The Complexity of Robust and Resilient k-Partition Problems

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Abstract

In this paper, we study a k-partition problem where a set of agents must be partitioned into a fixed number of k non-empty coalitions. The value of a partition is the sum of the pairwise synergies inside its coalitions. Firstly, we aim at computing a partition that is robust to failures from any set of agents with bounded size. Secondly, we focus on resiliency: when a set of agents fail, others can be moved to replace them. We settle the computational complexity of decision problem ROBUST-k-PART as complete for class Σ_2^P. We also conjecture that resilient k-partition is complete for class Σ_3^P under simultaneous replacements, and for class PSPACE under sequential replacements.

Model

We introduce relevant notation for k-partition problems. Given $i \in \mathbb{N}_{\geq 1}$, set $[i]$ are integers $\{1, 2, \ldots, i\}$. Set $N = \{1, \ldots, n\}$ are the agents. Given fixed integer $k \in \mathbb{N}_{\geq 1}$, a k-partition $\pi = \{C_1, \ldots, C_k\}$ of set N is a collection of coalitions $C_i \subseteq N$ such that $\forall C, D \in \pi,\ C \cap D = \emptyset$ and $N = \bigcup_{i \in [k]} C_i$. No coalition shall be empty, otherwise we call π a partition. For agent $i \in N$, let $\pi(i)$ denote the coalition to which she belongs.

Synergy function $w : N \times N \to \mathbb{Z}$ maps any two agents $i, j \in N$ to their synergy $w(i, j) \in \mathbb{Z}$, the additional value they obtain for being in the same coalition. Function w satisfies $w(i, j) = w(j, i)$ and $w(i, i) = 0$ for any $i, j \in N$. The value that an agent i gives to a k-partition π is defined by $v_i(\pi) = \sum_{j \in \pi(i)} w(i, j)$, the sum of synergies between her and the other agents in the same coalition. It follows that the utilitarian value $v(\pi) = \frac{1}{2} \sum_{i \in N} v_i(\pi)$ can also be defined as $v(\pi) = \sum_{C \in \pi} \sum_{\{i,j\} \subseteq C} w(i, j)$, the sum of synergies between pairs in a same coalition. If some coalition is empty, then π is not a k-partition, but a partition: in that case, we extend v to $v(\pi) = -\infty$.

Any set $M \subseteq N$ of at most $m \in \mathbb{N}_{\geq 0}$ agents might fail. Given a k-partition π of set N, resulting partition π_{-M} of set $N \setminus M$ is defined as $\{C \setminus M \mid C \in \pi\}$ (and might contain empty coalitions). The robustness value of π is

$$v^-(\pi) = \min_{M \subseteq N, |M| \leq m} \{v(\pi_{-M})\}.$$

To obtain $v^-(\pi) \neq -\infty$, every coalition in π shall contain at least $m + 1$ agents, so that no coalition of π_{-M} is empty.

Once agents $M \subseteq N$ fail and $N \setminus M$ remain, replacements are possible, by moving any subset $R \subseteq N \setminus M$ of m other agents, with a replacement function $\rho : R \to \pi$. For agent $i \in R$, $\rho(i)$ is her new coalition. For coalition $C \in \pi$, $\rho(C)$ are the agents newly moved to C. Hence, given k-partition $\pi = \{C_1, \ldots, C_k\}$, set $M \subseteq N$ of m failing agents and replacements $R \subseteq N \setminus M, |R| \leq m$ and $\rho : R \to \pi$, the repaired partition $\pi_{-M}^{R,\rho}$ is $\{(C \cup \rho(C)) \setminus (M \cup R) \mid C \in \pi\}$. Given k-partition π, its resiliency value is

$$v^+(\pi) = \min_{M} \max_{R,\rho} \left\{ v\left(\pi_{-M}^{R,\rho}\right) \right\},$$

where $M \subseteq N$ has size $|M| \leq m$, $R \subseteq N \setminus M$ has size $|R| \leq m$, and $\rho : R \to \pi$ is a map. As long as $2m \leq n$, given a k-partition π and a failing set M, there always exists a replacement R, ρ that repairs partition π_{-M} into a k-partition $\pi_{-M}^{R,\rho}$; hence, one always has $v^+(\pi) \neq -\infty$.

In such a *simultaneous* replacement, all agents in M fail at once, after which all agents in R are moved by ρ. We shall only require from R, ρ that for every coalition $C \in \pi$, $(C \cup \rho(C)) \setminus (M \cup R)$ is non-empty. In a *sequential* replacement, there are up to m rounds. The idea is that in each round $t \in [m]$, one new agent f_t fails, and then one remaining agent r_t is moved to coalition $\rho_t(r_t)$. Let $M_t = \{f_1, \ldots, f_t\}$ denote the agents who failed up to round t. It satisfies $f_t \in N \setminus M_{t-1}$ and $|M_t| = t$. Let $R_t = \{\ldots, r_t\}$ and $\rho_t : R_t \setminus M_t \to \pi$ denote replacements up to round t. While $r_t \in N \setminus M_t$ always holds, a same agent can be moved in two rounds, and an agent who moved can fail later, so that only $|R_t \setminus M_t| \leq t$ is true, instead of $|R_t \setminus M_t| = t$. Crucially, on any round $t \in [m]$, $\pi_{-M_t}^{R_t,\rho_t}$ has no empty coalition. In other words, if failure f_t empties a coalition, then it shall be replaced by $r_t, \rho_t(r_t)$. A valid strategy σ maps any history $(M_t, R_{t-1}, \rho_{t-1})$ (failures set M_t and replacements R_{t-1}, ρ_{t-1}) to next replacement $(r_t, \rho_t(r_t)) = \sigma(M_t, R_{t-1}, \rho_{t-1})$, while never letting any coalition of $\pi_{-M_t}^{R_t,\rho_t}$ empty, for any round t. Strategy σ, for any sequence of failures $f_1, \ldots, f_m$ alternated with the replacements of σ, induces final replacements R_m, ρ_m.

We assume that the following concepts are common knowledge: decision problem, length function, polynomial-time many-to-one reduction, hardness, completeness and classes P, NP, coNP, Σ_2^P, Π_2^P, Σ_3^P, PH and PSPACE.

Definition 1. We study this sequence of decision problems:

- ROBUST-k-PART/VERIF
  Given n agents, synergies w, a number m of failing agents, a k-partition π and a threshold $\theta \in \mathbb{Z}$, does robustness value $v^-(\pi)$ satisfy $v^-(\pi) \geq \theta$?
- ROBUST-k-PART
  Given n agents, synergies w, a number m of failing agents and a threshold $\theta \in \mathbb{Z}$, is there a k-partition π with robustness value satisfying $v^-(\pi) \geq \theta$?
- SIMRES-k-PART/VERIF2
  Given n agents, synergies w, k-partition π, set $M \subseteq N$ of m failures and threshold $\theta \in \mathbb{Z}$, is there a replacement $R \subseteq N \backslash M, |R| \leq m, \rho : R \to \pi$ such that $v(\pi_{-M}^{R,\rho}) \geq \theta$?
- SIMRES-k-PART/VERIF
  Given n agents, synergies w, k-partition π and threshold $\theta \in \mathbb{Z}$, does resiliency value $v^+(\pi)$ satisfy $v^+(\pi) \geq \theta$?
- SIMRES-k-PART
  Given n agents, synergies w and threshold $\theta \in \mathbb{Z}$, is there a k-partition π with resiliency value $v^+(\pi) \geq \theta$?
- SEQRES-k-PART/STR
  Given n agents, synergies w, k-partition π and threshold $\theta \in \mathbb{Z}$, does a valid strategy σ exist such that for any sequence of failures $f_1, \dots, f_m$, the final replacements R_m, ρ_m induced by σ are such that $v(\pi_{-M_m}^{R_m,\rho_m}) \geq \theta$?
- SEQRES-k-PART
  Given n agents, synergies w and threshold $\theta \in \mathbb{Z}$, does a k-partition π and valid strategy σ exist such that for any sequence of failures $f_1, \dots, f_m$, the final replacements R_m, ρ_m induced by σ are such that $v(\pi_{-M_m}^{R_m,\rho_m}) \geq \theta$?

The Complexity of Robust k-Partition

In this section, we settle the computational complexity of robust k-partition as complete for the second level of the polynomial hierarchy.

Theorem 1. ROBUST-k-PART/VERIF *is coNP-complete.*
(It holds even for $k = 1$, synergies w in $\{0, 1\}$ and $\theta = 1$.)

Proof. Decision problem ROBUST-k-PART/VERIF, given n agents, synergies w, a number m of failing agents, a k-partition π and a threshold $\theta \in \mathbb{Z}$, asks whether:

$$\forall M \subseteq N, |M| \leq m, \quad v(\pi_{-M}) \geq \theta.$$

This problem is in class coNP, since for any no-instance, a failing set M such that $v(\pi_{-M}) \leq \theta - 1$ is a no-certificate verifiable in polynomial-time. We show coNP-hardness by complementary reduction from MINVERTEXCOVER. Let graph $G = (V, E)$ and threshold $m \in \mathbb{N}$ be any instance of MINVERTEXCOVER, which asks whether there exists a subset $U \subseteq V, |U| \leq m$ such that $\forall \{i, j\} \in E, i \in U$ or $j \in U$, i.e. every edge is covered by a vertex in U. We reduce it to a ROBUST-k-PART/VERIF instance with agents $N \equiv V$, synergies $w(i, j) \in \{0, 1\}$ equal to one if and only if $\{i, j\} \in E$ (otherwise zero) and threshold $\theta = 1$. Our k-partition π is the grand coalition ($k = 1$).

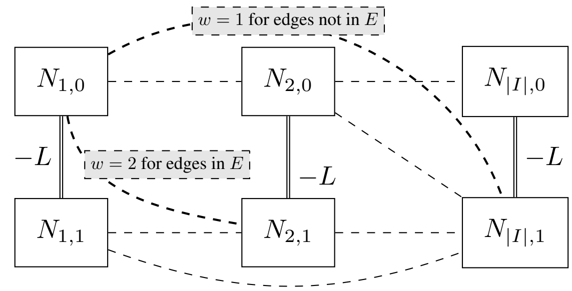

Figure 1: We reduce any instance of MAXMINVERTEXCOVER $G = (V, E)$ where $V = \bigcup_{i \in I}(V_{i,0} \cup V_{i,1})$ and m is a threshold, to the following instance of ROBUST-k-PART. Agents $N \equiv V$ are identified with vertices, hence can be partitionned the same into $N = \bigcup_{i \in I}(N_{i,0} \cup N_{i,1})$. We fix $k = 2$ coalitions and choose a large number L, e.g. $L = n^2$. For every $\{i, j\} \in \mathcal{P}_2(N)$, if $\{i, j\} \in E$, we define synergy $w(i, j) = 2$; otherwise if $\{i, j\} \notin E$, we define $w(i, j) = 1$; but for every $\ell \in I$ and every $(i, j) \in N_{\ell,0} \times N_{\ell,1}$, where we define synergy $w(i, j) = -L$. Up to $2m$ agents might fail, and the threshold is defined in the proof.

(yes⇒no) If there exists a vertex cover $U \subseteq V, |U| \leq m$, then failing set $M \equiv U$ is such that any synergy $w(i, j)$ equal to one has $i \in M$ or $j \in M$, hence disappears from π in π_{-M}, and value is $v(\pi_{-M}) \leq 0$.

(yes⇐no) If there is a failing set $M \subseteq N, |M| \leq m$ such that $v(\pi_{-M}) \leq 0$, then any synergy $w(i, j)$ equal to one has $i \in M$ or $j \in M$. Therefore, $U \equiv M$ is a vertex cover.

If $k > 1$, this result still holds (when coalitions are larger than any failing set). It suffices to copy the construction above into k identical coalitions and ask for at most km failing agents. The yes⇒no part is trivially the same. Concerning yes⇐no, even though a failing set of size at most km might be unequally distributed between the k identical coalitions, the coalition containing the smallest failing subset gives a vertex cover smaller than m. ☐

Theorem 2. ROBUST-k-PART *is Σ_2^P-complete.*
(It holds even for $k = 2$ coalitions and $w \in \{-n^2, 1, 2\}$)

Proof. Decision problem ROBUST-k-PART, given n agents, a number k of coalitions, synergies w, a number m of failures and a threshold $\theta \in \mathbb{Z}$, asks whether:

$$\exists\ k\text{-partition } \pi, \quad \forall\ M \subseteq N, |M| \leq m, \quad v(\pi_{-M}) \geq \theta.$$

It lies in class Σ_2^P, since for yes-instances, such a k-partition π is a certificate that can be verified by an NP-oracle on remaining coNP problem ROBUST-k-PART/VERIF. We show Σ_2^P-hardness by a complementary reduction from Π_2^P-complete problem MAXMINVERTEXCOVER, defined as follows. Given a finite graph $G = (V, E)$ which vertices are partitionned by a finite index set I into $V = \bigcup_{i \in I}(V_{i,0} \cup V_{i,1})$, for a function $p : I \to \{0, 1\}$, we define $V^{(p)} = \bigcup_{i \in I} V_{i,p(i)}$ and $G^{(p)} = G[V^{(p)}]$. Given a threshold

$m \in \mathbb{N}$, the problem asks whether:

$$\forall p : I \to \{0,1\}, \exists U \subseteq V^{(p)}, |U| \leq m, U \text{ vertex covers } G^{(p)}.$$

where "U vertex covers $G^{(p)}$" means $\forall \{u, v\} \in E[V^{(p)}]$, $u \in U$ or $v \in U$. Since edges between $V_{i,0}$ and $V_{i,1}$ are never relevant, we can shave E from them. One can assume that all $V_{i,j}$ sets have the same size, hence whatever p, set $V^{(p)}$ has constant size n (and $|V| = 2n$). The reduction is described in Figure 1. That construct is a no-instance when:

$$\forall 2\text{-partition } \pi, \exists M \subseteq N, |M| \leq 2m, v(\pi_{-M}) \leq f_{n,m}(m),$$

where $f_{n,m} : [0, 2m] \to [0, n^2]$ is defined later. First of all, this condition is trivially satisfied on 2-partitions π where two agents $(i, j) \in N_{\ell,0} \times N_{\ell,1}$ are in the same coalition and value $v(\pi_{-\emptyset})$ incurs synergy $w(i, j) = -L$. Hence, the interesting part of this condition is on the other 2-partitions: the *proper* 2-partitions $\pi = \{C_1, C_2\}$, that can be characterized by a function $q : I \to \{0, 1\}$ such that coalitions are $C_1 = \bigcup_{i \in I} N_{i,q(i)}$ and $C_2 = \bigcup_{i \in I} N_{i,1-q(i)}$, and $|C_1| = |C_2| = n$. Since the remaining synergies are positive, we can focus on failures of size $|M| = 2m$. Function $f_{n,m}$ maps $x \in [0, 2m]$ to the number of in-coalition agent pairs in a proper 2-partition $\pi = \{C_1, C_2\}$ when $2m$ agents fail: x in C_1 and $2m - x$ in C_2.

$$\begin{aligned} f_{n,m}(x) &= n(n-1)/2 - \sum\nolimits_{i=1}^{x} (n-i) \\ &+ n(n-1)/2 - \sum\nolimits_{j=1}^{2m-x} (n-j) \\ &= g_{n,m} + x(x - 2m), \end{aligned}$$

where $g_{n,m}$ is constant w.r.t. x. It is a strictly convex function with minimum $x = m$, hence for an integer $x \neq m$, inequality $f_{n,m}(m) + 1 \leq f_{n,m}(x)$ holds. Assuming all remaining synergies have value 1 (instead of 1 or 2) yields $f_{n,m}(x)$ as a lower bound on $v(\pi_{-M})$. Therefore, the main condition can only be satisfied by *balanced* failures $M = M_1 \cup M_2$ such that $M_1 \subseteq C_1$, $M_2 \subseteq C_2$ and $|M_1| = |M_2| = m$. We can now proceed with the proof.

(yes⇒no) Any subgraph $G^{(p)}$ admits a vertex cover $U \subseteq V^{(p)}$ with size $|U| \leq m$. Let us show that any proper 2-partition $\pi = \{C_1, C_2\}$ (characterized by function $q : I \to \{0, 1\}$) can be failed to $f_{n,m}(m)$. Let $M_1 \subseteq C_1$ correspond to the vertex cover of subgraph $G^{(q)}$ and $M_2 \subseteq C_2$ to $G^{(1-q)}$. Then, failing set $M = M_1 \cup M_2$ has size $|M| \leq 2m$, is balanced, and any agent pair i, j of synergy two in π has i or j in M, thanks to the vertex covers: $v(\pi_{-M}) = f_{n,m}(m)$.

(yes⇐no) Any proper 2-partition $\pi = \{C_1, C_2\}$ (characterized by function $q : I \to \{0, 1\}$) admits a well balanced failing set $M = M_1 \cup M_2$ such that $v(\pi_{-M}) \leq f_{n,m}(m)$. Then it must be the case that M_1 (and M_2) covers all the agent pairs of synergy two in C_1 (resp. C_2) that correspond to the edges of $G^{(q)}$ (resp. $G^{(1-q)}$). Let any function $p : I \to \{0, 1\}$. Then the proper 2-partition characterized by function p provides vertex cover $U \equiv M_1$ for $G^{(p)}$. □

The Complexity of Resilient k-Partition

We conjecture Theorems 3-7:

Theorem 3. SIMRES-k-PART/VERIF2 *is NP-complete.*

Theorem 4. SIMRES-k-PART/VERIF *is* Π_2^P*-complete.*

Theorem 5. SIMRES-k-PART *is* Σ_3^P*-complete.*

Theorem 6. SEQRES-k-PART/STR *is PSPACE-complete.*

Theorem 7. SEQRES-k-PART *is PSPACE-complete.*

Related Work

Partitioning of a set into (non-empty) subsets may also be referred as coalition structure formation of a set of agents into coalitions. When a number of coalitions k is required and there are synergies between vertices/agents, this problem is referred as k-cut, or k-way partition, where one minimizes the weight of edges/synergies between the coalitions, or maximizes it inside the coalitions. For positive weights and $k \geq 3$, this problem is NP-complete (Dahlhaus et al. 1992), when one vertex is fixed in each coalition. For positive weights and fixed k, a polynomial-time $O(n^{k^2} T(n, m))$ algorithm exists (Goldschmidt and Hochbaum 1994), when no vertex is fixed in coalitions, and where $T(n, m)$ is the time to find a minimum (s, t) cut on a graph with n vertices and m edges. When not too many negative synergies exist (that is, negative edges can be covered by $O(\log(n))$ vertices), an optimal k-partition can be computed in polynomial-time (Sless et al. 2018). Various formulations of the *robust* and *resilient* problem studied were initially proposed by (Kraus and Yokoo 2017). The complexity results in this paper result from an original work in July and August 2018, between Anisse Ismaili and Emi Watanabe.

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