Doppler Effect for Non-uniformly Moving Source, Observer, and Mirror

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Abstract

Doppler effect has been applied to study the accelerating universe and to daily life. We believe that to describe non-uniform moving system, physics laws should include terms of acceleration/deceleration for accuracy and completeness. In this article, we study systematically the effects of non-uniform motion of either source, or observer, or mirror on Doppler effect, called it the Extended Doppler effect. We show that, indeed, the non-uniform motion not only shifts wavelength/frequency but also changes wavelength/frequency and Extended Doppler shift continuously. There is no transformation between non-uniform moving reference frames. We derived Extended Doppler effect for non-uniform moving source, observer, and mirror based on (1) the speed of light wave is the same for all non-uniform moving frames; (2) geometric optical principles hold. We show that, for slow non-uniform motion, the shifts of wavelength/frequency depend on the relative velocity and relative acceleration/deceleration.

Key words: Doppler effect, Doppler frequency, Doppler shift, non-uniform motion, extended Doppler effect, Extended Doppler frequency

Content

1. Introduction

The classical Doppler effect is a shift in wavelength/frequency, which is caused by velocity of uniformly moving either source, or observer, or mirror. Doppler shift of wavelength/frequency is applied fruitfully in many fields including astronomy and Doppler radar. In the 1998, Scientists reported a revolutionary discovery that the expansion of the universe is accelerating [1]. In the 2017, for explaining the accelerating universe, Doppler effect and Hubble law is extended to include acceleration [2]. It is shown that acceleration of either source, or observer, or both does affect the shifts of wavelength/frequency. In automobile application, to avoid accident more effectively, it is important to detect the change of car's velocity, i.e., acceleration/deceleration.

We believe that to describe an accelerating system, physical laws must include terms of acceleration/deceleration for accuracy and completeness. It is nature to expect to extend Doppler's velocity-shift relation to include acceleration/deceleration. Note for accelerating/deceleration systems, both Galileo transformation and Lorentz transformation do not hold. Thus, we don't take into account effects of Special Relativity in this article, even we study light wave. One needs to distinguish whether a source is moving or an observer is moving, or both are moving. We introduce a coordinate system in which both an observer and a source are accelerating/decelerating arbitrarily. We restrict to (1) low speed of source, observer and mirror; (2) constant acceleration/deceleration; (3) the speed of light is the same for all of accelerating/decelerating sources, observers and mirrors.

Doppler effect does not change with time. On the contrary, we show that the shift of wavelength/frequency due to acceleration/deceleration does change with time. The time changes of shifts are important in applications of the accelerating expansion of universe and of automatic drive of cars. In latter application, finding time change can predict the trend of movement of cars for better prevention of collision.

2. Extended Doppler Effect of Non-Uniformly Moving Source and/or Observer

The mechanism of Doppler effect is that when the source is emitting, each successive wave crest is emitted from a position farther or closer to the observer than the previous wave crest. Therefore, each wave crest takes longer or shorter time to reach the observer than the previous one, i.e., increasing or reducing the wavelength. We argue that, for an accelerating/decelerating moving source, the same mechanism holds.

First lets summarize notations. Velocity of receding motion is defined as positive, and velocity of approaching movement is defined as negative. Subscript "o" and "s" denote the parameters related to observer and source respectively. Subscript "ED" denotes the Extended Doppler parameters. We derive shifts of wavelength/frequency, and time change of those shifts for each special situation.

2.1. Source is non-uniformly moving, Observer is stationary

2.1.1. Source is receding with acceleration/deceleration

Case (1): Source is receding with acceleration. Considering a situation that a wave source is receding from an observer with velocity (slower than the speed of wave), v_s, and acceleration, a_s. The speed of light is $c = \frac{\lambda_s}{\Delta t_s}$, which is a constant for both the source and the observer. Where λ_s is emitted wavelength and measured in the source coordinate system. Δt_s is the time interval that wave takes to move one wavelength λ_s, $\Delta t_s \equiv (t_{s2} - t_{s1}) = {}^{\lambda_s}/_{c}$. Considering two consecutive wave crests #1 and #2 emitted at time t_{s1} and t_{s2} respectively. During the time interval Δt_s, the source moves a distance D_s,

$$D_s = v_s \Delta t_s + \frac{1}{2} a_s (\Delta t_s)^2. \quad (1)$$

Thus, when the source is moving away from the observer, the wavelength is stretched from λ_s to λ_0,

$$\lambda_0 = \lambda_s + D_s. \quad (2)$$

We obtain observed wavelength λ_o, Extended Doppler shift cz_{ED}, frequency f_o, and Extended Doppler frequency $f_{ED} \equiv f_o - f_s$, respectively,

$$\lambda_o = \lambda_s \left[1 + \frac{v_s}{c} + \frac{a_s \lambda_s}{2c^2}\right], \quad (3)$$

$$cz_{ED} \equiv \frac{\lambda_o}{\lambda_s} - 1 = v_s + a_s \left(\frac{\lambda_s}{2c}\right). \quad (4)$$

$$f_o = \frac{f_s}{1 + \frac{v_s}{c} + \frac{a_s \lambda_s}{2c^2}}, \quad (5)$$

$$f_{ED} = -\frac{\frac{v_s}{c} + a_s\left(\frac{\lambda_s}{2c^2}\right)}{1 + \frac{v_s}{c} + \frac{a_s \lambda_s}{2c^2}} f_s. \quad (6)$$

Indeed the acceleration of a receding source does shift the wavelength/frequency. The first terms on the right hand sides of Eq. (4) is Doppler redshift; the second term is redshift due to acceleration, denoted as “Acceleration-redshift”.

Time Changes. A significant difference between Doppler effect and Extended Doppler effect is that the Extended Doppler frequency and Extended Doppler shift are time dependent. Taking derivative with respect to time, we obtain,

$$\frac{c}{f_s} \frac{df_{ED}}{dt} = -\frac{a_s}{\left[1 + \frac{v_s}{c} + \frac{a_s \lambda_s}{2c^2}\right]^2}, \quad (7)$$

$$c \frac{dz_{ED}}{dt} = a_s. \quad (8)$$

From the definitions of Extended Doppler frequency and Extended Doppler shift, we have respectively

$$\frac{df_o}{dt} = \frac{df_{ED}}{dt}, \quad (9)$$

$$\frac{c}{\lambda_s}\frac{d\lambda_o}{dt} = c\frac{dz_{ED}}{dt}. \tag{10}$$

Note, therefore, in the rest of the article, we will not write formulas of $\frac{df_o}{dt}$ and $\frac{d\lambda_o}{dt}$.

Approximation. For slow moving source, keeping the first order terms, we obtain,

$$f_o \approx f_s\left(1 - \frac{v_s}{c} - \frac{a_s\lambda_s}{2c^2}\right), \tag{5a}$$

$$f_{ED} \approx -\left[\frac{v_s}{c} + a_s\left(\frac{\lambda_s}{2c^2}\right)\right] f_s, \tag{6a}$$

$$\frac{c}{f_s}\frac{df_{ED}}{dt} \approx -a_s. \tag{7a}$$

The acceleration of a source may be measured by detecting the time change of shifts of wavelength/frequency.

Case (2): Source is receding with deceleration. For this case, the formulas of observed wavelength λ_o, Extended Doppler shift cz_{ED}, frequency f_o, Extended Doppler frequency f_{ED}, time change of cz_{ED} and f_{ED}, have the same form as that of Case (1), but replacing a_s with $-d_s$. The deceleration of a source may be measured by detecting the time change of shifts of wavelength/frequency.

2.1.2. Source is approaching with acceleration/deceleration

Case (1): Source is approaching with acceleration. During the time interval Δt_s, the source moves a distance $D_s = v_s\left(\frac{\lambda_s}{c}\right) + \frac{1}{2}a_s\left(\frac{\lambda_s}{c}\right)^2$. We obtain,

$$\lambda_o = \lambda_s - D_s = \lambda_s\left[1 - \frac{v_s}{c} - \frac{a_s\lambda_s}{2c^2}\right], \tag{11}$$

$$cz_{ED} = -v_s - a_s\left(\frac{\lambda_s}{2c}\right), \tag{12}$$

$$f_o = \frac{f_s}{1 - \frac{v_s}{c} - \frac{a_s\lambda_s}{2c^2}}, \tag{13}$$

$$f_{ED} = \frac{\frac{v_s}{c} + \frac{a_s\lambda_s}{2c^2}}{1 - \frac{v_s}{c} - \frac{a_s\lambda_s}{2c^2}} f_s. \tag{14}$$

Time Change. Taking derivative with respect to time, we obtain,

$$c\frac{dz_{ED}}{dt} = -a_s. \tag{15}$$

$$\frac{c}{f_s}\frac{df_o}{dt} = \frac{a_s}{\left[1 - \frac{v_s}{c} - \frac{a_s\lambda_s}{2c^2}\right]^2}. \tag{16}$$

Approximation. For slow moving source, keeping first order terms, we obtain,

$$f_o \approx f_s\left(1 + \frac{v_s}{c} + \frac{a_s\lambda_s}{2c^2}\right), \tag{13a}$$

$$f_{ED} \approx \left(\frac{v_s}{c} + \frac{a_s\lambda_s}{2c^2}\right) f_s. \tag{14a}$$

$$\frac{c}{f_s}\frac{df_{ED}}{dt} \approx a_s, \tag{16a}$$

Case (2): Source is approaching with Deceleration. For this case, the formulas of observed wavelength λ_o, Extended Doppler shift cz_{ED}, frequency f_o, Extended Doppler frequency f_{ED}, time change of cz_{ED} and f_{ED}, have the same form as that for acceleration, but replacing a_s with $-d_s$.

2.2. Observer is non-uniformly moving, Source is stationary

2.2.1. Observer is receding with acceleration/deceleration

Case (1): Observer is receding with acceleration. Considering two consecutive wave crests #1 and #2 received by the observer at time t_{o1} and t_{o2} respectively; and $\Delta t_o \equiv (t_{o2} - t_{o1}) = \lambda_o/c$. The wave velocity is: $c = \frac{\lambda_o}{\Delta t_o}$. Where λ_o is the observed wavelength measured in the observer coordinate system. Δt_o is the time interval that wave takes to move one wavelength λ_o. During the time interval Δt_o, the observer moves a distance

$$D_o = v_o\left(\frac{\lambda_o}{c}\right) + \frac{1}{2}a_o\left(\frac{\lambda_o}{c}\right)^2. \quad (17)$$

The wavelength is,

$$\lambda_o = \lambda_s + D_o = \lambda_s + v_o\left(\frac{\lambda_o}{c}\right) + \frac{1}{2}a_o\left(\frac{\lambda_o}{c}\right)^2. \quad (18)$$

For simplicity, we resolve Eq. (30) by the following approach. We obtain,

$$\lambda_o = \frac{\lambda_s}{1-\left(\frac{v_o}{c}\right)-\frac{a_o\lambda_o}{2\,c^2}}, \quad (19)$$

$$f_o = f_s\left[1-\left(\frac{v_o}{c}\right)-\frac{a_o}{2}\frac{\lambda_o}{c^2}\right]. \quad (20)$$

$$f_{ED} = -f_s\left[\frac{v_o}{c}+\frac{a_o}{2}\frac{\lambda_o}{c^2}\right]. \quad (21)$$

$$cz_{ED} = \frac{v_o+\frac{a_o\lambda_o}{2c}}{1-\left(\frac{v_o}{c}\right)-\frac{a_o\lambda_o}{2\,c^2}}. \quad (22)$$

Time Change. Taking derivative with respect to time, we obtain,

$$\frac{c}{f_s}\frac{df_{ED}}{dt} = -a_o. \quad (23)$$

$$c\frac{dz_{ED}}{dt} = \frac{a_o}{\left[1-\left(\frac{v_o}{c}\right)-\frac{a_o\lambda_o}{2\,c^2}\right]^2}. \quad (24)$$

Approximation. For slowly moving observer, keeping first order terms and taking $\lambda_o/c \approx \lambda_s/c$, we obtain

$$\lambda_o \approx \lambda_s\left(1+\frac{v_o}{c}+\frac{a_o\lambda_s}{2c^2}\right). \quad (19a)$$

$$f_o \approx f_s\left[1-\left(\frac{v_o}{c}\right)-\frac{a_o}{2}\frac{\lambda_s}{c^2}\right]. \quad (20a)$$

$$f_{ED} \approx -f_s\left[\frac{v_o}{c}+\frac{a_o}{2}\frac{\lambda_s}{c^2}\right]. \quad (21a)$$

$$cz_{ED} \approx v_o + \frac{a_o\lambda_s}{2c}. \quad (22a)$$

$$c\frac{dz_{ED}}{dt} \approx a_o. \quad (23a)$$

It is straightforwardly to show that for slow motion, the results obtained above are the same as that obtained from directly resolve Eq. (18) as a non-linear equation. Throughout this article，we use the above approach.

Case (2): Observer is receding with Deceleration. For this case, the formulas of observed wavelength λ_o, Extended Doppler shift cz_{ED}, frequency f_o, Extended Doppler frequency f_{ED}, time change of cz_{ED} and f_{ED}, have the same form as that for acceleration, but replacing a_o with $-d_o$.

2.2.2. Observer is approaching with acceleration/deceleration

Case (1): Observer is approaching with acceleration. The observer moves a distance D_o,

$$D_o = v_o\left(\frac{\lambda_o}{c}\right) + \frac{1}{2}a_o\left(\frac{\lambda_o}{c}\right)^2. \tag{25}$$

We obtain,

$$\lambda_o = \lambda_s - D_o = \frac{\lambda_s}{1+\left(\frac{v_o}{c}\right)+\frac{a_o\lambda_o}{2\,c^2}}, \tag{26}$$

$$f_o = f_s\left[1 + \left(\frac{v_o}{c}\right) + \frac{a_o}{2}\frac{\lambda_o}{c^2}\right], \tag{27}$$

$$f_{ED} = f_s\left[\left(\frac{v_o}{c}\right) + \frac{a_o}{2}\frac{\lambda_o}{c^2}\right]. \tag{28}$$

$$cz_{ED} = -\frac{v_o + \frac{a_o\lambda_o}{2c}}{1+\left(\frac{v_o}{c}\right)+\frac{a_o\lambda_o}{2\,c^2}}. \tag{29}$$

Time Change. Taking time derivative, we obtain,

$$\frac{c}{f_s}\frac{df_{ED}}{dt} = a_o. \tag{30}$$

$$c\frac{dz_{ED}}{dt} = -\frac{a_o}{\left[1+\left(\frac{v_o}{c}\right)+\frac{a_o\lambda_o}{2\,c^2}\right]^2}. \tag{31}$$

Approximation. For slow moving observer, keeping first order terms and taking $\frac{\lambda_o}{c} \approx \frac{\lambda_s}{c}$, we obtain

$$\lambda_o \approx \lambda_s\left[1 - \left(\frac{v_o}{c}\right) - \frac{a_o}{2}\frac{\lambda_s}{c^2}\right], \tag{26a}$$

$$f_o \approx f_s\left[1 + \left(\frac{v_o}{c}\right) + \frac{a_o}{2}\frac{\lambda_s}{c^2}\right], \tag{27a}$$

$$f_{ED} \approx f_s\left[\left(\frac{v_o}{c}\right) + \frac{a_o}{2}\frac{\lambda_s}{c^2}\right]. \tag{28a}$$

$$cz_{ED} \approx -v_o - \frac{a_o\lambda_s}{2c}. \tag{29a}$$

$$c\frac{dz_{ED}}{dt} \approx -a_o. \tag{31a}$$

Case (2): Observer is approaching with Deceleration. For this case, the formulas of observed wavelength λ_o, Extended Doppler shift cz_{ED}, frequency f_o, Extended Doppler frequency f_{ED}, time change of cz_{ED} and f_{ED}, have the same form as that for acceleration, but replacing a_o with $-d_o$.

2.3. Source is moving, Observer is moving

In this section, we will show that, for slow motion, the effects of the movement of both a source and an observer on the shifts of wavelength/frequency are the same. Namely, the shift of wavelength/ frequency is dependent on the relative speed and relative acceleration between the source and observer.

2.3.1. Source and Observer are receding with acceleration/deceleration

There are four cases.

Case (1): Source and Observer are receding with acceleration. First, the movement of a source shifts the wavelength/frequency. Then the shifted wavelength/frequency is shifted further due to the movement of the observer. Following above arguments and notations, the moving distances are, respectively,

$$D_s = v_s \left(\frac{\lambda_s}{c}\right) + \frac{1}{2} a_s \left(\frac{\lambda_s}{c}\right)^2, \tag{32a}$$

$$D_o = v_o \left(\frac{\lambda_o}{c}\right) + \frac{1}{2} a_o \left(\frac{\lambda_o}{c}\right)^2. \tag{32b}$$

Let's consider separately: the wavelength shifted by the movement of the source is $\lambda_1 = \lambda_s + D_s$. To the observer, the incident wavelength is λ_1 and $\lambda_o = \lambda_1 + D_o$. We obtain

$$\lambda_o = \frac{1+\frac{v_s}{c}+\frac{a_s\lambda_s}{2c^2}}{1-\frac{v_o}{c}-\frac{a_o\lambda_o}{2c^2}} \lambda_s, \tag{33}$$

$$f_o = \frac{1-\frac{v_o}{c}-\frac{a_o\lambda_o}{2c^2}}{1+\frac{v_s}{c}+\frac{a_s\lambda_s}{2c^2}} f_s. \tag{34}$$

$$f_{ED} = -\frac{(v_o+v_s)/c+(a_o\lambda_o+a_s\lambda_s)/2c^2}{1+\frac{v_s}{c}+\frac{a_s\lambda_s}{2c^2}} f_s. \tag{35}$$

$$cz_{ED} = \frac{v_s+v_o+\frac{1}{2c}[a_s\lambda_s+a_o\lambda_o]}{1-\frac{v_o}{c}-\frac{a_o\lambda_o}{2c^2}}. \tag{36}$$

Time Change. Taking time derivative, we obtain,

$$\frac{c}{f_s}\frac{df_{ED}}{dt} = \frac{-a_o}{1+\frac{v_s}{c}+\frac{a_s\lambda_s}{2c^2}} - \frac{\left[1-\frac{v_o}{c}-\frac{a_o\lambda_o}{2c^2}\right]a_s}{\left[1+\frac{v_s}{c}+\frac{a_s\lambda_s}{2c^2}\right]^2}, \tag{37}$$

$$c\frac{dz_{ED}}{dt} = \frac{a_s+a_o}{1-\frac{v_o}{c}-\frac{a_o\lambda_o}{2c^2}} + \frac{\left[v_s+v_o+\frac{1}{2c}(a_s\lambda_s+a_o\lambda_o)\right]}{\left[1-\frac{v_o}{c}-\frac{a_o\lambda_o}{2c^2}\right]^2}\frac{a_o}{c}. \tag{38}$$

Approximation. For slow motion, keeping the first order terms and taking $\lambda_o/c \approx \lambda_s/c$, we obtain

$$\lambda_o \approx \left[1+\frac{v_s+v_o}{c}+\frac{a_s+a_o}{2c^2}\lambda_s\right]\lambda_s, \tag{33a}$$

$$f_o \approx \left[1-\frac{v_s+v_o}{c}-\frac{a_s+a_o}{2c^2}\lambda_s\right]f_s, \tag{34a}$$

$$f_{ED} \approx -\left[\frac{v_o+v_s}{c}+\frac{a_o+a_s}{2c^2}\lambda_s\right]f_s, \tag{35a}$$

$$cz_{ED} \approx v_s + v_o + \frac{1}{2c}[a_s+a_o]\lambda_s. \tag{36a}$$

$$\frac{c}{f_s}\frac{df_{ED}}{dt} \approx -(a_s+a_o), \tag{37a}$$

$$c\frac{dz_{ED}}{dt} \approx a_s + a_o. \tag{38a}$$

Eq. (33a-38a) implies that for slow motion, the effects of the movements of a source and an observer on wavelength/frequency are equivalent.

Case (2): Source is receding with acceleration, Observer is receding with decelerating. For this case, the formulas of observed wavelength λ_o, Extended Doppler shift cz_{ED}, frequency f_o, Extended Doppler frequency f_{ED}, time change of cz_{ED} and f_{ED}, have the same form as that for Case (1), but replacing a_o with $-d_o$.

Case (3): Source is receding with deceleration, Observer is receding with acceleration. For this case, the formulas of observed wavelength λ_o, Extended Doppler shift cz_{ED}, frequency f_o, Extended Doppler frequency f_{ED}, time change of cz_{ED} and f_{ED}, have the same form as that for Case (1), but replacing a_s with $-d_s$.

Case (4): Source and Observer are receding with deceleration. For this case, the formulas of observed wavelength λ_o, Extended Doppler shift cz_{ED}, frequency f_o, Extended Doppler frequency f_{ED}, time change of cz_{ED} and f_{ED}, have the same form as that for Case (1), but replacing a_s and a_o with $-d_s$ and $-d_o$.

2.3.2. Source is receding with acceleration/deceleration, Observer is approaching with acceleration/deceleration

There are four cases.

Case (1): Source is receding with acceleration, Observer is approaching with acceleration.

For this case, the moving distances of the source and observer are respectively,

$$D_s = v_s\left(\frac{\lambda_s}{c}\right) + \frac{1}{2}a_s\left(\frac{\lambda_s}{c}\right)^2, \tag{39a}$$

$$D_o = v_o\left(\frac{\lambda_o}{c}\right) + \frac{1}{2}a_o\left(\frac{\lambda_o}{c}\right)^2. \tag{39b}$$

The wavelength is $\lambda_o = \lambda_s + D_s - D_o$. We obtain,

$$\lambda_o = \frac{1+\frac{v_s}{c}+\frac{a_s\lambda_s}{2c^2}}{1+\frac{v_o}{c}+\frac{a_o\lambda_o}{2c^2}}\lambda_s, \tag{40}$$

$$f_o = \frac{1+\frac{v_o}{c}+\frac{a_o\lambda_o}{2c^2}}{1+\frac{v_s}{c}+\frac{a_s\lambda_s}{2c^2}} f_s. \tag{41}$$

$$f_{ED} = f_s\frac{\left(\frac{v_o}{c}-\frac{v_s}{c}\right)+\left(\frac{a_o\lambda_o}{2c^2}-\frac{a_s\lambda_s}{2c^2}\right)}{1+\frac{v_s}{c}+\frac{a_s\lambda_s}{2c^2}}. \tag{42}$$

$$cz_{ED} = \frac{v_s - v_o + \frac{1}{2c}[a_s\lambda_s - a_o\lambda_o]}{1+\frac{v_o}{c}+\frac{a_o\lambda_o}{2c^2}}. \tag{43}$$

Time Change. Taking derivative with respect to time, we obtain,

$$\frac{c}{f_s}\frac{df_{ED}}{dt} = \frac{a_o}{1+\frac{v_s}{c}+\frac{a_s\lambda_s}{2c^2}} - \frac{a_s\left[1+\frac{v_o}{c}+\frac{a_o\lambda_o}{2c^2}\right]}{\left[1+\frac{v_s}{c}+\frac{a_s\lambda_s}{2c^2}\right]^2}, \tag{44}$$

$$c\frac{dz_{ED}}{dt} = \frac{a_s - a_o}{1+\frac{v_o}{c}+\frac{a_o\lambda_o}{2c^2}} - \frac{\left[v_s - v_o + \frac{1}{2c}(a_s\lambda_s - a_o\lambda_o)\right]}{c\left[1+\frac{v_o}{c}+\frac{a_o\lambda_o}{2c^2}\right]^2} a_o. \tag{45}$$

<u>Approximation</u>. For slow motion, keeping the first order terms and taking $\lambda_o/c \approx \lambda_s/c$, we obtain

$$\lambda_o \approx \left[1 + \frac{v_s - v_o}{c} + \frac{a_s - a_o}{2c^2}\lambda_s\right]\lambda_s, \tag{40a}$$

$$f_o \approx \left[1 - \frac{v_s - v_o}{c} - \frac{a_s - a_o}{2c^2}\lambda_s\right] f_s. \tag{41a}$$

$$f_{ED} \approx -\frac{f_s}{c}\left[(v_s - v_o) + \frac{\lambda_s}{2c}(a_s - a_o)\right]. \tag{42a}$$

$$cz_{ED} \approx v_s - v_o + \frac{\lambda_s}{2c}[a_s - a_o]. \tag{43a}$$

$$\frac{c}{f_s}\frac{df_{ED}}{dt} \approx -(a_s - a_o), \tag{44a}$$

$$c\frac{dz_{ED}}{dt} \approx a_s - a_o. \tag{45a}$$

<u>Case (2): Source is receding with acceleration, Observer is approaching with deceleration.</u>

For this case, the formulas of observed wavelength λ_o, Extended Doppler shift cz_{ED}, frequency f_o, Extended Doppler frequency f_{ED}, time change of cz_{ED} and f_{ED}, have the same form as that for Case (1), but replacing a_o with $-d_o$.

<u>Case (3): Source is receding with deceleration, Observer is approaching with acceleration.</u>

For this case, the formulas of observed wavelength λ_o, Extended Doppler shift cz_{ED}, frequency f_o, Extended Doppler frequency f_{ED}, time change of cz_{ED} and f_{ED}, have the same form as that for Case (1), but replacing a_s with $-d_s$.

<u>Case (4): Source is receding with deceleration, Observer is approaching with deceleration.</u>

For this case, the formulas of observed wavelength λ_o, Extended Doppler shift cz_{ED}, frequency f_o, Extended Doppler frequency f_{ED}, time change of cz_{ED} and f_{ED}, have the same form as that for Case (1), but replacing a_s and a_o with $-d_s$ and $-d_o$, respectively.

2.3.3. <u>Source is approaching with acceleration/deceleration, Observer is receding with acceleration/deceleration</u>

There are four cases.

<u>Case (1): Source is approaching with acceleration, Observer is receding with acceleration.</u>

Source's and observer's moving distances are, respectively,

$$D_s = v_s\left(\frac{\lambda_s}{c}\right) + \frac{1}{2}a_s\left(\frac{\lambda_s}{c}\right)^2, \tag{46a}$$

$$D_o = v_o\left(\frac{\lambda_o}{c}\right) + \frac{1}{2}a_o\left(\frac{\lambda_o}{c}\right)^2. \tag{46b}$$

The observed wavelength is changed, $\lambda_o = \lambda_s - D_s + D_o$. We obtain,

$$\lambda_o = \frac{1-\frac{v_s}{c}-\frac{a_s\lambda_s}{2c^2}}{1-\frac{v_o}{c}-\frac{a_o\lambda_o}{2c^2}}\lambda_s, \tag{47}$$

$$f_o = \frac{1-\frac{v_o}{c}-\frac{a_o\lambda_o}{2c^2}}{1-\frac{v_s}{c}-\frac{a_s\lambda_s}{2c^2}} f_s. \tag{48}$$

$$f_{ED} = -\frac{(v_o - v_s)/c + (a_o\lambda_o - a_s\lambda_s)/2c^2}{1-\frac{v_s}{c}-\frac{a_s\lambda_s}{2c^2}} f_s. \tag{49}$$

$$cz_{ED} = -\frac{v_s - v_o + \frac{1}{2c}[a_s\lambda_s - a_o\lambda_o]}{1-\frac{v_o}{c}-\frac{a_o\lambda_o}{2c^2}}. \tag{50}$$

Time Change. taking time derivative, we obtain,

$$\frac{c}{f_s}\frac{df_{ED}}{dt} = \frac{-a_o}{1-\frac{v_s}{c}-\frac{a_s\lambda_s}{2c^2}} + \frac{\left[1-\frac{v_o}{c}-\frac{a_o\lambda_o}{2c^2}\right]a_s}{\left[1-\frac{v_s}{c}-\frac{a_s\lambda_s}{2c^2}\right]^2}, \tag{51}$$

$$c\frac{dz_{ED}}{dt} = -\frac{a_s - a_o}{1-\frac{v_o}{c}-\frac{a_o\lambda_o}{2c^2}} - \frac{\left[v_s - v_o + \frac{1}{2c}(a_s\lambda_s - a_o\lambda_o)\right]}{\left[1-\frac{v_o}{c}-\frac{a_o\lambda_o}{2c^2}\right]^2}\frac{a_o}{c}. \tag{52}$$

Approximation. For slow motion and acceleration, keeping the first order terms and taking $\lambda_o/c \approx \lambda_s/c$, we obtain

$$\lambda_o \approx \left[1 - \frac{v_s - v_o}{c} - \frac{a_s - a_o}{2c^2}\lambda_s\right]\lambda_s, \tag{47a}$$

$$f_o \approx \left[1 + \frac{v_s - v_o}{c} + \frac{a_s - a_o}{2c^2}\lambda_s\right]f_s, \tag{48a}$$

$$f_{ED} \approx -\left[\frac{v_o - v_s}{c} + \frac{a_o - a_s}{2c^2}\lambda_s\right]f_s, \tag{49a}$$

$$cz_{ED} \approx v_o - v_s + \frac{1}{2c}[a_o - a_s]\lambda_s. \tag{50a}$$

$$\frac{c}{f_s}\frac{df_{ED}}{dt} \approx -(a_o - a_s), \tag{51a}$$

$$c\frac{dz_{ED}}{dt} \approx a_o - a_s. \tag{52a}$$

Case (2): Source is approaching with acceleration, Observer is receding with deceleration.

For this case, the formulas of observed wavelength λ_o, Extended Doppler shift cz_{ED}, frequency f_o, Extended Doppler frequency f_{ED}, time change of cz_{ED} and f_{ED}, have the same form as that for Case (1), but replacing a_o with $-d_o$.

Case (3): Source is approaching with deceleration, Observer is receding with acceleration.

For this case, the formulas of observed wavelength λ_o, Extended Doppler shift cz_{ED}, frequency f_o, Extended Doppler frequency f_{ED}, time change of cz_{ED} and f_{ED}, have the same form as that for Case (1), but replacing a_s with $-d_s$.

Case (4): Source is approaching with deceleration, Observer is receding with deceleration.

For this case, the formulas of observed wavelength λ_o, Extended Doppler shift cz_{ED}, frequency f_o, Extended Doppler frequency f_{ED}, time change of cz_{ED} and f_{ED}, have the same form as that for Case (1), but replacing a_s and a_o with $-d_s$ and $-d_o$, respectively.

2.3.4. Source and observer are approaching with acceleration/deceleration

There are four cases.

Case (1): Both approaching with acceleration. Source's and observer's moving distances are, respectively,

$$D_s = v_s\left(\frac{\lambda_s}{c}\right) + \frac{1}{2}a_s\left(\frac{\lambda_s}{c}\right)^2, \tag{53a}$$

$$D_o = v_o\left(\frac{\lambda_o}{c}\right) + \frac{1}{2}a_o\left(\frac{\lambda_o}{c}\right)^2. \tag{54b}$$

The wavelength is changed, $\lambda_o = \lambda_s - D_s - D_o$. We obtain,

$$\lambda_o = \frac{1-\frac{v_s}{c}-\frac{a_s\lambda_s}{2c^2}}{1+\frac{v_o}{c}+\frac{a_o\lambda_o}{2c^2}}\lambda_s, \tag{55}$$

$$f_o = \frac{1+\frac{v_o}{c}+\frac{a_o\lambda_o}{2c^2}}{1-\frac{v_s}{c}-\frac{a_s\lambda_s}{2c^2}}f_s, \tag{56}$$

$$cf_{ED} = \frac{(v_o+v_s)+(a_o\lambda_o+a_s\lambda_s)/(2c)}{1-\frac{v_s}{c}-\frac{a_s\lambda_s}{2c^2}}f_s, \tag{57}$$

$$cz_{ED} = -\frac{v_s+v_o+\frac{1}{2c}[a_s\lambda_s+a_o\lambda_o]}{1+\frac{v_o}{c}+\frac{a_o\lambda_o}{2c^2}}, \tag{58}$$

$$\frac{c}{f_s}\frac{df_{ED}}{dt} = \frac{a_o}{1-\frac{v_s}{c}-\frac{a_s\lambda_s}{2c^2}} + \frac{\left[1+\frac{v_o}{c}+\frac{a_o\lambda_o}{2c^2}\right]a_s}{\left[1-\frac{v_s}{c}-\frac{a_s\lambda_s}{2c^2}\right]^2}, \tag{59}$$

$$c\frac{dz_{ED}}{dt} = -\frac{a_s+a_o}{1+\frac{v_o}{c}+\frac{a_o\lambda_o}{2c^2}} - \frac{\left[v_s+v_o+\frac{1}{2c}(a_s\lambda_s+a_o\lambda_o)\right]}{\left[1+\frac{v_o}{c}+\frac{a_o\lambda_o}{2c^2}\right]^2}\frac{a_o}{c}, \tag{60}$$

$$\lambda_o \approx \left[1-\frac{v_s+v_o}{c}-\frac{a_s+a_o}{2c^2}\lambda_s\right]\lambda_s, \tag{55a}$$

$$f_o \approx \left[1+\frac{v_s+v_o}{c}+\frac{a_s+a_o}{2c^2}\lambda_s\right]f_s, \tag{56a}$$

$$f_{ED} \approx \left[\frac{v_o+v_s}{c}+\frac{a_o+a_s}{2c^2}\lambda_s\right]f_s, \tag{57a}$$

$$cz_{ED} \approx -(v_o+v_s)-\frac{1}{2c}[a_s+a_o]\lambda_s. \tag{58a}$$

$$\frac{c}{f_s}\frac{df_{ED}}{dt} \approx (a_o+a_s), \tag{59a}$$

$$c\frac{dz_{ED}}{dt} \approx -(a_o+a_s). \tag{60a}$$

Case (2): Source is approaching with acceleration, Observer is approaching with deceleration

For this case, the formulas of observed wavelength λ_o, Extended Doppler shift cz_{ED}, frequency f_o, Extended Doppler frequency f_{ED}, time change of cz_{ED} and f_{ED}, have the same form as that for Case (1), but replacing a_o with $-d_o$.

Case (3): Source is approaching with deceleration, Observer is approaching with acceleration.

For this case, the formulas of observed wavelength λ_o, Extended Doppler shift cz_{ED}, frequency f_o, Extended Doppler frequency f_{ED}, time change of cz_{ED} and f_{ED}, have the same form as that for Case

(1), but replacing a_s with $-d_s$.

Case (4): Source is approaching with deceleration, Observer is approaching with deceleration.

For this case, the formulas of observed wavelength λ_o, Extended Doppler shift cz_{ED}, frequency f_o, Extended Doppler frequency f_{ED}, time change of cz_{ED} and f_{ED}, have the same form as that for Case (1), but replacing a_s and a_o with $-d_s$ and $-d_o$, respectively.

3. Doppler Effect of Light from Non-uniformly Moving Mirror

By employing the Lorentz transformation to convert from a reference frame where the mirror is stationary to the reference frame where the mirror is in uniform motion, Einstein derived the Doppler formula from a uniformly moving mirror. However Lorentz transformation cannot be applied to a non-uniformly moving frame. We will derive the extended Doppler formula for a non-uniformly moving mirror for a special case that an observer and a source are at the same location and are at rest to each other (we call it the source/observer), and that the moving directions of source/observer and the mirror are along the same axis. We assume that, during the reflection, the wavelength is the same.

Notations: Subscripts "OS" and "M" denotes the quantities of source/observer and mirror respectively.

3.1. Source/Observer is moving, mirror is stationary

3.1.1. Source/Observer is receding with acceleration/deceleration

There are two cases.

Case (1): Source/Observer is receding with acceleration. We separate the process into two phases.

First Phase.

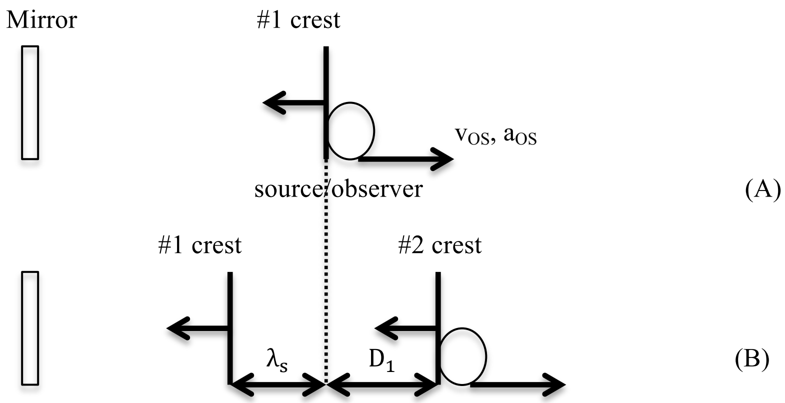

Fig. 1

Consider wavelength before reflected by mirror (Fig. 1 (A) and (B)). The source/observer recedes from the mirror with velocity v_{OS} and acceleration a_{OS} and emits wave of wavelength λ_s.

This case is equivalent to the case that a source is receding from an observer. Now the mirror plays the role of "observer". The source/observer moves a distance during the emission,

$$\mathrm{D}_1 = \mathrm{v}_{\mathrm{OS}}\frac{\lambda_s}{c} + \frac{1}{2}\mathrm{a}_{\mathrm{OS}}\left(\frac{\lambda_s}{c}\right)^2. \tag{61a}$$

The wavelength arrived at the mirror is $\lambda_1 = \lambda_s + \mathrm{D}_1 = \lambda_s + \mathrm{v}_{\mathrm{OS}}\frac{\lambda_s}{c} + \frac{1}{2}\mathrm{a}_{\mathrm{OS}}\left(\frac{\lambda_s}{c}\right)^2$. We assume that during the reflection, the wavelengths keep the same before and after.

Second Phase. Consider wavelength after reflecting by mirror. This case is equivalent to the case an observer is receding from a source where the mirror plays the role of "source" that "emitting" the wave of wavelength λ_1 (Fig. 1(C). When crest #2 reaches the source/observer, the source/observer moves a distance D_2 (Fig. 1(D),

$$\mathrm{D}_2 = \mathrm{v}_{\mathrm{OS}}\frac{\lambda_o}{c} + \frac{1}{2}\mathrm{a}_{\mathrm{OS}}\left(\frac{\lambda_o}{c}\right)^2. \tag{61b}$$

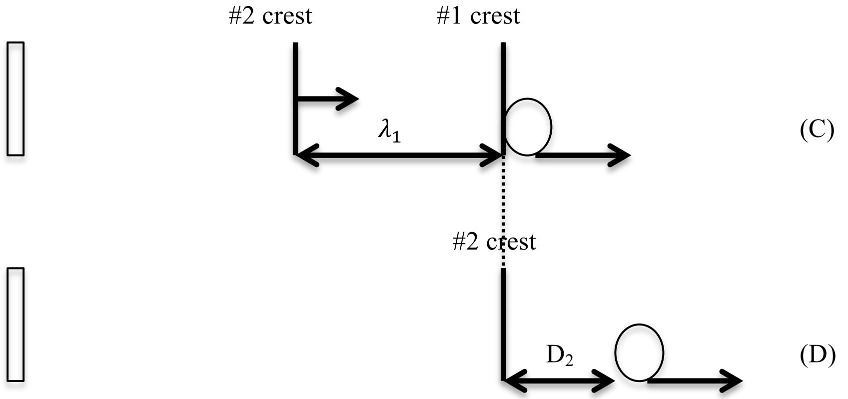

Fig. 1

The received wavelength is $\lambda_o = \lambda_1 + \mathrm{D}_2$, we obtain,

$$\lambda_o = \lambda_s \frac{1+\frac{\mathrm{v}_{\mathrm{OS}}}{c}+\frac{\mathrm{a}_{\mathrm{OS}}\lambda_s}{2c^2}}{1-\frac{\mathrm{v}_{\mathrm{OS}}}{c}-\frac{\mathrm{a}_{\mathrm{OS}}\lambda_o}{2c^2}}. \tag{62}$$

$$\mathrm{f}_o = \mathrm{f}_s \frac{1-\frac{\mathrm{v}_{\mathrm{OS}}}{c}-\frac{\mathrm{a}_{\mathrm{OS}}\lambda_o}{2c^2}}{1+\frac{\mathrm{v}_{\mathrm{OS}}}{c}+\frac{\mathrm{a}_{\mathrm{OS}}\lambda_s}{2c^2}}, \tag{63}$$

$$c\mathrm{f}_{\mathrm{ED}} = -\left\{\frac{2\mathrm{v}_{\mathrm{OS}}+\frac{\mathrm{a}_{\mathrm{OS}}(\lambda_o+\lambda_s)}{2c}}{1+\frac{\mathrm{v}_{\mathrm{OS}}}{c}+\frac{\mathrm{a}_{\mathrm{OS}}\lambda_s}{2c^2}}\right\}\mathrm{f}_s, \tag{64}$$

$$c\mathrm{z}_{\mathrm{ED}} = \frac{2\mathrm{v}_{\mathrm{OS}}+\frac{\mathrm{a}_{\mathrm{OS}}(\lambda_s+\lambda_o)}{2c}}{1-\frac{\mathrm{v}_{\mathrm{OS}}}{c}-\frac{\mathrm{a}_{\mathrm{OS}}\lambda_o}{2c^2}}, \tag{65}$$

Time change. Taking time derivative, we obtain,

$$\frac{c}{\mathrm{f}_s}\frac{d\mathrm{f}_{\mathrm{ED}}}{dt} = \frac{-\mathrm{a}_{\mathrm{OS}}}{1+\frac{\mathrm{v}_{\mathrm{OS}}}{c}+\frac{\mathrm{a}_{\mathrm{OS}}\lambda_s}{2c^2}} - \frac{\left[1-\frac{\mathrm{v}_{\mathrm{OS}}}{c}-\frac{\mathrm{a}_{\mathrm{OS}}\lambda_o}{2c^2}\right]\mathrm{a}_{\mathrm{OS}}}{\left[1+\frac{\mathrm{v}_{\mathrm{OS}}}{c}+\frac{\mathrm{a}_{\mathrm{OS}}\lambda_s}{2c^2}\right]^2}, \tag{66}$$

$$c\frac{d\mathrm{z}_{\mathrm{ED}}}{dt} = \frac{2\mathrm{a}_{\mathrm{OS}}}{1-\frac{\mathrm{v}_{\mathrm{OS}}}{c}-\frac{\mathrm{a}_{\mathrm{OS}}\lambda_o}{2c^2}} - \frac{\left[2\mathrm{v}_{\mathrm{OS}}+\frac{\mathrm{a}_{\mathrm{OS}}(\lambda_s+\lambda_o)}{2c}\right]\left[-\frac{\mathrm{a}_{\mathrm{OS}}}{c}\right]}{\left[1-\frac{\mathrm{v}_{\mathrm{OS}}}{c}-\frac{\mathrm{a}_{\mathrm{OS}}\lambda_o}{2c^2}\right]^2}, \tag{67}$$

Approximation. for slow moving source/observer, keeping first order terms only and taking $\frac{\lambda_s}{c} \approx \frac{\lambda_2}{c}$, we obtain the received wavelength and frequency,

$$\lambda_o \approx \lambda_s \left[1 + 2\frac{v_{OS}}{c} + \frac{a_{OS}}{c^2}\lambda_s\right], \tag{62a}$$

$$f_o \approx f_s \left[1 - 2\frac{v_{OS}}{c} - \frac{a_{OS}}{c^2}\lambda_s\right], \tag{63a}$$

$$\frac{c}{f_s} f_{ED} \approx -\left[2v_{OS} + a_{OS}\frac{\lambda_s}{c}\right], \tag{64a}$$

$$cz_{ED} \approx 2v_{OS} + \frac{a_{OS}\lambda_s}{c}, \tag{65a}$$

$$\frac{c}{f_s}\frac{df_{ED}}{dt} \approx -2a_{OS}, \tag{66a}$$

$$c\frac{dz_{ED}}{dt} \approx 2a_{OS}. \tag{67a}$$

Case (2): Source/Observer receding with deceleration. For this case, the formulas of observed wavelength λ_o, Extended Doppler shift cz_{ED}, frequency f_o, Extended Doppler frequency f_{ED}, time change of cz_{ED} and f_{ED}, have the same form as that for Case (1), but replacing a_{os} with $-d_{os}$.

3.1.2. Source/Observer is approaching with acceleration/deceleration

There are two cases.

Case (1): Source/Observer is approaching with acceleration. We separate into two phases.

First Phase. The source/observer moves a distance during the emission,

$$D_1 = v_{OS}\frac{\lambda_s}{c} + \frac{1}{2}a_{os}\left(\frac{\lambda_s}{c}\right)^2. \tag{68a}$$

The wavelength arrived at the mirror is $\lambda_1 = \lambda_s - D_1$.

Second Phase. When crest #2 reaches the source/observer, the source/observer moves a distance D_2,

$$D_2 = v_{OS}\frac{\lambda_o}{c} + \frac{1}{2}a_{os}\left(\frac{\lambda_o}{c}\right)^2, \tag{68b}$$

and $\lambda_o = \lambda_1 - D_2$. We obtain,

$$\lambda_o = \lambda_s \frac{1 - \frac{v_{OS}}{c} - \frac{a_{OS}\lambda_s}{2c^2}}{1 + \frac{v_{OS}}{c} + \frac{a_{OS}\lambda_o}{2c^2}}. \tag{69}$$

$$f_o = f_s \frac{1 + \frac{v_{OS}}{c} + \frac{a_{OS}\lambda_o}{2c^2}}{1 - \frac{v_{OS}}{c} - \frac{a_{OS}\lambda_s}{2c^2}}, \tag{70}$$

$$\frac{c}{f_s} f_{ED} = \frac{2v_{OS} + \frac{a_{OS}(\lambda_o + \lambda_s)}{2c}}{1 - \frac{v_{OS}}{c} - \frac{a_{OS}\lambda_s}{2c^2}}, \tag{71}$$

$$cz_{ED} = -\frac{2v_{OS} + \frac{a_{OS}(\lambda_s + \lambda_o)}{2c}}{1 + \frac{v_{OS}}{c} + \frac{a_{OS}\lambda_o}{2c^2}}, \tag{72}$$

$$\frac{df_{ED}}{dt} = f_s\left\{\frac{\frac{a_{OS}}{c}}{1 - \frac{v_{OS}}{c} - \frac{a_{OS}\lambda_s}{2c^2}} + \frac{\left[1 + \frac{v_{OS}}{c} + \frac{a_{OS}\lambda_o}{2c^2}\right]\frac{a_{OS}}{c}}{\left[1 - \frac{v_{OS}}{c} - \frac{a_{OS}\lambda_s}{2c^2}\right]^2}\right\}, \tag{73}$$

$$c\frac{dz_{ED}}{dt} = -\left\{\frac{2a_{OS}}{1 + \frac{v_{OS}}{c} + \frac{a_{OS}\lambda_o}{2c^2}} - \frac{\left[2v_{OS} + \frac{a_{OS}(\lambda_s + \lambda_o)}{2c}\right]\left[\frac{a_{OS}}{c}\right]}{\left[1 + \frac{v_{OS}}{c} + \frac{a_{OS}\lambda_o}{2c^2}\right]^2}\right\}, \tag{74}$$

$\lambda_o \approx \lambda_s \left[1 - 2\frac{v_{OS}}{c} - \frac{a_{OS}}{c^2}\lambda_s\right],$ (69a)

$f_o \approx f_s \left[1 + 2\frac{v_{OS}}{c} + \frac{a_{OS}}{c^2}\lambda_s\right],$ (70a)

$\frac{c}{f_s} f_{ED} \approx 2v_{OS} + a_{OS}\frac{\lambda_s}{c},$ (71a)

$cz_{ED} \approx -\left[2\frac{v_{OS}}{c} + \frac{a_{OS}\lambda_s}{c^2}\right],$ (72a)

$\frac{c}{f_s}\frac{df_{ED}}{dt} \approx 2a_{OS},$ (73a)

$c\frac{dz_{ED}}{dt} \approx -2a_{OS}.$ (74a)

Case (2): Source/Observer is approaching with deceleration. For this case, the formulas of observed wavelength λ_o, Extended Doppler shift cz_{ED}, frequency f_o, Extended Doppler frequency f_{ED}, time change of cz_{ED} and f_{ED}, have the same form as that for Case (1), but replacing a_{os} with $-d_{os}$.

3.2. Mirror is moving, Source/Observer is stationary

3.2.1. Mirror is receding with acceleration/deceleration

There are two cases. A mirror is moving at a velocity "v_M" and acceleration "a_M" to the right (Fig. 2).

Case (**1): Mirror receding with acceleration**

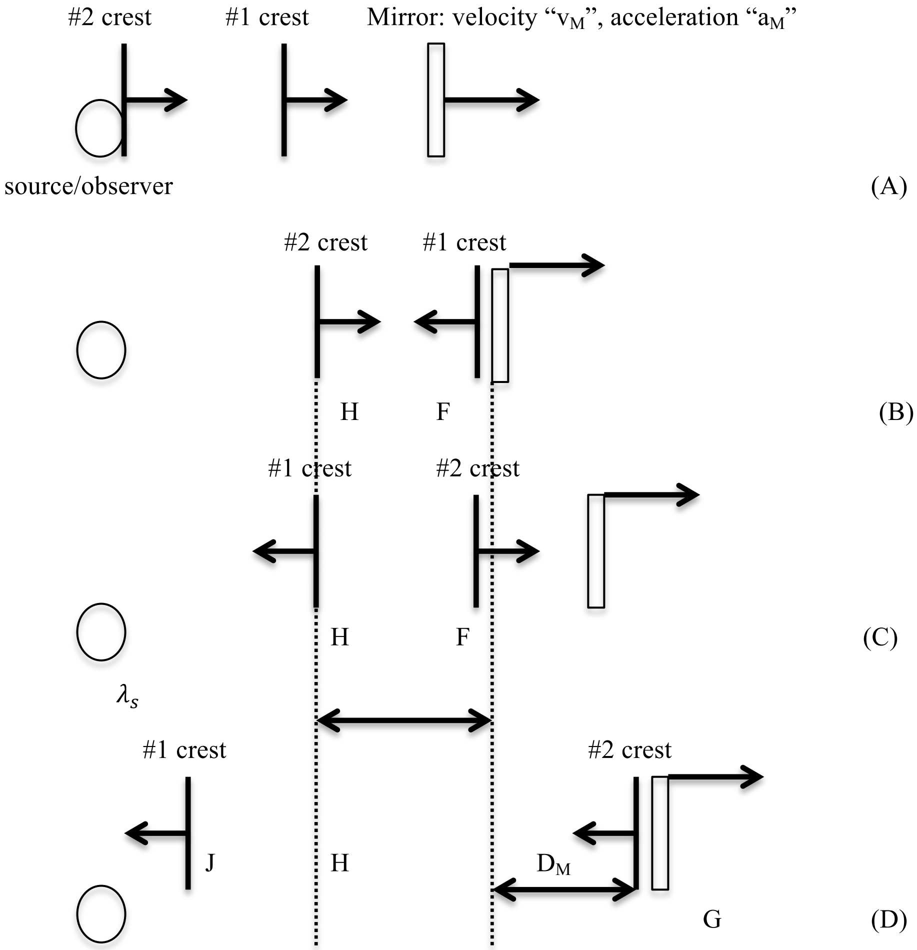

Fig. 2

The source emits a light wave incident on the mirror that reflects light. Wave crest #1 and #2 are emitted towards to the mirror (Fig. 2(A). Crest #1 reaches the mirror at position F and reflected back to source/observer while crest #2 at position H as shown in Fig. 2(B). The distance HF is equal to the wavelength λ_s. For this simplest situation, we assume that the angles of incidence and reflection of the light is equal to each other. Fig. 2(C) shows that when crest #2 reaches the position F, crest #1 reaches to the position H, and the mirror moves away from the position F. In Fig. 2(D) crest #2 moves an additional distance to reach the mirror at the position G and reflected back towards to the source/observer. Denote the distance FG ($=$ HJ) as D_M, then

$$D_M = v_M \Delta t_M + \frac{1}{2} a_M (\Delta t_M)^2 = v_M \frac{\lambda_o}{c} + \frac{1}{2} a_M \left(\frac{\lambda_o}{c}\right)^2 \quad (75)$$

and $\lambda_o = \lambda_s + 2D_M$. We obtain

$$\lambda_o = \frac{\lambda_s}{1 - 2\frac{v_M}{c} - \frac{a_M}{c^2}\lambda_o}, \quad (76)$$

$$f_o = f_s \left[1 - 2\frac{v_M}{c} - \frac{a_M}{c^2}\lambda_o\right], \quad (77)$$

$$f_{ED} = -f_s \left[2\frac{v_M}{c} + \frac{a_M}{c^2}\lambda_o\right], \quad (78)$$

$$cz_{ED} = \frac{2v_M + \frac{a_M}{c}\lambda_o}{1 - 2\frac{v_M}{c} - \frac{a_M}{c^2}\lambda_o}, \quad (79)$$

Time Derivative. Taking time derivative, we obtain,

$$\frac{c}{f_s}\frac{df_{ED}}{dt} = -2a_M, \quad (80)$$

$$c\frac{dz_{ED}}{dt} = \frac{2a_M}{1 - 2\frac{v_M}{c} - \frac{a_M}{c^2}\lambda_o} - \frac{\left[2v_M + \frac{a_M}{c}\lambda_o\right]\left[-2\frac{a_M}{c}\right]}{\left[1 - 2\frac{v_M}{c} - \frac{a_M}{c^2}\lambda_o\right]^2}, \quad (81)$$

Approximation. For slow motion, keep the first order terms only and taking $\frac{\lambda_o}{c} \approx \frac{\lambda_s}{c}$, we obtain,

$$\lambda_o \approx \lambda_s \left[1 + 2\frac{v_M}{c} + \frac{a_M}{c^2}\lambda_s\right], \quad (76a)$$

$$f_o \approx f_s \left[1 - 2\frac{v_M}{c} - \frac{a_M}{c^2}\lambda_s\right], \quad (77a)$$

$$f_{ED} \approx -f_s \left[2\frac{v_M}{c} + \frac{a_M}{c^2}\lambda_s\right], \quad (78a)$$

$$cz_{ED} \approx 2v_M + \frac{a_M}{c}\lambda_s. \quad (79a)$$

$$c\frac{dz_{ED}}{dt} \approx 2a_M. \quad (81a)$$

Case (2): Mirror is receding with Deceleration. For this case, the formulas of observed wavelength λ_o, Extended Doppler shift cz_{ED}, frequency f_o, Extended Doppler frequency f_{ED}, time change of cz_{ED} and f_{ED}, have the same form as that for Case (1), but replacing a_M with $-d_M$.

3.2.2. Mirror is approaching with acceleration/deceleration

There are two cases.

Case (1): Mirror is approaching with acceleration. By simply replacing $\lambda_o = \lambda_s + 2D_M$ with $\lambda_o = \lambda_s - 2D_M$ in formulas of section 3.2.1, where

$$D_M = v_M \frac{\lambda_o}{c} + \frac{1}{2} a_M \left(\frac{\lambda_o}{c}\right)^2, \tag{82}$$

we obtain

$$\lambda_o = \frac{\lambda_s}{1+2\frac{v_M}{c}+\frac{a_M}{c^2}\lambda_o}, \tag{83}$$

$$f_o = f_s \left[1 + 2\frac{v_M}{c} + \frac{a_M}{c^2}\lambda_o\right], \tag{84}$$

$$f_{ED} = f_s \left[2\frac{v_M}{c} + \frac{a_M}{c^2}\lambda_o\right], \tag{85}$$

$$cz_{ED} = -\frac{2v_M+\frac{a_M}{c}\lambda_o}{1+2\frac{v_M}{c}+\frac{a_M}{c^2}\lambda_o}, \tag{86}$$

Time Derivative. Taking time derivative, we obtain,

$$\frac{c}{f_s}\frac{df_{ED}}{dt} = 2a_M, \tag{87}$$

$$c\frac{dz_{ED}}{dt} = \frac{-2a_M}{1+2\frac{v_M}{c}+\frac{a_M}{c^2}\lambda_o} + \frac{\left[2v_M+\frac{a_M}{c}\lambda_o\right]\left[2\frac{a_M}{c}\right]}{\left[1+2\frac{v_M}{c}+\frac{a_M}{c^2}\lambda_o\right]^2}, \tag{88}$$

Approximation. For slow motion, keep the first order terms only and taking $\lambda_o \approx \lambda_s$, we obtain,

$$\lambda_o \approx \lambda_s \left[1 - 2\frac{v_M}{c} - \frac{a_M}{c^2}\lambda_s\right], \tag{83a}$$

$$f_o \approx f_s \left[1 + 2\frac{v_M}{c} + \frac{a_M}{c^2}\lambda_s\right], \tag{84a}$$

$$f_{ED} \approx f_s \left[2\frac{v_M}{c} + \frac{a_M}{c^2}\lambda_s\right], \tag{85a}$$

$$cz_{ED} \approx -2v_M - \frac{a_M}{c}\lambda_s. \tag{86a}$$

$$c\frac{dz_{ED}}{dt} \approx -2a_M. \tag{88a}$$

Case (2): Mirror is approaching with deceleration. For this case, the formulas of observed wavelength λ_o, Extended Doppler shift cz_{ED}, frequency f_o, Extended Doppler frequency f_{ED}, time change of cz_{ED} and f_{ED}, have the same form as that for Case (1), but replacing a_M with $-d_M$.

3.3. Source/Observer is moving, Mirror is moving

The mirror moves with velocity v_M and acceleration/deceleration, a_M/d_M. The source/observer moves with velocity v_s and acceleration/deceleration, a_s/d_s. Let's set a coordinate system, in which both a mirror and a source/observer are accelerating separately.

3.3.1. Source/Observer and Mirror are receding with acceleration/deceleration

There are four cases. For each case, we separate the process into three phases.

Case (1): Source/Observer and Mirror receding with acceleration.

First phase:

The wavelength become λ_1,

$$\lambda_1 = \lambda_s + D_1 = \lambda_s + v_s\frac{\lambda_s}{c} + \frac{1}{2}a_s\left(\frac{\lambda_s}{c}\right)^2 = \lambda_s\left[1 + \frac{v_s}{c} + \frac{a_s\lambda_s}{2c^2}\right].$$

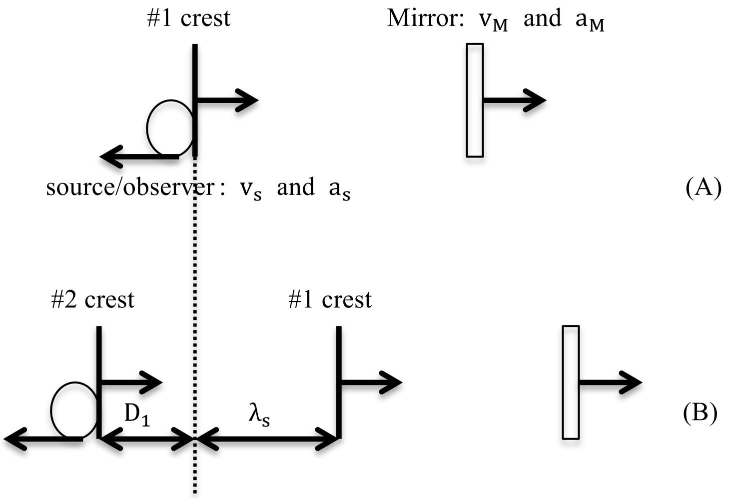

Fig. 3

<u>Second phase</u>: The incident wave of wavelength λ_1 reflected by moving mirror. In this phase the motion of the source/observer has no effect on the wavelength.

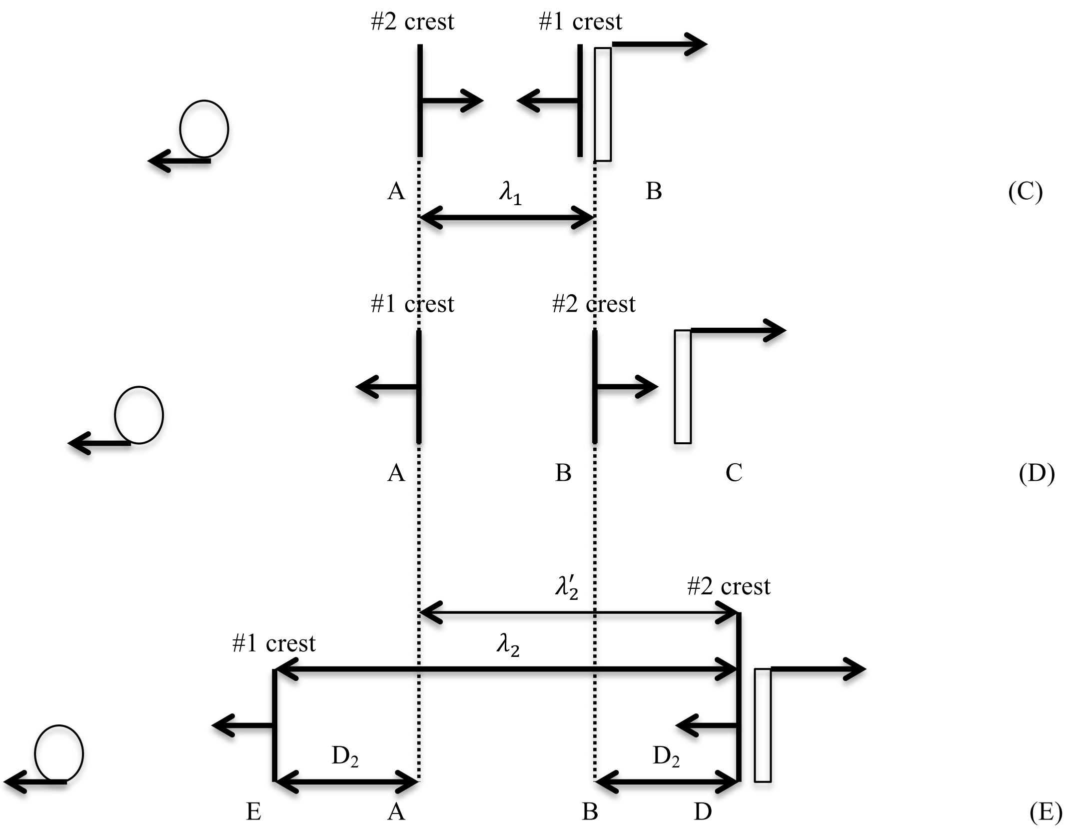

Fig. 3

Crest #1 reaches the mirror at point B and reflected back towards the source/observer while crest #2 reaches at the point A (Fig. 3(C)). When crest #2 reaches the point B, crest #1 reaches the point A, and the mirror moves to the point C (Fig. 3(D)). Crest #2 needs to move an extra distance D_2 to reaches the mirror at the point D and reflected back, while crest #1 moves to the point E (Fig. 3(E)), so $BD = EA = D_2$ and $\lambda_2' = \lambda_1 + D_2$. We have

$$D_2 = v_M \frac{\lambda_2'}{c} + \frac{1}{2} a_M \left(\frac{\lambda_2'}{c}\right)^2 \text{ and } \lambda_2' = \frac{\lambda_s\left[1+\frac{v_S}{c}+\frac{a_S\lambda_S}{2c^2}\right]}{1-\frac{v_M}{c}-\frac{1}{2}a_M\frac{\lambda_2'}{c^2}}.$$

The wavelength of the reflected wave is

$$\lambda_2 = \lambda_2' + D_2 = \frac{\lambda_s\left[1+\frac{v_S}{c}+\frac{a_S\lambda_S}{2c^2}\right]+\left[v_M\frac{\lambda_2'}{c}+\frac{1}{2}a_M\left(\frac{\lambda_2'}{c}\right)^2\right]\left[1-\frac{v_M}{c}-\frac{1}{2}a_M\frac{\lambda_2'}{c^2}\right]}{1-\frac{v_M}{c}-\frac{1}{2}a_M\frac{\lambda_2'}{c^2}}.$$

Third phase: The receding source/observer receives the reflected wave of wavelength λ_o. Fig. 3(F)) shows crest #1 reaches the source/observer at the point F. When crest #2 reaches the point F, crest #1 reaches the point K, and the source/observer has moved to the point H (Fig. 3(G)). Crest #2 reaches the source/observer at the point J, while crest #1 reaches the point L, while the observer has moved a distance $D_3 = FJ$,

$$D_3 = v_s\left(\frac{\lambda_o}{c}\right) + \frac{1}{2}a_s\left(\frac{\lambda_o}{c}\right)^2.$$

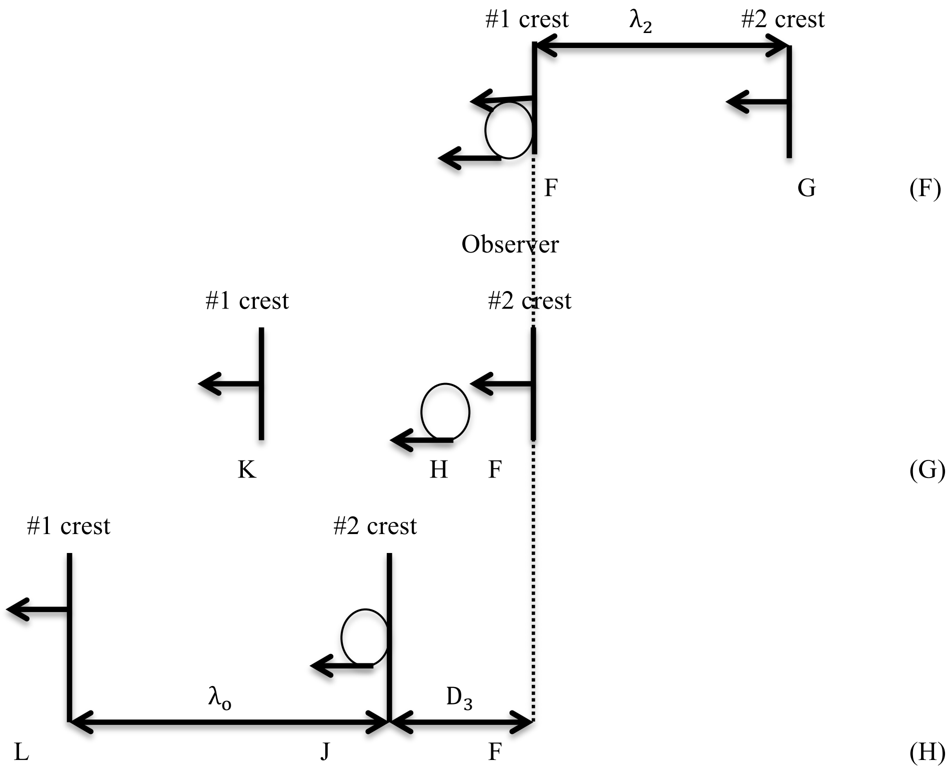

Fig. 3

Therefore the wavelength is stretched from λ_2 to λ_o, $\lambda_o = \lambda_2 + D_3$,

$$\lambda_o = \frac{\lambda_s\left[1+\frac{v_S}{c}+\frac{a_S\lambda_S}{2c^2}\right]+\left[v_M\frac{\lambda_2'}{c}+\frac{1}{2}a_M\left(\frac{\lambda_2'}{c}\right)^2\right]\left[1-\frac{v_M}{c}-\frac{1}{2}a_M\frac{\lambda_2'}{c^2}\right]}{\left[1-\frac{v_M}{c}-\frac{1}{2}a_M\frac{\lambda_2'}{c^2}\right]\left[1-\frac{v_S}{c}-\frac{a_S}{2c^2}\lambda_o\right]}, \tag{89}$$

$$z_{ED} = \frac{\lambda_s\left[1+\frac{v_S}{c}+\frac{a_S\lambda_S}{2c^2}\right]+\left[v_M\frac{\lambda_2'}{c}+\frac{1}{2}a_M\left(\frac{\lambda_2'}{c}\right)^2\right]\left[1-\frac{v_M}{c}-\frac{1}{2}a_M\frac{\lambda_2'}{c^2}\right]}{\lambda_s\left[1-\frac{v_M}{c}-\frac{1}{2}a_M\frac{\lambda_2'}{c^2}\right]\left[1-\frac{v_S}{c}-\frac{a_S}{2c^2}\lambda_o\right]} - 1, \tag{90}$$

$$f_o = \frac{\left[1-\frac{v_M}{c}-\frac{1}{2}a_M\frac{\lambda_2'}{c^2}\right]\left[1-\frac{v_s}{c}-\frac{a_s}{2c^2}\lambda_o\right]}{\lambda_s\left[1+\frac{v_s}{c}+\frac{a_s\lambda_s}{2c^2}\right]+\left[v_M\frac{\lambda_2'}{c}+\frac{1}{2}a_M\left(\frac{\lambda_2'}{c}\right)^2\right]\left[1-\frac{v_M}{c}-\frac{1}{2}a_M\frac{\lambda_2'}{c^2}\right]}, \quad (91)$$

$$f_{ED} = \frac{\left[1-\frac{v_M}{c}-\frac{1}{2}a_M\frac{\lambda_2'}{c^2}\right]\left[1-\frac{v_s}{c}-\frac{a_s}{2c^2}\lambda_o\right]}{\lambda_s\left[1+\frac{v_s}{c}+\frac{a_s\lambda_s}{2c^2}\right]+\left[v_M\frac{\lambda_2'}{c}+\frac{1}{2}a_M\left(\frac{\lambda_2'}{c}\right)^2\right]\left[1-\frac{v_M}{c}-\frac{1}{2}a_M\frac{\lambda_2'}{c^2}\right]} - f_s, \quad (92)$$

Time Derivative. The expressions of time derivatives of above parameters are complicated and not necessary for slow moving source/observer and mirror. Thus we only take time derivative of the approximation of those parameters as shown below.

Approximation. For slow motion, we can keep first order terms and take $\frac{\lambda_s}{c} \approx \frac{\lambda_2}{c} \approx \frac{\lambda_o}{c} \approx \frac{\lambda_2'}{c}$, then obtain,

$$\lambda_o \approx \lambda_s\left[1 + 2\frac{v_s+v_M}{c} + \frac{a_s+a_M}{c^2}\lambda_s\right], \quad (89a)$$

$$cz_{ED} \approx 2(v_s + v_M) + (a_M + a_s)\frac{\lambda_s}{c}, \quad (90a)$$

$$f_o \approx f_s\left[1 - 2\frac{v_M+v_s}{c} - (a_M + a_s)\frac{\lambda_s}{c^2}\right], \quad (91a)$$

$$f_{ED} \approx -f_s\left\{2\frac{v_M+v_s}{c} + (a_M + a_s)\frac{\lambda_s}{c^2}\right\}, \quad (92a)$$

$$\frac{c}{f_s}\frac{df_{ED}}{dt} \approx -2(a_s + a_M), \quad (93)$$

$$c\frac{dZ_{ED}}{dt} \approx 2(a_s + a_M). \quad (94)$$

Case (2): Source/Observer is receding with acceleration, Mirror is receding with deceleration

For this case, the formulas of observed wavelength λ_o, Extended Doppler shift cz_{ED}, frequency f_o, Extended Doppler frequency f_{ED}, time change of cz_{ED} and f_{ED}, have the same form as that for Case (1), but replacing a_M with $-d_M$.

Case（3）: Source/Observer is receding with deceleration, Mirror is receding with acceleration

For this case, the formulas of observed wavelength λ_o, Extended Doppler shift cz_{ED}, frequency f_o, Extended Doppler frequency f_{ED}, time change of cz_{ED} and f_{ED}, have the same form as that for Case (1), but replacing a_{os} with $-d_{os}$.

Case（4）: Source/Observer and Mirror are receding with deceleration.

For this case, the formulas of observed wavelength λ_o, Extended Doppler shift cz_{ED}, frequency f_o, Extended Doppler frequency f_{ED}, time change of cz_{ED} and f_{ED}, have the same form as that for Case (1), but replacing a_{os} and a_M with $-d_{os}$ and $-d_M$.

3.3.2. Source/Observer is receding with acceleration/deceleration, Mirror is approaching with acceleration/deceleration

There are four cases.

Case (1): Source/Observer is receding with acceleration, Mirror is approaching with acceleration.

Following the same argument above, for slow motion, we take $\frac{\lambda_s}{c} \approx \frac{\lambda_2}{c} \approx \frac{\lambda_o}{c} \approx \frac{\lambda_2'}{c}$, obtain,

$$\lambda_o \approx \lambda_s\left[1 + 2\frac{v_s - v_M}{c} + \frac{a_s - a_M}{c^2}\lambda_s\right], \tag{95}$$

$$cz_{ED} \approx 2(v_s - v_M) + (a_s - a_M)\frac{\lambda_s}{c}, \tag{96}$$

$$f_o \approx f_s\left[1 - 2\frac{v_s - v_M}{c} - (a_s - a_M)\frac{\lambda_s}{c^2}\right], \tag{97}$$

$$f_{ED} \approx -f_s\left\{2\frac{v_s - v_M}{c} + (a_s - a_M)\frac{\lambda_s}{c^2}\right\}, \tag{98}$$

$$\frac{c}{f_s}\frac{df_{ED}}{dt} \approx -2(a_s - a_M), \tag{99}$$

$$c\frac{dZ_{ED}}{dt} \approx 2(a_s - a_M). \tag{100}$$

Case (2): Source/Observer is receding with acceleration, Mirror is approaching with deceleration

For this case, the formulas of observed wavelength λ_o, Extended Doppler shift cz_{ED}, frequency f_o, Extended Doppler frequency f_{ED}, time change of cz_{ED} and f_{ED}, have the same form as that for Case (1), but replacing a_M with $-d_M$.

Case (3): Source/Observer is receding with deceleration, Mirror is approaching with acceleration

For this case, the formulas of observed wavelength λ_o, Extended Doppler shift cz_{ED}, frequency f_o, Extended Doppler frequency f_{ED}, time change of cz_{ED} and f_{ED}, have the same form as that for Case (1), but replacing a_{os} with $-d_{os}$.

Case (4): Source/Observer is receding with deceleration, Mirror is approaching with deceleration

For this case, the formulas of observed wavelength λ_o, Extended Doppler shift cz_{ED}, frequency f_o, Extended Doppler frequency f_{ED}, time change of cz_{ED} and f_{ED}, have the same form as that for Case (1), but replacing a_{os} and a_M with $-d_{os}$ and $-d_M$, respectively.

3.3.3. Source/Observer is approaching with acceleration/deceleration, Mirror is receding with acceleration/deceleration

There are four cases.

Case (1): Source/Observer is approaching with acceleration, Mirror is receding with acceleration

For slow motion, we obtain,

$$\lambda_o \approx \lambda_s\left[1 - 2\frac{v_s - v_M}{c} - \frac{a_s - a_M}{c^2}\lambda_s\right], \tag{101}$$

$$cz_{ED} \approx -2(v_s - v_M) - (a_s - a_M)\frac{\lambda_s}{c}, \tag{102}$$

$$f_o \approx f_s\left[1 + 2\frac{v_s - v_M}{c} + (a_s - a_M)\frac{\lambda_s}{c^2}\right], \tag{103}$$

$$f_{ED} \approx f_s\left\{2\frac{v_s - v_M}{c} + (a_s - a_M)\frac{\lambda_s}{c^2}\right\}, \tag{104}$$

$$\frac{c}{f_s}\frac{df_o}{dt} \approx 2(a_s - a_M), \tag{105}$$

$$c\frac{dZ_{ED}}{dt} \approx -2(a_s - a_M). \tag{106}$$

Case (2): Source/Observer is approaching with acceleration, Mirror is receding with deceleration

For this case, the formulas of observed wavelength λ_o, Extended Doppler shift cz_{ED}, frequency f_o, Extended Doppler frequency f_{ED}, time change of cz_{ED} and f_{ED}, have the same form as that for Case (1), but replacing a_M with $-d_M$.

Case (3): Source/Observer is approaching with deceleration, Mirror is receding with acceleration

For this case, the formulas of observed wavelength λ_o, Extended Doppler shift cz_{ED}, frequency f_o, Extended Doppler frequency f_{ED}, time change of cz_{ED} and f_{ED}, have the same form as that for Case (1), but replacing a_{os} with $-d_{os}$.

Case (4): Source/Observer is approaching with deceleration, Mirror is receding with deceleration

For this case, the formulas of observed wavelength λ_o, Extended Doppler shift cz_{ED}, frequency f_o, Extended Doppler frequency f_{ED}, time change of cz_{ED} and f_{ED}, have the same form as that for Case (1), but replacing a_{os} and a_M with $-d_{os}$ and $-d_M$, respectively.

3.3.4. Source and Mirror are approaching with acceleration/deceleration

There are four cases.

Case (1): Source and Mirror are approaching with acceleration.

$$\lambda_o \approx \lambda_s\left[1 - 2\frac{v_s+v_M}{c} - \frac{a_s+a_M}{c^2}\lambda_s\right], \tag{107}$$

$$cz_{ED} \approx -2(v_s+v_M) - (a_s+a_M)\frac{\lambda_s}{c}, \tag{108}$$

$$f_o \approx f_s\left[1 + 2\frac{v_s+v_M}{c} + (a_s+a_M)\frac{\lambda_s}{c^2}\right], \tag{109}$$

$$f_{ED} \approx f_s\left\{2\frac{v_s+v_M}{c} + (a_s+a_M)\frac{\lambda_s}{c^2}\right\}, \tag{110}$$

$$\frac{c}{f_s}\frac{df_o}{dt} \approx 2(a_s+a_M), \tag{111}$$

$$c\frac{dZ_{ED}}{dt} \approx -2(a_s+a). \tag{112}$$

Case (2): Source/Observer is approaching with acceleration, Mirror is approaching with deceleration. For this case, the formulas of observed wavelength λ_o, Extended Doppler shift cz_{ED}, frequency f_o, Extended Doppler frequency f_{ED}, time change of cz_{ED} and f_{ED}, have the same form as that for Case (1), but replacing a_M with $-d_M$.

Case (3): Source/Observer is approaching with deceleration, Mirror is approaching with acceleration. For this case, the formulas of observed wavelength λ_o, Extended Doppler shift cz_{ED}, frequency f_o, Extended Doppler frequency f_{ED}, time change of cz_{ED} and f_{ED}, have the same form as that for Case (1), but replacing a_{os} with $-d_{os}$.

Case (4): Source/Observer and Mirror are approaching with deceleration. For this case, the formulas of observed wavelength λ_o, Extended Doppler shift cz_{ED}, frequency f_o, Extended Doppler frequency f_{ED}, time change of cz_{ED} and f_{ED}, have the same form as that for Case (1), but replacing a_{os} and a_M with $-d_{os}$ and $-d_M$, respectively.

4. Conclusion and Discussion

We study systematically the effects of acceleration and deceleration of moving source, observer, and mirror on wavelength and frequency and show that acceleration and deceleration causes shifts of wavelength and frequency, denoted as "Acceleration-deceleration-shift". The total shift, denoted as the Extended Doppler shift, is a summation of Doppler shift and Acceleration-deceleration-shift. Moreover acceleration and deceleration changes Doppler shift and total shift continuously, which implies that one can measure the time changes of both wavelength/frequency and shift of them to detect acceleration and/or deceleration of either source, or observer, or mirror/reflector. Also we show that only for slow moving case, the wavelength/frequency and shift are determined by the relative velocity and relative acceleration/deceleration.

We summarize the Extended Doppler shift formulas for the simplest cases of the non-uniformly moving observer, source/observer, and Mirror/reflector variations. The formulas for the more complex cases of non-uniformly motions are out scope of this article.

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