GRADSHTEYN and RYZHIK Page 578 , Integral 4.371.1

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Abstract.
Some remarks on the integral 4.371.1 in G&R tables of integrals.

En la Referencia [3] (Gradshteyn & Ryzhik , 7th Ed.), pagina 578, aparece la integral:

$$\int_0^\infty \frac{\ln x}{\cosh x}\,dx = \pi \ln\left(\frac{\Gamma(3/4)}{\Gamma(1/4)}\sqrt{2\pi}\right) \tag{1}$$

La integral (1) fue evaluada por C.J. Malmsten en 1842. Algunas integrales equivalentes son:

$$\frac{1}{2}\int_0^\infty \frac{\ln x}{\cosh x}\,dx = \int_1^\infty \frac{\ln\ln x}{1+x^2}\,dx = \int_0^1 \frac{\ln(-\ln x)}{1+x^2}\,dx = \int_{\pi/4}^{\pi/2} \ln\ln\tan x\,dx = \frac{\pi}{2}\ln\left(\frac{\Gamma(3/4)}{\Gamma(1/4)}\sqrt{2\pi}\right) \tag{2}$$

En esta nota exploramos la integral:

$$I = \int_0^1 \frac{-\ln x}{\cosh x}\,dx = \int_0^1 \frac{1}{\cosh x}\ln\frac{1}{x}\,dx \tag{3}$$

$$I = 0.95138341823661468132651717 91... \tag{4}$$

$$I = \cfrac{1}{1+\cfrac{1}{19+\cfrac{1}{1+\dots}}} = [0,1,19,1,1,3,8,1,1,7,1,2,6,1,3,\dots] \tag{5}$$

Algunas observaciones sobre la integral: $I = -\int_0^1 \frac{\ln x}{\cosh x}\,dx$

Entry 1:

$$I = -\int_0^1 \frac{\ln x}{\cosh x}\,dx = -\int_0^1 \frac{\ln(1-x)}{\cosh(1-x)}\,dx \tag{6}$$

Entry 2:

$$I = \sum_{n=0}^\infty \frac{(-1)^n E_n}{(2n)!(2n+1)^2} \tag{7}$$

En (7) $E_n = \{1,1,5,61,1385,50521,\dots\}$, son los números de Euler.

Entry 3:

$$I=\int_0^1 \frac{-\ln x}{\cosh x}dx=2\sum_{n=0}^{\infty}\frac{(-1)^n\,\Gamma(0,2n+1)}{2n+1}-\pi\ln\left(\frac{\Gamma(3/4)}{\Gamma(1/4)}\sqrt{2\pi}\right) \tag{8}$$

En (8) $\Gamma(0,2n+1)$, es la función gamma incompleta:

$$\Gamma(0,2n+1)=\frac{e^{-2n-1}}{2n+1+}\,\frac{1}{1+}\,\frac{1}{2n+1+}\,\frac{2}{1+}\,\frac{2}{2n+1+}\,\frac{3}{1+\ldots} \tag{9}$$

Entry 4:

$$I=\int_1^{\infty}\frac{\ln x}{x^2}\operatorname{sech}\frac{1}{x}\,dx \tag{10}$$

$$I=\int_0^{\infty}\frac{x\,e^{-x}}{\cosh e^{-x}}dx \tag{11}$$

$$I=-\int_0^{\tanh 1}\frac{\ln\tanh^{-1}x}{\sqrt{1-x^2}}dx \tag{12}$$

$$I=\int_0^{\cosh 1}\frac{-\ln\cosh^{-1}x}{x\sqrt{x^2-1}}dx=-\int_0^{\cosh 1}\frac{\ln\ln\left(x+\sqrt{x^2-1}\right)}{x\sqrt{x^2-1}}dx \tag{13}$$

$$I=\int_0^{\sinh 1}\frac{-\ln\sinh^{-1}x}{1+x^2}dx=-\int_0^{\sinh 1}\frac{\ln\ln\left(x+\sqrt{x^2+1}\right)}{1+x^2}dx \tag{14}$$

$$I=\int_0^1\frac{\tan^{-1}\sinh x}{x}dx \tag{15}$$

$$I=4\int_0^1\frac{\tanh^{-1}x}{(1+x)^2}\operatorname{sech}\left(\frac{1-x}{1+x}\right)dx \tag{16}$$

$$I=1-\int_0^1\frac{1}{x}\tan^{-1}\left(\frac{\tan x-\sinh x}{1+\tan x\sinh x}\right)dx \tag{17}$$

$$I=-2\int_1^e\frac{\ln\ln x}{1+x^2}dx \tag{18}$$

$$I=-2\int_{e^{-1}}^1\frac{\ln(-\ln x)}{1+x^2}dx \tag{19}$$

$$I=-4\int_0^1\frac{x\ln x}{\cosh x^2}dx \tag{20}$$

$$I=4\int_0^{\infty}\frac{x\,e^{-2x}}{\cosh e^{-2x}}dx \tag{21}$$

$$I=-n^2\int_0^1\frac{x^{n-1}\ln x}{\cosh x^n}dx \quad ,n>0 \tag{22}$$

Entry 5.

$$I\sim\int_0^N\frac{x\,e^{-x}}{\cosh e^{-x}}dx+\left(1+N\right)e^{-N} \quad ,N\to\infty \tag{23}$$

Entry 6.

$$I=\int_0^N\frac{x\,e^{-x}}{\cosh e^{-x}}dx+\sum_{n=0}^{\infty}\frac{\left(-1\right)^n E_n\left(\left(2n+1\right)N+1\right)e^{-\left(2n+1\right)N}}{\left(2n\right)!\left(2n+1\right)^2} \quad ,N\geq 0 \tag{24}$$

Entry 7.

$$I=2\sum_{n=1}^{\infty}n\tan^{-1}\left(\frac{e^{-e^{-n-1}}-e^{-e^{-n}}}{1+e^{-e^{-n}\left(1+e^{-1}\right)}}\right)+\sum_{n=0}^{\infty}e^{-n}\int_0^1\frac{xe^{-x}}{\cosh\left(e^{-x-n}\right)}dx \tag{25}$$

Entry 8.

$$I=\int_0^1\frac{-\ln x}{\cosh x}dx=\int_0^{\infty}e^{-x\cosh e^{-x\cosh e^{-x...}}}dx=$$

$$=\int_0^{\infty}\exp\left(-x\cosh\left(\exp\left(-x\cosh\left(\exp\left(-x...\right)\right)\right)\right)\right)dx \tag{26}$$

Entry 9. Para $0<a\leq 1$, se tiene

$$I=\int_0^a\frac{-\ln x}{\cosh x}dx+2\ln a\tan^{-1}e^a+2\int_a^1\frac{\tan^{-1}e^x}{x}dx \tag{27}$$

$$I=\int_0^a\frac{-\ln x}{\cosh x}dx-2\ln a\tan^{-1}e^{-a}-2\int_a^1\frac{\tan^{-1}e^{-x}}{x}dx \tag{28}$$

$$I=-\frac{\pi}{2}\ln a+\int_0^a\frac{\tan^{-1}\sinh x}{x}dx-\int_a^1\frac{1}{x}\tan^{-1}\frac{1}{\sinh x}dx \tag{29}$$

Entry 10.

$$I=\frac{\pi\gamma}{2}+2\sum_{n=1}^{\infty}\frac{\left(-1\right)^n\left(1-e^{-2n-1}\right)\ln\left(2n+1\right)}{2n+1}+2\sum_{n=0}^{\infty}\frac{\left(-1\right)^n}{2n+1}\int_{2n+1}^{\infty}e^{-x}\ln x\,dx \tag{30}$$

donde γ es la constante de Euler-Mascheroni:

$$\gamma=\lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}-\ln n\right)=0.577215... \tag{31}$$

Entry 11.

$$I=2\ln 2\tan^{-1}\left(\frac{\sqrt{e}-1}{\sqrt{e}+1}\right)+\sum_{n=0}^{\infty}\frac{\left(-1\right)^n E_n 2^{-2n-1}}{\left(2n\right)!\left(2n+1\right)^2}+\frac{1}{\cosh 1}\sum_{n=0}^{\infty}\frac{2^{-n-2}}{n+2}c_n \tag{32}$$

donde

$$c_0 = 1 \quad , c_n = \frac{1}{n+1} - \frac{1}{e+e^{-1}} \sum_{k=1}^{n} \frac{(-1)^k \left(e + (-1)^k e^{-1}\right)}{k!} c_{n-k} \,, \quad n \in \mathbb{N} \tag{33}$$

Entry 12.

$$I = 2\tan^{-1}\left(\frac{1-e^{-e^{-1}}}{1+e^{-e^{-1}}}\right) + \sum_{n=0}^{\infty} \frac{(-1)^n E_n e^{-2n-1}}{(2n)!(2n+1)^2} + \int_0^1 \frac{x e^{-x}}{\cosh e^{-x}} dx \tag{34}$$

Entry 13.

$$I = i \sum_{n=0}^{\infty} (-1)^n \left(Li_2\left(1 + \frac{2i}{(2n+1)\pi}\right) - Li_2\left(1 - \frac{2i}{(2n+1)\pi}\right)\right) \tag{35}$$

donde $i = \sqrt{-1}$, y

$$Li_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2} \quad , |x| \le 1 \tag{36}$$

Entry 14.

$$I = (\ln 2)\sin^{-1}\tanh 1 - \int_0^{\tanh 1} \frac{1}{\sqrt{1-x^2}} \ln\ln\left(\frac{1+x}{1-x}\right) dx \tag{37}$$

$$I = (\ln 2)\sin^{-1}\tanh 1 - \int_1^{e^2} \frac{\ln\ln x}{\sqrt{x}(1+x)} dx \tag{38}$$

Entry 15.

$$I = \lim_{n\to\infty} \sum_{k=1}^{\infty} \frac{\tan^{-1}\sinh(k/n)}{k} \tag{39}$$

Entry 16.

$$I = \sum_{n=0}^{\infty} \frac{1}{(2n)!(2n+1)^2} - \int_0^{\infty} x e^{-x} \tanh e^{-x} \sinh e^{-x} \, dx \tag{40}$$

$$I = \sum_{n=0}^{\infty} \frac{1}{(2n)!(2n+1)^2} + \int_0^1 \ln x \tanh x \sinh x \, dx \tag{41}$$

Entry 17.

$$\begin{aligned} I = {} & \sum_{n=0}^{\infty} \frac{1}{(2n)!(2n+1)^2} - 2\sum_{n=1}^{\infty} n \cosh\left(\frac{e^{-n}(1+e^{-1})}{2}\right) \sinh\left(\frac{e^{-n}(1-e^{-1})}{2}\right) + \\ & + \sum_{n=1}^{\infty} n \tan^{-1}\left(\frac{\sinh e^{-n} - \sinh e^{-n-1}}{1 + \sinh e^{-n} \sinh e^{-n-1}}\right) - \sum_{n=0}^{\infty} e^{-n} \int_0^1 x e^{-x} \tanh e^{-n-x} \sinh e^{-n-x} \, dx \end{aligned} \tag{42}$$

Entry 18.

$$I = 1 + 2\sum_{n=0}^{\infty} \binom{2n}{n} \frac{2^{-2n}}{n+1} \int_0^1 x \ln x \left(\tanh x^2\right)^{2n+2} dx \tag{43}$$

Entry 19.

$$I=4\sum_{n=1}^{\infty}\frac{1}{n}\int_0^1\frac{\left(1-x^x\right)^n}{\cosh x^2}dx \tag{44}$$

$$I=8\sum_{n=0}^{\infty}\frac{1}{2n+1}\int_0^1\frac{1}{\cosh x^2}\left(\frac{1-x^x}{1+x^x}\right)^{2n+1}dx \tag{45}$$

Entry 20. Si $f\left(n\right)=-\int_0^1 e^{-(2n+1)x}\ln x\,dx \quad ,n=0,1,2,3,\ldots$, entonces

$$I=\frac{4}{3+e^{-2}}\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{1+e^{-2}}{3+e^{-2}}\right)^{n-k}\left(\frac{2}{3+e^{-2}}\right)^k f\left(k\right) \tag{46}$$

Entry 21.

$$I+\frac{e^{-1}}{\cosh e^{-1}}=-\int_0^{e^{-1}}\frac{\ln x}{\cosh x}dx+\int_0^{\operatorname{sech} e^{-1}} e^{-x\cosh e^{-x\cosh e^{-x\cdots}}}dx \tag{47}$$

Entry 22. Si $i=\sqrt{-1}$, entonces

$$I=\sum_{n=1}^{\infty}\frac{n\,e^{-n}}{\cosh e^{-n}}+\int_0^{\infty}\frac{x}{e^{2\pi x}-1}\left(\frac{e^{ix}}{\cosh e^{ix}}+\frac{e^{-ix}}{\cosh e^{-ix}}\right)dx \tag{48}$$

Referencias

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