More Sine Function at Rational Argument, Product of Gamma Functions and Infinite Product Representation

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"It is the spirit that quickeneth; the flesh profiteth nothing: the words that I speak unto you, they are spirit, and they are life." - John 6:63.

Abstract. I derived an identity involving gamma functions and sine function at rational argument; hence, the representation of infinite product arose.

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1. Introduction

In present paper, I attain the following identity

$$\frac{q}{p\pi}\sin\left(\frac{p\pi}{q}\right)=\frac{\left(1-\frac{p}{q}\right)_{q-1}\left(1+\frac{p}{q}\right)_{q-1}}{\Gamma\left(q-\frac{p}{q}\right)\Gamma\left(q+\frac{p}{q}\right)},$$

which enabled me to prove the infinite product below

$$\frac{q^3}{(q^4-p^2)p\pi}\sin\left(\frac{p\pi}{q}\right)=\prod_{j=1}^{\infty}\frac{(j^2\,q^2-p^2)[q^2(j+q)^2-p^2]}{q^2(j+1)^2[q^2(j+q-1)^2-p^2]}$$
$$=\prod_{j=1}^{\infty}\frac{(qj+q^2-p)(qj+q^2+p)(q^2j^2-p^2)}{q^2(j+1)^2(qj+q^2-q-p)(qj+q^2-q+p)},$$

more specifically, I get

$$\frac{8}{15\pi}=\prod_{j=1}^{\infty}\frac{(2j-1)(2j+5)}{4(j+1)^2}$$
$$=\frac{1\cdot 7}{4\cdot 2^2}\cdot\frac{3\cdot 9}{4\cdot 3^2}\cdot\frac{5\cdot 11}{4\cdot 4^2}\cdot\frac{7\cdot 13}{4\cdot 5^2}\cdot\frac{9\cdot 15}{4\cdot 6^2}\cdot\frac{11\cdot 17}{4\cdot 7^2}\cdot\ldots;$$

$$\frac{27\sqrt{3}}{160\pi}==\prod_{j=1}^{\infty}\frac{(3j+8)(3j+10)(9j^2-1)}{9(j+1)^2(3j+5)(3j+7)}$$
$$=\frac{11\cdot 13\cdot 8}{9\cdot 2^2\cdot 8\cdot 10}\cdot\frac{14\cdot 16\cdot 35}{9\cdot 3^2\cdot 11\cdot 13}\cdot\frac{17\cdot 19\cdot 80}{9\cdot 4^2\cdot 14\cdot 16}\cdot\frac{20\cdot 22\cdot 143}{9\cdot 5^2\cdot 17\cdot 19}\cdot\frac{23\cdot 25\cdot 224}{9\cdot 6^2\cdot 20\cdot 22}\cdot\ldots;$$

$$\frac{32\sqrt{2}}{255\pi}=\prod_{j=1}^{\infty}\frac{(4j+15)(4j+17)(16j^2-1)}{16(j+1)^2(4j+11)(4j+13)}$$
$$=\frac{19\cdot 21\cdot 15}{16\cdot 2^2\cdot 15\cdot 17}\cdot\frac{23\cdot 25\cdot 63}{16\cdot 3^2\cdot 19\cdot 21}\cdot\frac{27\cdot 29\cdot 143}{16\cdot 4^2\cdot 23\cdot 25}\cdot\frac{31\cdot 33\cdot 255}{16\cdot 5^2\cdot 27\cdot 29}\cdot\frac{35\cdot 37\cdot 399}{16\cdot 6^2\cdot 31\cdot 33}\cdot\ldots;$$

$$\frac{32\sqrt{2}}{741\pi}=\prod_{j=1}^{\infty}\frac{(4j+13)(4j+19)(16j^2-9)}{16(j+1)^2(4j+9)(4j+15)}$$
$$=\frac{17\cdot 23\cdot 7}{16\cdot 2^2\cdot 13\cdot 19}\cdot\frac{21\cdot 27\cdot 55}{16\cdot 3^2\cdot 17\cdot 23}\cdot\frac{25\cdot 31\cdot 135}{16\cdot 4^2\cdot 21\cdot 27}\cdot\frac{29\cdot 35\cdot 247}{16\cdot 5^2\cdot 25\cdot 31}\cdot\frac{33\cdot 39\cdot 391}{16\cdot 6^2\cdot 29\cdot 35}\cdot\ldots;$$

$$\frac{125}{1248\pi}\sqrt{\frac{5}{2}-\frac{\sqrt{5}}{2}}=\prod_{j=1}^{\infty}\frac{(5j+24)(5j+26)(25j^2-1)}{25(j+1)^2(5j+19)(5j+21)}$$
$$=\frac{29\cdot 31\cdot 24}{25\cdot 2^2\cdot 24\cdot 26}\cdot\frac{34\cdot 36\cdot 99}{25\cdot 3^2\cdot 29\cdot 31}\cdot\frac{39\cdot 41\cdot 224}{25\cdot 4^2\cdot 34\cdot 36}\cdot\frac{44\cdot 46\cdot 399}{25\cdot 5^2\cdot 39\cdot 41}\cdot\frac{49\cdot 51\cdot 624}{25\cdot 6^2\cdot 44\cdot 46}\cdot\ldots$$

and so on.

2. Preliminary

I use the following classical formula, [1, Section 12.13; 2], which is a Corollary of the Weierstrass infinite product representation for the gamma function:

Corollary 2.1. *If k is a positive integer and $a_1+a_2+\ldots+a_k=b_1+b_2+\ldots+b_k$, where the a_j and b_j are complex numbers and no b_j is zero or a negative integer, then*

$$\prod_{\ell=0}^{\infty}\frac{(\ell+a_1)\cdot\ldots\cdot(\ell+a_k)}{(\ell+b_1)\cdot\ldots\cdot(\ell+b_k)}=\frac{\Gamma(b_1)\cdot\ldots\cdot\Gamma(b_k)}{\Gamma(a_1)\cdot\ldots\cdot\Gamma(a_k)}. \tag{2.1}$$

Proof. See [1, Section 12.13]. ☐

3. The Main Theorem

3.1. The sine function at rational argument and the product of gamma functions.

Theorem 3.1. *If p and q are positive integers and $p \leqslant q$, then*

$$\frac{q}{p\pi}\sin\left(\frac{p\pi}{q}\right)\prod_{\nu=1}^{q-1}\left(1-\frac{p^2}{\nu^2 q^2}\right)^{-1}=\frac{\Gamma^2(q)}{\Gamma\left(q-\frac{p}{q}\right)\Gamma\left(q+\frac{p}{q}\right)}, \tag{3.1}$$

where $\Gamma(z)$ denotes the gamma function and $\sin(z)$ denotes the sine function.

Proof. . Consider the Euler's infinite product representation for sine function [3, p. 321]

$$\frac{\sin(\pi z)}{\pi z}=\prod_{\nu=1}^{\infty}\left(1-\frac{z^2}{\nu^2}\right). \tag{3.2}$$

Let $z=p/q$ in (3.2), with $p\in\mathbb{Z}^+$ and $q\in\mathbb{Z}_{>1}$, and encounter

$$\frac{q\sin\left(\frac{p\pi}{q}\right)}{p\pi}=\prod_{\nu=1}^{\infty}\left(1-\frac{p^2}{\nu^2 q^2}\right). \tag{3.3}$$

Divide both members of (3.3) by $\prod_{\nu=1}^{q-1}(1-p^2/\nu^2 q^2)$

$$\begin{aligned}\frac{q\sin\left(\frac{p\pi}{q}\right)}{p\pi}\prod_{\nu=1}^{q-1}\left(1-\frac{p^2}{\nu^2 q^2}\right)^{-1}&=\frac{\prod_{\nu=1}^{\infty}\left(1-\frac{p^2}{\nu^2 q^2}\right)}{\prod_{\nu=1}^{q-1}\left(1-\frac{p^2}{\nu^2 q^2}\right)}\\ &=\prod_{\nu=q}^{\infty}\left(1-\frac{p^2}{\nu^2 q^2}\right)=\frac{\Gamma^2(q)}{\Gamma\left(q-\frac{p}{q}\right)\Gamma\left(q+\frac{p}{q}\right)}\\ \Rightarrow\frac{q}{p\pi}\sin\left(\frac{p\pi}{q}\right)\prod_{\nu=1}^{q-1}\left(1-\frac{p^2}{\nu^2 q^2}\right)^{-1}&=\frac{\Gamma^2(q)}{\Gamma\left(q-\frac{p}{q}\right)\Gamma\left(q+\frac{p}{q}\right)},\end{aligned}$$

which is the desired result. □

Example 3.2. Set $p=1$ and $q=2$ in Theorem 3.1

$$\frac{8}{3\pi}=\frac{\Gamma^2(2)}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{5}{2}\right)}.$$

Example 3.3. Set $p=1$ and $q=3$ in Theorem 3.1

$$\frac{243\sqrt{3}}{140\pi}=\frac{\Gamma^2(3)}{\Gamma\left(\frac{8}{3}\right)\Gamma\left(\frac{10}{3}\right)}.$$

Corollary 3.4.

$$\sin\left(\frac{p\pi}{q}\right)=\frac{p\pi}{q}\cdot\frac{\left(1-\frac{p}{q}\right)_{q-1}\left(1+\frac{p}{q}\right)_{q-1}}{\Gamma\left(q-\frac{p}{q}\right)\Gamma\left(q+\frac{p}{q}\right)} \tag{3.4}$$

and

$$\sin\left(\frac{p\pi}{q}\right)=\frac{p\pi}{q}\cdot\frac{1}{\Gamma\left(1-\frac{p}{q}\right)\Gamma\left(1+\frac{p}{q}\right)}, \tag{3.5}$$

where $\sin(z)$ *denotes the sine function and* $\Gamma(z)$ *denotes the gamma function.*

Proof. I easily calculate

$$\prod_{\nu=1}^{q-1}\left(1-\frac{p^2}{\nu^2q^2}\right)^{-1}=\frac{\Gamma^2(q)}{\left(1-\frac{p}{q}\right)_{q-1}\left(1+\frac{p}{q}\right)_{q-1}}. \tag{3.6}$$

From Theorem 3.1 and (3.6), it follows that

$$\frac{q}{p\pi}\sin\left(\frac{p\pi}{q}\right)=\frac{\left(1-\frac{p}{q}\right)_{q-1}\left(1+\frac{p}{q}\right)_{q-1}}{\Gamma\left(q-\frac{p}{q}\right)\Gamma\left(q+\frac{p}{q}\right)}. \tag{3.7}$$

The definition of the Pochhammer's symbol [4] is

$$(x)_n\overset{\text{def.}}{=}\frac{\Gamma(x+n)}{\Gamma(x)}. \tag{3.8}$$

From (3.7) and (3.8), I conclude that

$$\frac{q}{p\pi}\sin\left(\frac{p\pi}{q}\right)=\frac{1}{\Gamma\left(1-\frac{p}{q}\right)\Gamma\left(1+\frac{p}{q}\right)},$$

which are the desired results. □

3.2. News infinite product representations for the sine function at rational argument.

Corollary 3.5. *If p and q are positive integers and $p \leqslant q$, then*

$$\frac{q}{p\pi}\sin\left(\frac{p\pi}{q}\right)\prod_{\ell=1}^{q-1}\left(1-\frac{p^2}{\ell^2 q^2}\right)^{-1}=\prod_{\ell=0}^{\infty}\left(1-\frac{p^2}{q^2(\ell+q)^2}\right). \tag{3.9}$$

where $\sin(z)$ *denotes the sine function.*

Proof. Note that

$$q+q=q-\frac{p}{q}+q+\frac{p}{q},$$

satisfy the condition $a_1+a_2=b_1+b_2$; $k=2$ is a positive integer; the a_j and b_j are complex numbers and no b_j is zero or a negative integer. From Corollary 2.1 and the right hand side of the Theorem 3.1, it follows that

$$\begin{aligned}\frac{\Gamma^2(q)}{\Gamma\left(q-\frac{p}{q}\right)\Gamma\left(q+\frac{p}{q}\right)}=&\prod_{\ell=0}^{\infty}\frac{\left(\ell+q-\frac{p}{q}\right)\left(\ell+q+\frac{p}{q}\right)}{(\ell+q)(\ell+q)}\\ =&\prod_{\ell=0}^{\infty}\left(1-\frac{p^2}{(q\ell+q^2)^2}\right).\end{aligned} \tag{3.10}$$

From Theorem 3.1 and (3.10), I conclude that

$$\frac{q}{p\pi}\sin\left(\frac{p\pi}{q}\right)\prod_{\nu=1}^{q-1}\left(1-\frac{p^2}{\nu^2 q^2}\right)^{-1}=\prod_{\ell=0}^{\infty}\left(1-\frac{p^2}{q^2(\ell+q)^2}\right). \tag{3.11}$$

Replace ν by ℓ in (3.11) and this completes the proof. □

Example 3.6. Set $p=1$ and $q=2$ in Corollary 3.5

$$\begin{aligned}\frac{8}{3\pi}=&\prod_{\ell=0}^{\infty}\left(1-\frac{1}{4(\ell+2)^2}\right)\\ =&\left(1-\frac{1}{4\cdot 2^2}\right)\left(1-\frac{1}{4\cdot 3^2}\right)\left(1-\frac{1}{4\cdot 4^2}\right)\left(1-\frac{1}{4\cdot 5^2}\right)\left(1-\frac{1}{4\cdot 6^2}\right)\left(1-\frac{1}{4\cdot 7^2}\right)\cdots\\ =&\frac{15}{16}\cdot\frac{35}{36}\cdot\frac{63}{64}\cdot\frac{99}{100}\cdot\frac{143}{144}\cdot\frac{195}{196}\cdot\ldots\end{aligned}$$

Example 3.7. Set $p=1$ and $q=3$ in Corollary 3.5

$$\begin{aligned}\frac{243\sqrt{3}}{140\pi}=&\prod_{\ell=0}^{\infty}\left(1-\frac{1}{9(\ell+3)^2}\right)\\ =&\left(1-\frac{1}{9\cdot 3^2}\right)\left(1-\frac{1}{9\cdot 4^2}\right)\left(1-\frac{1}{9\cdot 5^2}\right)\left(1-\frac{1}{9\cdot 6^2}\right)\left(1-\frac{1}{9\cdot 7^2}\right)\left(1-\frac{1}{9\cdot 8^2}\right)\cdots\\ =&\frac{80}{81}\cdot\frac{143}{144}\cdot\frac{224}{225}\cdot\frac{323}{324}\cdot\frac{440}{441}\cdot\frac{575}{576}\cdot\ldots\end{aligned}$$

Example 3.8. Set $p=1$ and $q=4$ in Corollary 3.5

$$\frac{32768\sqrt{2}}{15015\pi}=\prod_{\ell=0}^{\infty}\left(1-\frac{1}{16(\ell+4)^2}\right)$$
$$=\left(1-\frac{1}{16\cdot 4^2}\right)\left(1-\frac{1}{16\cdot 5^2}\right)\left(1-\frac{1}{16\cdot 6^2}\right)\left(1-\frac{1}{16\cdot 7^2}\right)\left(1-\frac{1}{16\cdot 8^2}\right)\left(1-\frac{1}{16\cdot 9^2}\right)\dots$$
$$=\frac{255}{256}\cdot\frac{399}{400}\cdot\frac{575}{576}\cdot\frac{783}{784}\cdot\frac{1023}{1024}\cdot\frac{1295}{1296}\cdot\dots$$

Example 3.9. Set $p=3$ and $q=4$ in Corollary 3.5

$$\frac{32768\sqrt{2}}{17325\pi}=\prod_{\ell=0}^{\infty}\left(1-\frac{9}{16(\ell+4)^2}\right).$$
$$=\left(1-\frac{9}{16\cdot 4^2}\right)\left(1-\frac{9}{16\cdot 5^2}\right)\left(1-\frac{9}{16\cdot 6^2}\right)\left(1-\frac{9}{16\cdot 7^2}\right)\left(1-\frac{9}{16\cdot 8^2}\right)\left(1-\frac{9}{16\cdot 9^2}\right)\dots$$
$$=\frac{247}{256}\cdot\frac{391}{400}\cdot\frac{567}{576}\cdot\frac{775}{784}\cdot\frac{1015}{1024}\cdot\frac{1287}{1296}\cdot\dots$$

Example 3.10. Set $p=1$ and $q=5$ in Corollary 3.5

$$\frac{1953125}{737352\pi}\sqrt{\frac{5}{2}-\frac{\sqrt{5}}{2}}=\prod_{\ell=0}^{\infty}\left(1-\frac{1}{25(\ell+5)^2}\right)$$
$$=\left(1-\frac{1}{25\cdot 5^2}\right)\left(1-\frac{1}{25\cdot 6^2}\right)\left(1-\frac{1}{25\cdot 7^2}\right)\left(1-\frac{1}{25\cdot 8^2}\right)\left(1-\frac{1}{25\cdot 9^2}\right)\left(1-\frac{1}{25\cdot 10^2}\right)\dots$$
$$=\frac{624}{625}\cdot\frac{899}{900}\cdot\frac{1224}{1225}\cdot\frac{1599}{1600}\cdot\frac{2024}{2025}\cdot\frac{2499}{2500}\cdot\dots$$

Corollary 3.11. *If p and q are positive integers and $p\leqslant q$, then*

$$\begin{aligned}\frac{q^3}{(q^4-p^2)p\pi}\sin\left(\frac{p\pi}{q}\right)&=\prod_{j=1}^{\infty}\frac{[q^2(j+q)^2-p^2](q^2j^2-p^2)}{q^2(j+1)^2[q^2(j+q-1)^2-p^2]}\\&=\prod_{j=1}^{\infty}\frac{(qj+q^2-p)(qj+q^2+p)(q^2j^2-p^2)}{q^2(j+1)^2(qj+q^2-q-p)(qj+q^2-q+p)},\end{aligned}\tag{3.12}$$

where $\sin(z)$ *denotes the sine function.*

Proof. I know the Euler's infinite product representation for the gamma function [5, p. 33] given by

$$\Gamma(z)=\frac{1}{z}\prod_{j=1}^{\infty}\frac{\left(1+\frac{1}{j}\right)^z}{\left(1+\frac{z}{j}\right)},\tag{3.13}$$

which converges for all complex numbers z, except the negative integers.

In [6, p. 3, Theorem 3], I derived the following infinite product representation for the Pochhammer's symbol

$$(\ell)_n=\prod_{j=1}^{\infty}\frac{\left(1+\frac{1}{j}\right)^n}{\left(1+\frac{n}{j+\ell-1}\right)}. \tag{3.14}$$

From Theorem (3.4), (3.13) and (3.14), I conclude that

$$\begin{aligned}
&\sin\left(\frac{p\pi}{q}\right)\\
&=\frac{p\pi}{q}\cdot\frac{\left[\prod_{j=1}^{\infty}\frac{\left(1+\frac{1}{j}\right)^{q-1}}{\left(1+\frac{q-1}{j+1-\frac{p}{q}-1}\right)}\right]\left[\prod_{j=1}^{\infty}\frac{\left(1+\frac{1}{j}\right)^{q-1}}{\left(1+\frac{q-1}{j+1+\frac{p}{q}-1}\right)}\right]}{\left[\frac{1}{q-\frac{p}{q}}\prod_{j=1}^{\infty}\frac{\left(1+\frac{1}{j}\right)^{q-\frac{p}{q}}}{\left(1+\frac{q-\frac{p}{q}}{j}\right)}\right]\left[\frac{1}{q+\frac{p}{q}}\prod_{j=1}^{\infty}\frac{\left(1+\frac{1}{j}\right)^{q+\frac{p}{q}}}{\left(1+\frac{q+\frac{p}{q}}{j}\right)}\right]}\\
&=\frac{p\pi}{q}\cdot\frac{q^4-p^2}{q^2}\cdot\frac{\left[\prod_{j=1}^{\infty}\frac{\left(1+\frac{1}{j}\right)^{q-1}}{\left(1+\frac{q-1}{j-\frac{p}{q}}\right)}\right]\left[\prod_{j=1}^{\infty}\frac{\left(1+\frac{1}{j}\right)^{q-1}}{\left(1+\frac{q-1}{j+\frac{p}{q}}\right)}\right]}{\left[\prod_{j=1}^{\infty}\frac{\left(1+\frac{1}{j}\right)^{q-\frac{p}{q}}}{\left(1+\frac{q-\frac{p}{q}}{j}\right)}\right]\left[\prod_{j=1}^{\infty}\frac{\left(1+\frac{1}{j}\right)^{q+\frac{p}{q}}}{\left(1+\frac{q+\frac{p}{q}}{j}\right)}\right]}\\
&=\frac{p\pi}{q}\cdot\frac{q^4-p^2}{q^2}\cdot\prod_{j=1}^{\infty}\frac{\frac{\left(1+\frac{1}{j}\right)^{q-1}}{\left(1+\frac{q-1}{j-\frac{p}{q}}\right)}\cdot\frac{\left(1+\frac{1}{j}\right)^{q-1}}{\left(1+\frac{q-1}{j+\frac{p}{q}}\right)}}{\frac{\left(1+\frac{1}{j}\right)^{q-\frac{p}{q}}}{\left(1+\frac{q-\frac{p}{q}}{j}\right)}\cdot\frac{\left(1+\frac{1}{j}\right)^{q+\frac{p}{q}}}{\left(1+\frac{q+\frac{p}{q}}{j}\right)}} \qquad (3.15)\\
&=\frac{(q^4-p^2)p\pi}{q^3}\cdot\prod_{j=1}^{\infty}\frac{\left(1-\frac{p^2}{j^2\,q^2}\right)}{\left(1+\frac{1}{j}\right)^2}\left(\frac{1}{1-\frac{q}{q(j+q)-p}}\right)\left(\frac{1}{1-\frac{q}{q(j+q)+p}}\right)\\
&\Rightarrow\frac{q^3}{(q^4-p^2)p\pi}\sin\left(\frac{p\pi}{q}\right)\\
&=\prod_{j=1}^{\infty}\frac{\left(1-\frac{p^2}{j^2\,q^2}\right)}{\left(1+\frac{1}{j}\right)^2}\left(\frac{1}{1-\frac{q}{q(j+q)-p}}\right)\left(\frac{1}{1-\frac{q}{q(j+q)+p}}\right)\\
&=\prod_{j=1}^{\infty}\frac{(j^2\,q^2-p^2)(qj+q^2-p)(qj+q^2+p)}{q^2(j+1)^2(qj+q^2-q-p)(qj+q^2-q+p)}\\
&=\prod_{j=1}^{\infty}\frac{(j^2\,q^2-p^2)[q(j+q)-p][q(j+q)+p]}{q^2(j+1)^2[q(j+q-1)-p][q(j+q-1)+p]}\\
&=\prod_{j=1}^{\infty}\frac{(q^2j^2-p^2)[q^2(j+q)^2-p^2]}{q^2(j+1)^2[q^2(j+q-1)^2-p^2]},
\end{aligned}$$

which are the desired results. □

Example 3.12. Set $p=1$ and $q=2$ in Corollary 3.11

$$\frac{8}{15\pi}=\prod_{j=1}^{\infty}\frac{(2j-1)(2j+5)}{4(j+1)^2}$$
$$=\frac{1\cdot 7}{4\cdot 2^2}\cdot\frac{3\cdot 9}{4\cdot 3^2}\cdot\frac{5\cdot 11}{4\cdot 4^2}\cdot\frac{7\cdot 13}{4\cdot 5^2}\cdot\frac{9\cdot 15}{4\cdot 6^2}\cdot\frac{11\cdot 17}{4\cdot 7^2}\cdot\cdots$$

Example 3.13. Set $p=1$ and $q=3$ in Corollary 3.11

$$\frac{27\sqrt{3}}{160\pi}==\prod_{j=1}^{\infty}\frac{(3j+8)(3j+10)(9j^2-1)}{9(j+1)^2(3j+5)(3j+7)}$$
$$=\frac{11\cdot 13\cdot 8}{9\cdot 2^2\cdot 8\cdot 10}\cdot\frac{14\cdot 16\cdot 35}{9\cdot 3^2\cdot 11\cdot 13}\cdot\frac{17\cdot 19\cdot 80}{9\cdot 4^2\cdot 14\cdot 16}\cdot\frac{20\cdot 22\cdot 143}{9\cdot 5^2\cdot 17\cdot 19}\cdot\frac{23\cdot 25\cdot 224}{9\cdot 6^2\cdot 20\cdot 22}\cdot\cdots$$

Example 3.14. Set $p=1$ and $q=4$ in Corollary 3.11

$$\frac{32\sqrt{2}}{255\pi}=\prod_{j=1}^{\infty}\frac{(4j+15)(4j+17)(16j^2-1)}{16(j+1)^2(4j+11)(4j+13)}$$
$$=\frac{19\cdot 21\cdot 15}{16\cdot 2^2\cdot 15\cdot 17}\cdot\frac{23\cdot 25\cdot 63}{16\cdot 3^2\cdot 19\cdot 21}\cdot\frac{27\cdot 29\cdot 143}{16\cdot 4^2\cdot 23\cdot 25}\cdot\frac{31\cdot 33\cdot 255}{16\cdot 5^2\cdot 27\cdot 29}\cdot\frac{35\cdot 37\cdot 399}{16\cdot 6^2\cdot 31\cdot 33}\cdot\cdots$$

Example 3.15. Set $p=3$ and $q=4$ in Corollary 3.11

$$\frac{32\sqrt{2}}{741\pi}=\prod_{j=1}^{\infty}\frac{(4j+13)(4j+19)(16j^2-9)}{16(j+1)^2(4j+9)(4j+15)}$$
$$=\frac{17\cdot 23\cdot 7}{16\cdot 2^2\cdot 13\cdot 19}\cdot\frac{21\cdot 27\cdot 55}{16\cdot 3^2\cdot 17\cdot 23}\cdot\frac{25\cdot 31\cdot 135}{16\cdot 4^2\cdot 21\cdot 27}\cdot\frac{29\cdot 35\cdot 247}{16\cdot 5^2\cdot 25\cdot 31}\cdot\frac{33\cdot 39\cdot 391}{16\cdot 6^2\cdot 29\cdot 35}\cdot\cdots$$

Example 3.16. Set $p=1$ and $q=5$ in Corollary 3.11

$$\frac{125}{1248\pi}\sqrt{\frac{5}{2}-\frac{\sqrt{5}}{2}}=\prod_{j=1}^{\infty}\frac{(5j+24)(5j+26)(25j^2-1)}{25(j+1)^2(5j+19)(5j+21)}$$
$$=\frac{29\cdot 31\cdot 24}{25\cdot 2^2\cdot 24\cdot 26}\cdot\frac{34\cdot 36\cdot 99}{25\cdot 3^2\cdot 29\cdot 31}\cdot\frac{39\cdot 41\cdot 224}{25\cdot 4^2\cdot 34\cdot 36}\cdot\frac{44\cdot 46\cdot 399}{25\cdot 5^2\cdot 39\cdot 41}\cdot\frac{49\cdot 51\cdot 624}{25\cdot 6^2\cdot 44\cdot 46}\cdot\cdots$$

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