Nash embedding and equilibrium in pure quantum states

Faisal Shah Khan

Quantum Computing Research Group, Applied Math & Sciences, Khalifa University, Abu Dhabi UAE[*]

Travis S. Humble

Quantum Computing Institute, Oak Ridge National Laboratory, Oak Ridge, Tennessee USA[†]

With respect to probabilistic mixtures of the strategies in non-cooperative games, quantum game theory provides guarantee of fixed-point stability, the so-called Nash equilibrium. This permits players to choose mixed quantum strategies that prepare mixed quantum states optimally under constraints. We show here that fixed-point stability of Nash equilibrium can also be guaranteed for pure quantum strategies via an application of the Nash embedding theorem, permitting players to prepare pure quantum states optimally under constraints.

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INTRODUCTION

As quantum technologies increase in scale and complexity, constrained optimization of the underlying quantum processes will also increase in importance. A good example is the case of a quantum Internet as a network of quantum devices that process and relay quantum information using teleportation and entanglement swapping. Connectivity constraints arise naturally when optimizing classical network resources [1, 2], and similar constraints manifest in the design and operation of quantum networks [3–5]. Similarly, quantum computers implementing fault-tolerant operations must optimize circuit design in order to minimize the decoherence due to the intrinsic physical noise in gate operation while also adhering to constraints in the circuit layout, scheduling, and parallelization. Finding optimal solutions to such constrained optimization problems are expected to maximize the efficiency and performance of quantum information technologies.

Non-cooperative game theory [6] studies optimization under constraints and can offer useful insights into the engineering of scalable quantum technologies such as optimal bounds on errors and their correction in quantum computations. Any quantum physical process modeled as a non-cooperative game describes a *quantum game* of the same type. The first instance of non-cooperative game-theoretic modeling of quantum physical processes appears to be the 1980 work of A. Blaquiere [7], where wave mechanics are considered as a two player, zero-sum (strictly competitive) differential game and a minimax result is established for certain quantum phys-

ical aspects. The more recent and more sustained game-theoretic treatment of quantum physical processes was initiated in 1999 with the work of Meyer [8]. Meyer's work considered quantum computational and quantum algorithmic aspects of quantum physics as non-cooperative games.

The year 1999 also saw the publication of the paper [9] by Eisert et al. in which a quantum informational model for the informational component of two players games was considered. This consideration was in the same spirit as the consideration of randomizing in a game which produces the so-called *mixed* game played with mixed strategies. The quantum informational model of Eisert et al. produces a *quantized* game. The inspiration for considering extensions of the informational aspect of games to larger domains comes from John Nash's famous theorem [10] in economics which not only innovates the solution concept of non-cooperative games as an equilibrium problem but, for probabilistic extensions of finite non-cooperative games, also guarantees its existence.

The promise of a Nash equilibrium solution is a foundational concept for game theory as it may be used to guarantee the behavior for the non-cooperating players. The relative simplicity of the proof of Nash's theorem for the existence of an equilibrium in mixed strategies in conventional games relies entirely on Kakutani's fixed-point theorem [11]. For quantum games, Meyer established the existence of Nash equilibrium in mixed strategies, which are modeled as mixed quantum states, using Glicksberg's [12] extension of Kakutani's fixed point theorem to topological vector spaces.

In this contribution, we note that the Kakutani fixed-point theorem does not apply directly to quantum games played with pure quantum strategies. But, one can use Nash's embedding of compact Riemannian manifolds into Euclidean space [13] (Nash's other, mathematically more famous theorem) and, under appropriate conditions, indirectly apply the Kakutani fixed-point theorem to guarantee Nash equilibrium in pure quantum strategies. We begin with a mathematically formal discussion of non-cooperative game theory and fixed-points.

NON-COOPERATIVE GAMES AND NASH EQUILIBRIUM

An N player, non-cooperative game in normal form is a function Γ

$$\Gamma : \prod_{i=1}^{N} S_i \longrightarrow O, \tag{1}$$

with the additional feature of the notion of non-identical preferences over the elements of the set of *outcomes* O, for every "player" of the game. The preferences are a pre-ordering of the elements of O, that is, for $l, m, n \in O$

$$m \preceq m, \quad \text{and} \quad l \preceq m \text{ and } m \preceq n \implies l \preceq n. \tag{2}$$

where the symbol $\preceq$ denotes "of less or equal preference". Preferences are typically quantified numerically for the ease of calculation of the payoffs. To this end, functions Γ_i are introduced which act as the *payoff function* for each player i magenta and typically map elements of O into the real numbers in a way that preserves the preferences of the players. That is, $\preceq$ is replaced with $\leq$ when analyzing the payoffs. The factor S_i in the domain of Γ is said to be the *strategy set* of player i, and a *play* of Γ is an n-tuple of strategies, one per player, producing a payoff to each player in terms of his preferences over the elements of O in the image of Γ.

A non-cooperative N-player quantum game in normal form arises from (1) when one introduces quantum physically relevant restrictions. We declare a pure (strategy) quantum game to be any unitary function

$$Q : \otimes_{i=1}^{N} \mathbb{C}P^{d_i} \longrightarrow \otimes_{i=1}^{N} \mathbb{C}P^{d_i} \tag{3}$$

where $\mathbb{C}P^{d_i}$ is the d_i-dimensional complex projective Hilbert space of pure quantum states. The latter are typically referred to as d-ary "quantum digits" or *qudits*. By analogy with mixed game extensions, where players' strategies are probability distributions over the elements of some set, the strategies of each player in a quantum game consist of quantum superpositions over the elements of a set of observable states in $\mathbb{C}P^{d_i}$. These strategic choices are then mapped by Q into $\otimes_{i=1}^{N} \mathbb{C}P^{d_i}$, over the elements of which the players have non-identical preferences defined using the overlap of two qudits as the payoff functions.

The overlap of two qudits is a complex number in general. This is in contrast to the more standard practice in classical game theory of defining payoff functions that map into the real numbers. Indeed, in the current context of pure strategy quantum games, the expected value of an observable computed after quantum measurement can be taken as the payoff function mapping into the set of real numbers. However, as we will show later in detail, the non-linearity of the expected value of an observable fails to guarantee Nash equilibrium whereas the linearity of the overlap does not.

Nash equilibrium is a play of Γ in which every player employs a strategy that is a best reply, with respects to his preferences over the outcomes, to the strategic choice of every other player. In other words, unilateral deviation from a Nash equilibrium by any one player in the form of a different choice of strategy will produce an outcome which is less preferred by that player than before. Following Nash, we say that a play p' of Γ *counters* another play p if $\Gamma_i(p') \geq \Gamma_i(p)$ for all players i, and that a self-countering play is an (Nash) equilibrium.

Let C_p denote the set of all the plays of Γ that counter p. Denote $\prod_{i=1}^{N} S_i$ by S for notational convenience, and note that $C_p \subset S$ and therefore $C_p \in 2^S$. Further note that the game Γ can be factored as

$$\Gamma : S \xrightarrow{\Gamma_C} 2^S \xrightarrow{E} O \tag{4}$$

where to any play p the map Γ_C associates its countering set C_p via the payoff functions Γ_i. The set-valued map Γ_C may be viewed as a preprocessing stage where players seek out a self-countering play, and if one is found, it is mapped to its corresponding outcome in O by the function E. The condition for the existence of a self-countering play, and therefore of a Nash equilibrium, is that Γ_C have a fixed point, that is, an element $p^* \in S$ such that $p^* \in \Gamma_C(p^*) = C_{p^*}$.

In a general set-theoretic setting for non-cooperative games, the map Γ_C may not have a fixed point. Hence, not all non-cooperative games will have a Nash equilibrium. However, according to Nash's theorem, when the S_i are finite and the game is extended to its *mixed* version, that is, the version in which randomization via probability distributions is allowed over the elements of all the S_i, as well as over the elements of O, then Γ_C has at least one fixed point and therefore at least one Nash equilibrium.

Formally, given a game Γ with finite S_i for all i, its mixed version is the product function

$$\Lambda : \prod_{i=1}^{N} \Delta(S_i) \longrightarrow \Delta(O) \qquad (5)$$

where $\Delta(S_i)$ is the set of probability distributions over the ith player's strategy set S_i, and the set $\Delta(O)$ is the set of probability distributions over the outcomes O. Payoffs are now calculated as *expected payoffs*, that is, weighted averages of the values of Γ_i, for each player i, with respect to probability distributions in $\Delta(O)$ that arise as the product of the plays of Λ. Denote the expected payoff to player i by the function Λ_i. Also, note that Λ restricts to Γ. In these games, at least one Nash equilibrium play is guaranteed to exist as a fixed point of Λ via Kakutani's fixed-point theorem.

Kakutani fixed-point theorem: *Let $S \subset \mathbb{R}^n$ be nonempty, compact, and convex, and let $F : S \to 2^S$ be an upper semi-continuous set-valued mapping such that $F(s)$ is non-empty, closed, and convex for all $s \in S$. Then there exists some $s^* \in S$ such that $s^* \in F(s^*)$.*

To see this, make $S = \prod_{i=1}^{N} \Delta(S_i)$. Then $S \subset \mathbb{R}^n$ and S is non-empty, bounded, and closed because it is a finite product of finite non-empty sets. The set S is also convex because its the convex hull of the elements of a finite set. Next, let C_p be the set of all plays of Λ that counter the play p. Then C_p is non-empty, closed, and convex. Further, $C_p \subset S$ and therefore $C_p \in 2^S$. Since Λ is a game, it factors according to (4)

$$\Lambda : S \xrightarrow{\Lambda_C} 2^S \xrightarrow{E_\Pi} \Delta(O) \qquad (6)$$

where the map Λ_C associates a play to its countering set via the payoff functions Λ_i. Since Λ_i are

all continuous, Λ_C is continuous. Further, $\Lambda_C(s)$ is non-empty, closed, and convex for all $s \in S$ (we will establish the convexity of $\Lambda_C(s)$ below; the remaining conditions are also straightforward to establish). Hence, Kakutani's theorem applies and there exists an $s^* \in S$ that counters itself, that is, $s^* \in \Lambda_C(s^*)$, and is therefore a Nash equilibrium. The function E_Π simply maps s^* to $\Delta(O)$ as the product probability distribution from which the Nash equilibrium expected payoff is computed for each player.

The convexity of the $\Lambda_C(s) = C_p$ is straight forward to show. Let $r, s \in C_p$. Then

$$\Lambda_i(r) \geq \Lambda_i(p) \quad \text{and} \quad \Lambda_i(s) \geq \Lambda_i(p) \qquad (7)$$

for all i. Now let $0 \leq \mu \leq 1$ and consider the convex combination $\mu r + (1 - \mu)s$ which we will show to be in C_p. First note that $\mu r + (1 - \mu)s \in S$ because S is the product of the convex sets $\Delta(S_i)$. Next, since the Λ_i are all linear, and because of the inequalities in (7) and the restrictions on the values of μ,

$$\Lambda_i(\mu r + (1 - \mu)s) = \mu \Lambda_i(r) + (1 - \mu)\Lambda_i(s) \geq \Lambda_i(p) \tag{8}$$

whereby $\mu r + (1 - \mu)s \in C_p$ and C_p is convex.

Going back to the game Γ in (1) defined in the general set-theoretic setting, Kakutani's theorem would apply to Γ if the conditions are right, that is, whenever the image set of Γ is pre-ordered and Γ_i is both linear and preserves the pre-order.

Kakutani's fixed-point theorem can be generalized to include subsets S of convex topological vector spaces, as was done by Glicksberg in [12]. Using Glicksberg's fixed-point theorem, one can show that Nash equilibrium exists in games where the strategy sets are infinite or possibly even uncountably infinite. As mentioned earlier in the Introduction, Meyer used Glicksberg's fixed-point theorem to establish the guarantee of Nash equilibrium in mixed quantum games. This is not surprising given that probabilistic mixtures form a convex structure, which is an essential ingredient for fixed-point theorems to hold on "flat" manifolds such as $\mathbb{R}^m$.

PURE QUANTUM GAMES AND NASH EQUILIBRIUM

No fixed-point theorem guarantee for Nash equilibrium in pure quantum strategies is known to exist in the literature. This is surprising perhaps given the

rich, albeit non-convex, mathematical structure of $\mathbb{C}P^n$. More precisely, $\mathbb{C}P^n$ has a compact Riemannian (Kahler in fact) manifold structure with positive sectional curvature with respect to the Fubini-Study metric [14]. We use the richness of the mathematical structure of $\mathbb{C}P^n$ here to produce a guarantee of Nash equilibrium in pure quantum strategies, under restrictive conditions, by invoking Nash's embedding theorem:

Nash embedding theorem: *For every compact Riemannian manifold M, there exists an isometric embedding of M into $\mathbb{R}^m$ for a suitably large m.*

The Nash embedding theorem tells us that $\mathbb{C}P^n$ is diffeomorphic to its image under a length preserving map into $\mathbb{R}^m$. The homeomorphism underlying this diffeomorphism allows us to treat $\mathbb{C}P^n$ and its image inside $\mathbb{R}^m$ as topologically equivalent; hence, we can treat $\mathbb{C}P^n$ as a sub-manifold S of $\mathbb{R}^m$ and look for a fixed-point guarantee for continuous set-valued functions F

$$\mathbb{C}P^n \overset{e}{\hookrightarrow} S \overset{F}{\longrightarrow} 2^S \qquad (9)$$

via Kakutani's fixed-point theorem, where e is the Nash embedding. To this end, recall that the Kakutani fixed-point theorem requires S to be compact and convex. Since e is a homeomorphism and $\mathbb{C}P^n$ is compact, S is compact in $\mathbb{R}^m$. However, S is not necessarily convex as homeomorphisms do not preserve convexity in general. Note also that the linearity of Nash embedding e would be insufficient to ensure convexity of S, for an element q of $\mathbb{C}P^n$ is equivalent to all scalar multiples λq, for $\lambda \neq 0$. This violates the definition of convexity, which requires the possibility that $\lambda = 0$.

But suppose for the moment that there exists a convex embedding S of $\mathbb{C}P^n$ into $\mathbb{R}^m$. For the Kakutani fixed-point theorem to be applicable to F, one encases S in a simplex Δ, establishes Kakutani's theorem on Δ via barycentric subdivision and the Brouwer fixed-point theorem [15] , and then constructs a *retract* function

$$R : \Delta \longrightarrow S \qquad (10)$$

that fixes the points of S, that is,

$$R(s) = s \qquad (11)$$

for all $s \in S$. One can visualize the action of R as projecting Δ onto S, possibly in a geometrically
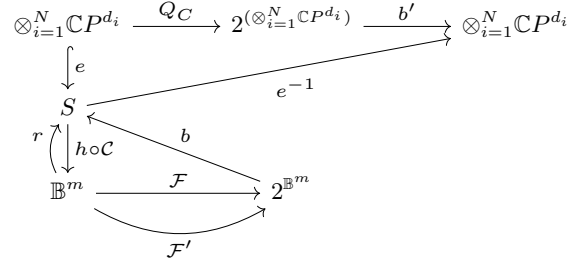

FIG. 1. Establishing the guarantee of Nash equilibrium in pure quantum strategies in a quantum game Q using Nash's embedding theorem. The functions b and b' take a fixed-point from the power set back to the set.

convoluted way, and projecting the points of S onto themselves. Next, one defines a set-valued function

$$F' : \Delta \longrightarrow 2^\Delta \qquad (12)$$

as $F'(s) = F(R(s))$; this function is upper-semicontinuous, and its fixed points lie in S and are therefore fixed points of F.

Therefore, whenever it is possible to construct e so that S is convex in some R^m, then as per the discussion in the preceding paragraph, Nash equilibrium in pure strategy quantum games is guaranteed. On the other hand, when S is not convex, for example when $S = e\left(\mathbb{C}P^1\right) = \mathbb{S}^2 \subset \mathbb{R}^3$, then one can extend to its convex hull $\mathrm{Conv}(S)$ (taking into account the conditions outlined in the Caveats section below) via the convex hull operation, call it $\mathcal{C}$, and note that by Caratheodory's theorem [16] $\mathrm{Conv}(S)$ is compact. It is a well-established topological fact that a non-empty, convex compact subset of $\mathbb{R}^m$ is homeomorphic to the closed unit ball

$$\mathbb{B}^m = \left\{ (x_1, \ldots, x_m) \in \mathbb{R}^m : \sum_{j=1}^{m} x_j^2 \leq 1 \right\}. \qquad (13)$$

Let h be the homeomorphism $\mathrm{Conv}(S) \cong \mathbb{B}^m$. The set-valued function F in (9) now has factors

$$S \overset{\mathcal{C}}{\rightarrow} \mathrm{conv}(S) \overset{h}{\rightarrow} \mathbb{B}^m \overset{\mathcal{F}}{\rightarrow} 2^{\mathbb{B}^m} \qquad (14)$$

with continuous $\mathcal{F}$.

Next, set the payoff function $\mathcal{F}_i = Q_i$ to be any linear function with the property that the elements of its image can be pre-ordered so that(7) and (8) may be invoked. This produces convex countering sets $\mathcal{F}(h(\mathcal{C}(e(q))))$ of the image $e(q)$ of a play q of

Q and guarantees a fixed point for $\mathcal{F}$. To ensure the existence of fixed-points of $\mathcal{F}$ over S, construct a retract

$$r : \mathbb{B}^m \longrightarrow S \qquad (15)$$

and define the set-valued function

$$\mathcal{F}'(v) = \mathcal{F}(r(v)) \qquad (16)$$

so that the fixed points of $\mathcal{F}'$ lie in S and hence are the fixed points of $\mathcal{F}$. Mapping the fixed points back to $\mathbb{C}P^n$ via e^{-1} gives a Nash equilibrium q^* in Q.

This construction is captured in FIG. 1 for the relevant n. Note that when S is convex, $\mathrm{Conv}(S) = S$ and our construction restricts to an application of the Kakutani fixed-point theorem. Therefore, up to the caveats mentioned in the following section, our construction guarantees Nash equilibrium in general. Identifying the values of m for which $\mathbb{C}P^n$ embeds as a convex subset into R^m appears to be an open problem to the best of our knowledge.

CAVEATS

The Nash equilibrium guarantee due to the construction in FIG. 1 comes with the conditions that S is non-empty and that S does not equal the boundary of $\mathrm{Conv}(S)$. We elaborate below.

Denote by ∂ the boundary of a non-empty set. Then, $h(\partial\mathrm{Conv}(S)) \cong \partial h(\mathrm{Conv}(S)) = \partial\mathbb{B}^m = \mathbb{S}^{m-1}$, where

$$\mathbb{S}^{m-1} = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{j=1}^{m} x_j^2 = 1 \right\} \qquad (17)$$

is the $(m-1)$-dimensional sphere. Again, it is well-established that no retract exists from $\mathbb{B}^m$ to $\mathbb{S}^{m-1}$ [17] (the no-retract theorem). Therefore, if the Nash embedding $S = \partial\mathrm{Conv}(S)$, then the construction in FIG. 1 will fail to guarantee Nash equilibrium. This is certainly true for $N = 1$ and $d_i = 1$ in the definition of a quantum game in (3), giving $\mathbb{C}P^1 = \mathbb{S}^2 = \partial\mathbb{B}^2$ (recall that $\mathbb{C}P^1$ *is* $\mathbb{S}^2$ and hence the latter is the natural Nash embedding) as the single player's strategy set. Fortunately, since we are concerned only with multiplayer games, $N \geq 2$.

In general, the elements of any set are the extreme points of its convex hull, that is, points x and y in the convex hull for which $\lambda x + (1-\lambda)y$ implies that either

$\lambda = 0$ or $\lambda = 1$. Moreover, the set of extreme points generally forms a proper subset of the boundary of a convex set. Hence,

$$S \subset \partial\mathrm{Conv}(S) \implies h(S) \subset \mathbb{S}^{m-1}. \qquad (18)$$

Next, to see that it is possible to have a retract from $\mathbb{B}^m$ to $h(S) \subset \mathbb{S}^{m-1}$, note that every continuous function from $\mathbb{B}^m$ to $\mathbb{S}^{m-1}$ moves at least one point of the $(m-1)$-sphere by the no-retract theorem. The extreme case where all the points move is undesirable, but any other situation gives a subset $\mathcal{S}$ of fixed points of $\mathbb{S}^{m-1}$ over which a retract from $\mathbb{B}^m$ may be constructed. Setting $h(S) = \mathcal{S}$ will give a class of retracts to serve as the function r in FIG. 1. For example, the function

$$f : (x_1, \ldots, x_m) \to \left(x_1, \ldots, x_{m-1}, \sqrt{1 - \sum_{j=1}^{m-1} x_j^2} \right) \qquad (19)$$

has the "upper hemisphere" as retract.

AN APPLICATION OF PURE STRATEGY QUANTUM GAMES

As an application of Nash equilibrium in pure quantum strategies, consider the problem of preparing a n qudit state. Due to decoherence errors, quantum states can deviate from some desired configuration. To model the state preparation and decoherence errors as a non-cooperative pure strategy quantum game, we introduce N notional players so that each player prefers the prepared state to be closest to his desired configuration. A state preparing quantum mechanism Q can now be viewed as a non-cooperative, pure strategy quantum game that maps a play of the game to a quantum superposition $q = \sum_{j=1}^{n} \alpha_j b_j \in \otimes_{i=1}^{n} \mathbb{C}P^{d_i}$ with $\sum |\alpha_j|^2 = 1$, and in which the payoffs are defined by the overlap of the prepared quantum state relative to the desired configuration of each player, or in other words, the inner-product of the two states. This linear payoff function is

$$Q_i(q) = \langle \psi_i, q \rangle \qquad (20)$$

with $\psi_i \in \otimes_{i=1}^{N} \mathbb{C}P^{d_i}$ the preferred state of the i-th player. The fact that the complex numbers can be pre-ordered, combined with the linearity of $Q_i(q)$,

allows one to invoke (7) and (8) to establish the convexity of $\mathcal{F}(h(\mathcal{C}(e(q)))) = C_{e(q)}$.

By an application of the Nash embedding theorem and the Kakutani fixed-point theorem, we conclude that this game has fixed-point guarantee of Nash equilibrium and that there is an optimal solution to the problem of preparing a pure quantum state under the constraint of decoherence. A similar game model can also be applied to parameter estimation problems, especially in the context of quantum logic gates.[18, 19].

In contrast, we may also consider the case of a quantum game in which the payoff is defined with respect to a physical observable, as is typically done in quantum game theory literature. A physical observable is represented by a linear Hermitian operator whose eigenstates define possible outcomes. We may define the expectation value of such an operator with respect to a prepared quantum state q as the corresponding payoff. The payoff function $\bar{Q}_i$ calculates the expected value of q to player i via

$$\bar{Q}_i(q) = \sum_{j=1}^{n} e_j |\alpha_j|^2 \qquad (21)$$

where the e_j are real numbers that numerically reflect the preferences of player i over observable states b_j of $\otimes_{i=1}^{n} \mathbb{C}P^{d_i}$ and $|\alpha_j|^2$ is the probability with which q measures as b_j. The payoff function $\bar{Q}_i$ is not linear in general. Hence, the convexity of $C_{e(q)}$ does not follow in general and neither does a fixed-point guarantee for the existence of Nash equilibrium. This result is consistent with the results based on the quantization schemes a la Eisert et al. which define payoffs via (21) and for which pure quantum strategy Nash equilibrium are known to not exist in general.

For details on how quantum games are realized experimentally, we refer the reader to [20]. For a complete game-theoretic model for designing two qubit quantum computational mechanisms at Nash equilibrium, we refer the reader to [21, 22].

FUTURE APPLICATIONS OF QUANTUM GAMES

State amplification quantum algorithms like the famous Grover's algorithm [23] may be viewed as non-cooperative games between a player and Nature.

Consider the quantum state representing the item being searched for as the preferred configuration of the player with control over x qubits, while Nature, with her y qubits, prefers anything but this configuration, with $x + y = n$. A multiplayer model is also possible. If the players' payoffs are given linearly as in (20), then the algorithm has a Nash equilibrium solution. If the payoffs are computed via (21), then Nash equilibrium may not exist.

With respect to mixed strategies, a natural application of quantum games would be in the areas of quantum communication or stochastic quantum processes where a coalition of players (Alice and Bob) engage in a non-cooperative way with the eavesdropper (Eve) [24]. Alice and Bob want to amplify privacy of the communication whereas Eve does not, and in fact may want to decrease it. If the Alice and Bob coalition and Eve try to achieve their respective outcomes via random quantum processes, then Glicksberg's theorem will guarantee a Nash equilibrium. With this guarantee in place, mechanism design methods can be adopted to find an equilibrium.

An important class of quantum games would be those that study equilibrium behavior of the subset of generalized quantum measurements on finite dimensional systems known as local operations and classical communication (LOCC), a set that is both compact and convex [25]. Because LOCC is significant in many quantum information processes as the natural class of operations, and given its compact and convex structures, constructing a non-cooperative finite quantum game model for it would be a worthwhile effort.

Adiabatic quantum computing [26–28] can potentially benefit from the quantum game model. An adiabatic quantum computation starts with a system of n qudits in its lowest energy state. A Hamiltonian H_I is constructed that corresponds to this lowest energy state and another Hamiltonian H_f is used to encode an objective function the solution of which is the minimum energy state of H_f. Finally, the actual adiabatic computation occurs as the interpolating Hamiltonian

$$H\left(s(t)\right) = s(t)H_I + (1 - s(t))H_f \qquad (22)$$

which is expected to adiabatically transform the lowest energy state of H_I to that of H_f as a function of the interpolating path $s(t)$ with respect to time t. For large enough t values, adiabaticity holds; on the other hand, t should be much smaller than its corre-

sponding value in classical computational processes for $H(s(t))$ to constitute a worthwhile effort.

Note that under exponentiation, $H(s(t))$ corresponds to a time-dependent unitary map from $\otimes_{i=1}^{n} \mathbb{C}P^{d_i}$ to itself that we can view as a zero-sum, non-cooperative quantum game. This quantum game has a notional player I that prefers an element of $\otimes_{i=1}^{n} \mathbb{C}P^{d_i}$ that corresponds to the lowest energy state of H_f. Player II or Nature, prefers anything but this element. Players I and II can be given access to any division of qubits to manipulate respectively via pure quantum strategies. Again, if the payoffs to the players are linear, then a Nash equilibrium is guaranteed.

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* faisal.khan@kustar.ac.ae

† humblets@ornl.gov

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