebraic integer. So if α is a vertex of a triangle having broken edges, then the application of transformation DA on α yields an algebraic integer β in another triangle having broken edges such that $DA(\alpha) = \beta$ (Fig. 4).

These two triangles are joined by three edges which represent generator D. Hence, if all the three vertices of a triangle with broken edges are labelled by algebraic integers, then the triangle joined with this triangle by edges also contain three algebraic integers. □

Proposition 4 *The triangles in diagram of S_3, generated by B and C, in the coset diagram for the action of Γ on $\mathbb{Q}(i, \sqrt{3})$, have the same number of algebraic integers.*

Proof Let $\alpha \in \mathbb{Q}(i, \sqrt{3})$ be an algebraic integer. By Theorem 1, $\alpha = \frac{1}{2}((a+bi) + (c+di)\sqrt{3})$, where $a \equiv d \pmod 2$, $b \equiv c \pmod 2$. Now $CB(\alpha) = C(B(\alpha)) = -1-\alpha = \frac{1}{2}\{(-(2+a)-bi)-(c+di)\sqrt{3}\}$. Then $a+2 \equiv d \pmod 2$ since $a \equiv d \pmod 2$. This implies that $CB(\alpha)$ is also an algebraic integer. Similarly, for other vertices of the triangle with unbroken edges, that is, if all the three vertices of a triangle with unbroken edges are labelled by algebraic integers, then the triangle with unbroken edges joined with this triangle by bold edges, also labelled by three algebraic integers. □

Proposition 5 *There are exactly four diagrams of A_4, which contains six algebraic integers in each diagram, in the orbit containing fixed points of A and C.*

Proof It is clear from Proposition 2 and Fig. 5 that one portion of the diagram containing the fixed points of generators A and C contains four vertices or two triangles,

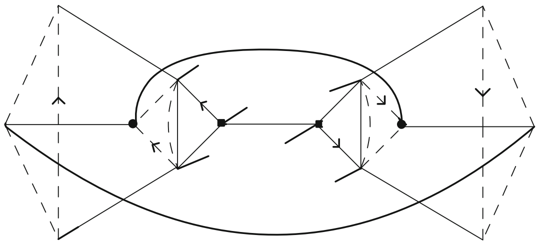


Fig. 5 Fragment of coset diagram

one having broken edges and the other having unbroken edges. Each of the triangles is labelled by three algebraic integers. By Proposition 3, the three algebraic integers of the triangle having broken edges are mapped to the triangle having broken edges of another diagram of A_4 by edges, whose vertices are also labelled by algebraic integers. By Proposition 4, the three algebraic integers of the triangle having unbroken edges are mapped to the triangle having unbroken edges of another diagram of A_4 by bold edges, whose vertices are also labelled by algebraic integers. Since CA maps an algebraic integer to an algebraic integer, so there are six algebraic integers in a diagram of A_4. Consider the same argument for another fragment having the fixed points of A and C. So this fragment is connected to two other diagrams of A_4 having six algebraic integers. So, in total there are four diagrams of A_4 which contain six algebraic integers. □

Remark 2 The fragment of a coset diagram for the action of Γ on $\mathbb{Q}(i, \sqrt{3})$ whose all vertices are labelled by algebraic integers.

Proposition 6 *If a triangle in a diagram of S_3 does not have any algebraic integers, then the other triangle joined with it by edges does not have any algebraic integers.*

Proof Let α, β and γ be vertices of a triangle having broken edges representing three cycles of generator A and they be not algebraic integers, where $\alpha, \beta, \gamma \in \mathbb{Q}(i, \sqrt{3})$. Let $\alpha = \frac{1}{e}\{(a + bi) + (c + di)\sqrt{3}\}$, where $a \not\equiv d \pmod 2$ or $b \not\equiv c \pmod 2$.

Let $a \not\equiv d \pmod 2$, $b \equiv c \pmod 2$ and $e = 2$. Then $DA(\alpha) = \frac{1}{2}\{(-a - (b - 2)i)$ $-(c + di)\sqrt{3}\}$. Since $a \not\equiv d \pmod 2$, so $DA(\alpha)$ is not an algebraic integer. Let $a \equiv d \pmod 2$, $b \not\equiv c \pmod 2$ and $e = 2$. Then $(b - 2) \not\equiv c \pmod 2$. So $DA(\alpha)$ is not an algebraic integer. Similarly, β_2 and γ_2 are not algebraic integers. □

Theorem 2 *The algebraic integers in the orbit containing the fixed points of A and C are of the form $\left\{\frac{(\pm k \pm li) \pm \sqrt{3}i}{2} : k \text{ is odd integer, } l \text{ is even integer}\right\}$ and $\{\frac{(\pm k \pm li) \pm \sqrt{3}}{2} : k \text{ is even and } l \text{ is odd integer}\}$.*

Proof The Fixed points of A and C are $\frac{i \pm \sqrt{3}}{2}$ and $\frac{-1 \pm \sqrt{3}i}{2}$ respectively. By applying CD and DC^2 repeatedly on $\frac{i \pm \sqrt{3}}{2}$, we get the series $\frac{\pm k + i \pm \sqrt{3}}{2}$ such that k is even. By applying A^2B or BA on $\frac{i\pm\sqrt{3}}{2}$, we get $\frac{li\pm\sqrt{3}}{2}$ such that l is odd. So we get the series $\frac{\pm k\pm li\pm\sqrt{3}}{2}$, where k is even and l is odd integer. By applying CD and DC^2 repeatedly on $\frac{-1\pm\sqrt{3}i}{2}$, we get $\frac{\pm k\mp\sqrt{3}i}{2}$ where k is odd. By applying A^2B or BA on $\frac{-1\pm\sqrt{3}i}{2}$, we get $\frac{-1\pm li\pm\sqrt{3}i}{2}$ such that l is even. By combining above two forms we get the series $\frac{\pm k\pm li\pm\sqrt{3}i}{2}$, where k is odd and l is even integer. □

Proposition 7 *In a diagram of A_4, algebraic integers are present in pairs.*

Proof If $\alpha=\frac{(a+bi)+(c+di)\sqrt{3}}{2}$ is an algebraic integer, where $a\equiv d(\bmod 2)$, $b\equiv c(\bmod 2)$, then $CA(\alpha)=-1+i-\alpha=\frac{\{-(2+a)+(2-b)i\}-(c+di)\sqrt{3}}{2}$. Since $d\equiv a+2(\bmod 2)$ and $c\equiv 2-b(\bmod 2)$, therefore $CA(\alpha)$ is also an algebraic integer. □

Proposition 8 *If there is one diagram of A_4 having two algebraic integers, then there are infinite diagrams of A_4 having two algebraic integers in an orbit.*

Proof Suppose α,β exist in a diagram of A_4, where α,β are algebraic integers. Then $CA(\alpha)=\beta$ or $CA(\beta)=\alpha$, where α is a vertex of a triangle representing three cycles of A as well as a vertex of a triangle representing three cycles of C. Thus, by Propositions 3 and 4, we get two different diagrams of A_4 each containing one algebraic integer. Application of transformation CA gives another algebraic integer in the same diagram of A_4. By applying DA and BC on β we get two more diagrams of A_4 each containing one algebraic integer and by the transformation CA, we get another algebraic integer in the same diagram of A_4. In the same way by applying transformations DA, BC and CA on other diagrams of A_4 which contain algebraic integers, we get infinite diagrams of A_4 which have two algebraic integers in them. □

Proposition 9 *The fragment of the coset diagram for the action of Γ on $\mathbb{Q}(i,\sqrt{3})$, containing fixed points of the generators B and D has six vertices and four of them are algebraic integers.*

Proof The fixed points of generators B and D are ± 1 and $\pm i$ respectively which lie in the orbit $\mathbb{Q}(i)$ of $\mathbb{Q}(i,\sqrt{3})$. The diagram containing the fixed points of both generators B and D, that is, -1 and i, have six vertices in total. Starting from -1 and applying transformations C and C^2, that is, $C(-1)=0$ and $C^2(-1)=\infty$. Now by considering i, $C(i)=(-1+i)$ and $C^2(i)=\frac{-1+i}{2}$. Also $A(-1+i)=-1$ and $A^2(-1+i)=\frac{-1+i}{2}$. We have $A(0)=i$ and $A^2(0)=\infty$.

So we get six vertices in total and $0,-1,-1+i$ and i are algebraic integers. □

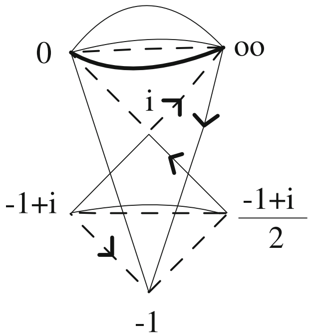


Fig. 6 Diagram having fixed points of generators B and D

Proposition 10 *There are three diagrams of A_4, in the orbit containing fixed points of generators B and D, which contain four algebraic integers.*

Proof By Proposition 9, the fragment containing fixed points of both generators B and D have four algebraic integers, namely $0, -1, -1+i$ and i. Each triangle with unbroken edges contains two algebraic integers as shown in Fig. 6. These two triangles are joined with two more diagrams of A_4 by bold edges, so in total we have three diagrams of A_4. By Proposition 4, each triangles of other diagrams of A_4 which are joined by bold edges, also have two algebraic integers. Since the transformation CA maps an algebraic integer to an algebraic integer, so both of the diagrams of A_4 contain four algebraic integers. □

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