Equivalent of Brocard's problem

Hajime Mashima

August 15, 2018

Abstract

Brocard's problem was presented by Henri Brocard in 1876 and 1885. $n! + 1 = m^2$. The number that satisfies this is called "Brown numbers" and three are known: $(n, m) = (4, 5), (5, 11), (7.71)$.

$$n! + 1 = m^2$$

$$(a + b)^2 = (a - b)^2 + 2^2 ab \tag{1}$$

$$(a - b)^2 = 1$$
$$a - b = \pm 1$$
$$a = b \pm 1$$

上式を (1) へ代入する。

$$((b \pm 1) + b)^2 = ((b \pm 1) - b)^2 + 2^2 (b \pm 1) b$$
$$(2b \pm 1)^2 = 1 + 4b(b \pm 1)$$
$$4b(b \pm 1) = (2b \pm 1)^2 - 1$$
$$n! = (2b \pm 1)^2 - 1$$
$$n! = 4b(b \pm 1)$$

$$4 \cdot 2 \cdot 3 = 2^3 (1 + 2)$$
$$4 \cdot 3 \cdot 4 = 2^3 (1 + 2 + 3)$$
$$4 \cdot 4 \cdot 5 = 2^3 (1 + 2 + 3 + 4)$$
$$4 \cdot 5 \cdot 6 = 2^3 (1 + 2 + 3 + 4 + 5)$$
$$\vdots$$
$$n! = 4b(b \pm 1) = 2^3 (1 + 2 + 3 + 4 + 5 + \cdots)$$

よって、同値問題は以下の等式を満たす n の存在である。

$$2^{-3} n! = 1 + 2 + 3 + 4 + 5 + \cdots$$

Brown numbers

$$2^{-3} \cdot 4! = 2^{-3} \cdot 24 = 3 = 1 + 2$$
$$2^{-3} \cdot 5! = 2^{-3} \cdot 120 = 15 = 1 + 2 + 3 + 4 + 5$$
$$2^{-3} \cdot 7! = 2^{-3} \cdot 5040 = 630 = 1 + 2 + 3 + 4 + 5 + \cdots + 35$$