The Exponential Function and its Infinite Product

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"It is the spirit that quickeneth; the flesh profiteth nothing: the words that I speak unto you, they are spirit, and they are life." - John 6:63.

ABSTRACT. I derive some infinite product representations for the exponential function.

1. INTRODUCTION

In present paper, I deduct the following infinite product representation for the exponential function

$$\frac{(z+1)^{z+1}}{(2e)^z} = \prod_{n=1}^{\infty} \frac{(n+2)^z (n+1)^{n+1} (n+z)^{n+z}}{(n+1)^z n^n (n+z+1)^{n+z+1}}$$
$$= \prod_{n=1}^{\infty} \frac{\left(1+\frac{1}{n}\right)^n \left(1+\frac{1}{n+1}\right)^z}{\left(1+\frac{z}{n+1}\right)\left(1+\frac{1}{n+z}\right)^{n+z}},$$

therefrom, I put some beautiful infinite products

$$\frac{2}{e} = \prod_{n=1}^{\infty} \frac{(n+1)^{2n+1}}{n^n (n+2)^{n+1}} = \left(\frac{2^3}{3^2}\right)\cdot\left(\frac{3^5}{2^8}\right)\cdot\left(\frac{2^{14}}{3^3\cdot 5^4}\right)\cdot\left(\frac{5^9}{2^{13}\cdot 3^5}\right)\cdot\ldots$$

and

$$\frac{27}{4e^2} = \prod_{n=1}^{\infty} \frac{(n+1)^{n-1}(n+2)^{n+4}}{n^n (n+3)^{n+3}} = \left(\frac{3^5}{2^8}\right)\cdot\left(\frac{2^{10}\cdot 3}{5^5}\right)\cdot\left(\frac{5^7}{2^2\cdot 3^9}\right)\cdot\left(\frac{3^8\cdot 5^3}{7^7}\right)\cdot\ldots;$$

as well as

$$\frac{z+1}{e^z} = \prod_{n=1}^{\infty} \frac{(n+1)^{n+1}(n+z)^n}{n^n (n+z+1)^{n+1}} = \prod_{n=1}^{\infty} \frac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{1}{n+z}\right)^n \left(1+\frac{z}{n+1}\right)},$$

thereout, I put other pretty infinite products

$$\frac{2}{e} = \prod_{n=1}^{\infty} \frac{(n+1)^{2n+1}}{n^n (n+2)^{n+1}} = \left(\frac{2^3}{3^2}\right)\cdot\left(\frac{3^5}{2^8}\right)\cdot\left(\frac{2^{14}}{3^3\cdot 5^4}\right)\cdot\left(\frac{5^9}{2^{13}\cdot 3^5}\right)\cdot\ldots$$

and

$$\frac{3}{e^2} = \prod_{n=1}^{\infty} \frac{(n+1)^{n+1}(n+2)^n}{n^n (n+3)^{n+1}} = \left(\frac{3}{2^2}\right)\cdot\left(\frac{2^2\cdot 3^3}{5^3}\right)\cdot\left(\frac{2^4\cdot 5^3}{3^7}\right)\cdot\left(\frac{3^4\cdot 5^5}{2^4\cdot 7^5}\right)\cdot\ldots$$

2. THE INFINITE PRODUCT REPRESENTATION FOR THE EXPONENTIAL FUNCTION

Theorem 2.1. *If* $z \in \mathbb{R}_{\geqslant 0}$, *then*

$$\begin{aligned}\frac{(z+1)^{z+1}}{(2e)^z} &= \prod_{n=1}^{\infty} \frac{(n+2)^z (n+1)^{n+1} (n+z)^{n+z}}{(n+1)^z n^n (n+z+1)^{n+z+1}} \\ &= \prod_{n=1}^{\infty} \frac{\left(1+\frac{1}{n}\right)^n \left(1+\frac{1}{n+1}\right)^z}{\left(1+\frac{z}{n+1}\right)\left(1+\frac{1}{n+z}\right)^{n+z}},\end{aligned} \tag{2.1}$$

where e^z *denotes the exponential function.*

Proof. In [1, p. 4, Corollary 8, (13)], I have a new integral representation for the natural logarithm function. Thereupon, I derive the following power series

$$\begin{aligned}\frac{\log x}{x-1} &= \int_0^{\infty} \frac{\mathrm{d}t}{(x+t)(1+t)} = \sum_{n=0}^{\infty} \int_n^{n+1} \frac{\mathrm{d}t}{(x+t)(1+t)} \\ &= \frac{1}{x-1} \sum_{n=0}^{\infty} \log\left[\frac{(n+2)(n+x)}{(n+1)(n+x+1)}\right] \\ &= \frac{1}{x-1} \log\left(\frac{2x}{x+1}\right) + \frac{1}{x-1} \sum_{n=1}^{\infty} \log\left[\frac{(n+2)(n+x)}{(n+1)(n+x+1)}\right] \\ &\Rightarrow \log x = \log\left(\frac{2x}{x+1}\right) \\ &+ \sum_{n=1}^{\infty} [\log(n+2) - \log(n+1) + \log(n+x) - \log(n+x+1)] \\ &\Rightarrow \log(x+1) - \log 2 \\ &= \sum_{n=1}^{\infty} [\log(n+2) - \log(n+1) + \log(n+x) - \log(n+x+1)].\end{aligned} \tag{2.2}$$

Integrate both members of the (2.2) from 0 at z with respect to x, as follows

$$\begin{aligned}&\int_0^z \log(x+1)\,\mathrm{d}x - z\log 2 \\ &= \sum_{n=1}^{\infty} \left[z\log(n+2) - z\log(n+1) + \int_0^z \log(n+x)\,\mathrm{d}x - \int_0^z \log(n+x+1)\,\mathrm{d}x\right].\end{aligned} \tag{2.3}$$

I easily calculate the integrals

$$\int_0^z \log(x+1)\,\mathrm{d}x = (z+1)\log(z+1) - z, \tag{2.4}$$

$$\int_0^z \log(n+x)\mathrm{d}x = (n+z)\log(n+z) - n\log n - z \tag{2.5}$$

and

$$-\int_0^z \log(n+x+1)\mathrm{d}x = -(n+z+1)\log(n+z+1) + (n+1)\log(n+1) + z. \tag{2.6}$$

Substitute the rigth hand side of (2.4), (2.5) and (2.6) into both members of (2.3), and obtain

$$(z+1)\log(z+1)-z-z\log 2=\sum_{n=1}^{\infty}\log\left[\frac{(n+2)^{z}(n+1)^{n+1}(n+z)^{n+z}}{(n+1)^{z}n^{n}(n+z+1)^{n+z+1}}\right]. \tag{2.7}$$

The exponentiation of (2.7) give us

$$\begin{aligned}\frac{(z+1)^{z+1}}{(2e)^{z}}&=\prod_{n=1}^{\infty}\frac{(n+2)^{z}(n+1)^{n+1}(n+z)^{n+z}}{(n+1)^{z}n^{n}(n+z+1)^{n+z+1}}\\&=\prod_{n=1}^{\infty}\frac{\left(1+\frac{1}{n}\right)^{n}\left(1+\frac{1}{n+1}\right)^{z}}{\left(1+\frac{z}{n+1}\right)\left(1+\frac{1}{n+z}\right)^{n+z}},\end{aligned}$$

which are the desired results. □

Example 2.2. Put $z=1$ in Theorem 2.1 and encounter

$$\frac{2}{e}=\prod_{n=1}^{\infty}\frac{(n+1)^{2n+1}}{n^{n}(n+2)^{n+1}}=\left(\frac{2^{3}}{3^{2}}\right)\cdot\left(\frac{3^{5}}{2^{8}}\right)\cdot\left(\frac{2^{14}}{3^{3}\cdot 5^{4}}\right)\cdot\left(\frac{5^{9}}{2^{13}\cdot 3^{5}}\right)\cdots$$

Example 2.3. Put $z=2$ in Theorem 2.1 and encounter

$$\frac{27}{4e^{2}}=\prod_{n=1}^{\infty}\frac{(n+1)^{n-1}(n+2)^{n+4}}{n^{n}(n+3)^{n+3}}=\left(\frac{3^{5}}{2^{8}}\right)\cdot\left(\frac{2^{10}\cdot 3}{5^{5}}\right)\cdot\left(\frac{5^{7}}{2^{2}\cdot 3^{9}}\right)\cdot\left(\frac{3^{8}\cdot 5^{3}}{7^{7}}\right)\cdots$$

Exercise 2.1. Prove that, for $z\in\mathbb{R}_{>0}$,

$$\prod_{n=1}^{\infty}\frac{(n+2)^{z}(n+1)^{n+1}(n+z)^{n+z}}{(n+1)^{z}n^{n}(n+z+1)^{n+z+1}}=(2e)^{-z}\exp(\zeta'(-1,z+2)-\zeta'(-1,z+1));$$

hence,

$$\zeta'(-1,z+1)-\zeta'(-1,z)=z\log z.$$

Corollary 2.4. *If $z\in\mathbb{R}_{\geqslant 0}$, then*

$$\frac{z+1}{e^{z}}=\prod_{n=1}^{\infty}\frac{(n+1)^{n+1}(n+z)^{n}}{n^{n}(n+z+1)^{n+1}}=\prod_{n=1}^{\infty}\frac{\left(1+\frac{1}{n}\right)^{n}}{\left(1+\frac{1}{n+z}\right)^{n}\left(1+\frac{z}{n+1}\right)}, \tag{2.8}$$

where e^{z} denotes the exponential function.

Proof. From left hand side of the Theorem 2.1, I get

$$\frac{(z+1)^{z+1}}{(2e)^{z}}=\frac{z+1}{e^{z}}\cdot\left(\frac{z+1}{2}\right)^{z}. \tag{2.9}$$

On the other hand, in [2, Lemma 1, p. 2], I find

$$\frac{a}{b}=\prod_{n=1}^{\infty}\frac{(a+n-1)(b+n)}{(a+n)(b+n-1)}, \tag{2.10}$$

provided to $a, b \in \mathbb{R}$ and $b \neq 0$.

From (2.10), I easily deduce that

$$\left(\frac{z+1}{2}\right)^z = \prod_{n=1}^{\infty} \frac{(z+n)^z(n+2)^z}{(z+n+1)^z(n+1)^z}. \tag{2.11}$$

Now, from Theorem 2.1, (2.9) and (2.11), it follows that

$$\begin{aligned}
\frac{z+1}{e^z} \cdot \prod_{n=1}^{\infty} \frac{(z+n)^z(n+2)^z}{(z+n+1)^z(n+1)^z} &= \prod_{n=1}^{\infty} \frac{(n+2)^z(n+1)^{n+1}(n+z)^{n+z}}{(n+1)^z n^n (n+z+1)^{n+z+1}} \\
\Rightarrow \frac{z+1}{e^z} &= \prod_{n=1}^{\infty} \frac{(z+n+1)^z(n+1)^z}{(z+n)^z(n+2)^z} \cdot \prod_{n=1}^{\infty} \frac{(n+2)^z(n+1)^{n+1}(n+z)^{n+z}}{(n+1)^z n^n (n+z+1)^{n+z+1}} \\
\Rightarrow \frac{z+1}{e^z} &= \prod_{n=1}^{\infty} \frac{(z+n+1)^z(n+1)^z(n+2)^z(n+1)^{n+1}(n+z)^{n+z}}{(z+n)^z(n+2)^z(n+1)^z n^n (n+z+1)^{n+z+1}} \\
\Rightarrow \frac{z+1}{e^z} &= \prod_{n=1}^{\infty} \frac{(n+1)^{n+1}(n+z)^n}{n^n(n+z+1)^{n+1}} = \prod_{n=1}^{\infty} \frac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{1}{n+z}\right)^n\left(1+\frac{z}{n+1}\right)},
\end{aligned}$$

which are the desired results. □

Example 2.5. Put $z = 1$ in Theorem 2.1 and encounter

$$\frac{2}{e} = \prod_{n=1}^{\infty} \frac{(n+1)^{2n+1}}{n^n(n+2)^{n+1}} = \left(\frac{2^3}{3^2}\right) \cdot \left(\frac{3^5}{2^8}\right) \cdot \left(\frac{2^{14}}{3^3 \cdot 5^4}\right) \cdot \left(\frac{5^9}{2^{13} \cdot 3^5}\right) \cdots$$

Example 2.6. Put $z = 2$ in Theorem 2.1 and encounter

$$\frac{3}{e^2} = \prod_{n=1}^{\infty} \frac{(n+1)^{n+1}(n+2)^n}{n^n(n+3)^{n+1}} = \left(\frac{3}{2^2}\right) \cdot \left(\frac{2^2 \cdot 3^3}{5^3}\right) \cdot \left(\frac{2^4 \cdot 5^3}{3^7}\right) \cdot \left(\frac{3^4 \cdot 5^5}{2^4 \cdot 7^5}\right) \cdots$$

REFERENCES

[1] Guedes, Edigles, *On the Natural Logarithm Function and its Applications*, March 7, 2015, **viXra:1503.0058**.

[2] Guedes, Edigles, *Infinite Product Representations for Binomial Coefficient, Pochhammer's Symbol, Newton's Binomial and Exponential Function*, June 27, 2016, **viXra:1611.0368**.