On the properties of the two-sided Laplace transform and the Riemann hypothesis

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Abstract

We will show interesting properties of two-sided Laplace transform, mainly of positive even functions. Further, we will also prove that the Laguerre inequalities and generalized Laguerre inequalities are true and finally, the Riemann hypothesis is true.

1. Introduction

Remark:

We are using $\frac{\partial^n}{\partial z^n}$, $F'(z)$, $F^{(n)}(z)$ and D_z^n as the differential operators and choosing the most suitable notation for the case.

We will begin with the definition of the two-sided Laplace transform.[1] The Laplace transform of a real function $f(t)$ is defined as:

$$F(z) = \int_{-\infty}^{\infty} f(t) \cdot e^{-zt} dt \tag{1}$$

and the inverse transform is:

$$f(t) = \frac{1}{i2\pi} \int_{x-i\infty}^{x+i\infty} F(z) \cdot e^{zt} dz \tag{2}$$

where $z = x + iy$ for x and y real.

If $f(t)$ is even, then $F(-z) = F(z)$ and we can write

$$F(z) = \int_{-\infty}^{\infty} f(t) \cdot e^{-zt} dt = \int_{-\infty}^{\infty} f(t) \cdot e^{zt} dt = \int_{-\infty}^{\infty} f(t) \cdot \cosh(zt)\, dt = 2\int_{0}^{\infty} f(t) \cdot \cosh(zt)\, dt \tag{3}$$

From (3), we have the power series expansion of $F(z)$, namely

$$F(z) = \sum_{n=0}^{\infty} a_{2n} \cdot z^{2n} = a_0 + a_2 z^2 + a_4 z^4 + \cdots \tag{4}$$

where $a_{2n} = \frac{1}{(2n)!} \int_{-\infty}^{\infty} t^{2n} \cdot f(t) dt$. Note that if $f(t)$ is non-negative, a_{2n} is positive for all n.

$F(z)$ can be separated into the real and imaginary part, namely, $F(z) = u(x, y) + iv(x, y)$. Since $z = x + iy$, if $f(t)$ is an even function, we have

[1] Since we are only dealing with the two-sided Laplace transform, the term “two-sided” will be omitted afterward.

$$F(z) = \int_{-\infty}^{\infty} f(t) \cdot \cosh(xt) \cdot \cos(yt)\, dt + i \int_{-\infty}^{\infty} f(t) \cdot \sinh(xt) \cdot \sin(yt)\, dt \tag{5}$$

Therefore,

$$u(x, y) = \int_{-\infty}^{\infty} f(t) \cdot \cosh(xt) \cdot \cos(yt)\, dt \tag{6}$$

$$v(x, y) = \int_{-\infty}^{\infty} f(t) \cdot \sinh(xt) \cdot \sin(yt)\, dt \tag{7}$$

Notice that $u(x, y)$ is an even function and $v(x, y)$ is an odd function for both x and y, hence $v(0, y) = v(x, 0) = 0$. Therefore, we have $F(z)$ on the real axis, denoted $F(x)$, and on the imaginary axis, denoted $F(iy)$ as follows

$$F(x) = \int_{-\infty}^{\infty} f(t) \cdot \cosh(xt)\, dt = u(x, 0) \tag{8}$$

The function $F(x)$ is increasing log-convex and has a unique minimum at $x = 0$.

$$F(iy) = \int_{-\infty}^{\infty} f(t) \cdot \cos(yt)\, dt = u(0, y) \tag{9}$$

Thus, $F(iy)$ is real if y is real and all derivatives of $F(iy)$ are real if $f(t)$ is non-negative and even and since $u(x, 0)$ and $u(0, y)$ are even, we have $\frac{\partial}{\partial x} u(x, 0)\Big|_{x=0} = 0$ and $\frac{\partial}{\partial y} u(0, y)\Big|_{y=0} = 0$.

Since $|F(\mathrm{z})| = \sqrt{u^2 + v^2}$ and $|F(\mathrm{z})|^2 = u^2 + v^2$, we have

$$|F(\mathrm{x} + \mathrm{iy})| = |F(\mathrm{x} - \mathrm{iy})| = |F(-\mathrm{x} + \mathrm{iy})| = |F(-\mathrm{x} - \mathrm{iy})| \tag{10}$$

$$|F(x + iy)|^2 = |F(x - iy)|^2 = |F(-x + iy)|^2 = |F(-x - iy)|^2 \tag{11}$$

The equalities (10) and (11) are valid if $f(t)$ is even.

Another definition is $\int_{-\infty}^{\infty} f(t) \cdot e^{-izt} dt$ which can be often seen in the literature. It can be obtained by replacing z to iz in (1), hence we have

$$F(iz) = \int_{-\infty}^{\infty} f(t) \cdot e^{-izt} dt \tag{12}$$

and if $f(t)$ is even, from (12), we have

$$F(iz) = \int_{-\infty}^{\infty} f(t) \cdot \cos(zt)\, dt = 2 \int_{0}^{\infty} f(t) \cdot \cos(zt)\, dt \tag{13}$$

Note that $F(iz)$ is only the rotated function of $F(z)$ by $\pi/2$, thus we get $|F(i\mathrm{z})|$ from $|F(\mathrm{z})|$ by swapping x and y if $f(t)$ is even. This is also valid for $|F(i\mathrm{z})|^2$.

2. The positive definiteness and co-positive definiteness

Now, we consider the Laplace transform of $f(t)$ which is non-negative as well as even, meaning $f(-t) = f(t)$ and $f(t) \geq 0$ for all t.

Definition 1: Real-valued positive definiteness and co-positive definiteness

For real θ, a function $\varphi(\theta)$ is positive semi-definite if and only if $\sum_{n=1}^{N}\sum_{k}^{N} c_n c_k^* \varphi(\theta_n - \theta_k) \geq 0$ for any N which is non-zero positive integer, any complex value c_n and any real value θ_n. The star * denotes the complex conjugate.

Similarly, a function $\varphi(\theta)$ is co-positive semi-definite if and only if $\sum_{n=1}^{N}\sum_{k}^{N} c_n c_k^* \varphi(\theta_n + \theta_k) \geq 0$

Definition 2: Complex-valued positive definiteness and co-positive definiteness

A function $\varphi(z)$ is complex-valued positive semi-definite if and only if $\sum_{n=1}^{N}\sum_{k}^{N} c_n c_k^* \varphi(z_n - z_k^*) \geq 0$ for any N which is non-zero positive integer, any complex value c_n and any complex value z_n. Similarly, a function $\varphi(z)$ is co-positive semi-definite if and only if $\sum_{n=1}^{N}\sum_{k}^{N} c_n c_k^* \varphi(z_n + z_k^*) \geq 0$

For the sake of convenience, we define the differential operator $D_\theta \equiv \frac{\partial}{\partial\theta}$, thus $D_\theta^n \varphi(\theta) = \frac{\partial^n}{\partial\theta^{\,n}}\varphi(\theta)$.

The positive semi-definite and co-positive semi-definite functions have many properties. Some of them are:

For the positive-definite functions:

i. If $\varphi(\theta)$ is positive semi-definite, then $(-1)^n D_\theta^{2n}\varphi(\theta)$ is also positive semi-definite.
ii. $\varphi(0) \geq 0$ and $|\varphi(\theta)| \leq \varphi(0)$.

For the co-positive-definite functions:

i. If $\varphi(\theta)$ is co-positive semi-definite, then $D_\theta^{2n}\varphi(\theta)$ is also co-positive semi-definite.
ii. $\varphi(\theta) \geq 0$ and hence $\varphi(0) \geq 0$. From the property of $\varphi(\theta) \geq 0$, the real-valued co-positive semi-definite functions are real when θ real.

The properties of complex-valued (co-)positive semi-definite functions are similar to of the real-valued ones.

Observing eq. (1), $F(z)$ is clearly complex-valued co-positive semi-definite if $f(t) \geq 0$, since $\sum_{n=1}^{N}\sum_{k}^{N} c_n c_k^* \mathrm{F}(z_n + z_k^*) = \int_{-\infty}^{\infty} f(t) \cdot \sum_{n=1}^{N}\sum_{k}^{N} c_n c_k^* e^{-(z_n + z_k^*)t}\, dt = \int_{-\infty}^{\infty} f(t) \cdot |\sum_{n=1}^{N} c_n e^{-z_n t}|^2 dt \geq 0$.

Similarly, from (12), it can be shown that $F(iz)$ is complex-valued positive semi-definite when $f(t) \geq 0$.

Theorem 1:

Assuming $f(t)$ is non-negative, not necessarily even, $F(z)$ is entire if $F(z)$ has no pole on the real axis.

<u>Proof</u>

From (1), we have $F(x+iy) = \int_{-\infty}^{\infty} f(t) \cdot e^{-xt} \cdot e^{-iyt} dt$, which is the Fourier transform of $f(t) \cdot e^{-xt}$. Since $f(t) \cdot e^{-xt} \geq 0$ when $f(t) \geq 0$, For fixed x, $F(x+iy)$ is positive semi-definite for y by the Bochner's theorem. Hence $F(x) \geq |F(\mathrm{x}+\mathrm{iy})|$ and if $F(x)$ does not have any pole, $|F(\mathrm{x}+\mathrm{iy})|$ does not have any pole. Therefore $F(\mathrm{z})$ is entire.

Theorem 2:

Assuming $f(t)$ is non-negative, not necessarily even, $F(iz)$ is entire if $F(iz)$ has no pole on the imaginary axis.

Proof

Since $F(iz)$ is the rotated function of $F(z)$ by $\pi/2$, the proof is straightforward from theorem 1.

We showed that $F(\mathrm{x}+\mathrm{iy})$ is positive semi-definite for y. About x, however, $F(\mathrm{x}+\mathrm{iy})$ is generally neither positive semi-definite nor co-positive semi-definite. Since $F(iy)$ can be negative, $F(\mathrm{x}+\mathrm{iy})$ cannot be positive semi-definite for x, and $F(\mathrm{x}+\mathrm{iy})$ is generally complex-valued, hence $F(\mathrm{x}+\mathrm{iy})$ cannot be co-positive semi-definite for x as well.

But how are $|F(x+iy)|$ and $|F(x+iy)|^2$? Since they are non-negative, they could be positive semi-definite or co-positive semi-definite. Firstly, observing of $F(x)$ is needed, since if $|F(x+iy)|$ is (co-)positive semi-definite for x, it must be (co-)positive semi-definite at any fixed y. Since $F(x)$ is a special case of $|F(x+iy)|$ when $y=0$, if $F(x)$ is neither positive semi-definite nor co-positive semi-definite, $|F(\mathrm{x}+\mathrm{iy})|$ and $|F(x+iy)|^2$ cannot be (co-)positive semi-definite.

From (1), we have $\mathrm{F}(\mathrm{x})=\int_{-\infty}^{\infty} f(t)\cdot e^{-xt}dt$. This is not positive semi-definite, but it is co-positive semi-definite if $f(t)\geq 0$. Therefore, $|F(x+iy)|$ and $|F(x+iy)|^2$ may be co-positive semi-definite for x, if $f(t)$ is non-negative.

Theorem 3:

If $f(t)$ is non-negative and even, $|F(z)|=\left|F\left(re^{i\theta}\right)\right|$ on the circle centered at the origin has maxima at x-axis, that is, $\theta=0$ and π. Moreover, $|F(iz)|=\left|F\left(ire^{i\theta}\right)\right|$ has maxima at iy-axis.

Proof

Since $F(x+iy)$ is positive semi-definite for y, $|F(x+iy)|\leq F(x)$ on the vertical line and hence, $\left|F\left(re^{i\theta}\right)\right|\leq F[r\cdot\cos(\theta)]$ and since $f(t)$ is non-negative and even, $F(x)$ has unique minimum at $x=0$ and is increasing when $x>0$, therefore $\left|F\left(re^{i\theta}\right)\right|$ has maximum at $\theta=0$. Similarly, if $x<0$, the maximum of $\left|F\left(re^{i\theta}\right)\right|$ is $\theta=\pi$.

$|F(iz)|$ is positive semi-definite for x on the horizontal line, and in the same way, we can prove that $\left|F\left(ire^{i\theta}\right)\right|$ has maxima at iy-axis.

3. The minima of $|F(\mathrm{z})|$

If $f(t)$ is even, from (8) we have $F(z)$ on the real axis

$$F(x)=\int_{-\infty}^{\infty} f(t)\cdot\cosh(xt)dt$$

Let a_0 be $\int_{-\infty}^{\infty} f(t)dt$ and define $G(x)=F(x)/a_0$ to normalize $F(x)$, i.e. $F(0)=1$.Therefore, we have

$$G(x) = \frac{1}{a_0} \int_{-\infty}^{\infty} f(t) \cdot \cosh(xt)\, dt \tag{14}$$

where $a_0 = \int_{-\infty}^{\infty} f(t) dt$.

Further, if $f(t)$ is also non-negative, by the mean-value theorem of integration, we have

$$G(x) = \frac{1}{a_0} \cosh[P(x)] \cdot \int_{-\infty}^{\infty} f(t) dt = \cosh[P(x)] \tag{15}$$

where $P(x)$ is a function depending on x and $f(t)$ which is $arcosh[G(x)]$. By letting $x \mapsto z$, we have

$$G(z) = \cosh[P(z)] \tag{16}$$

where $P(z) = \ arcosh[G(z)]$.

Since $|cosh(z)|$ has the global minimum on the imaginal axis of the horizontal line as well as on a circle centered at the origin, $|cosh[P(z)]|$ has, at least, local minima when $P(z)$ is purely imaginary. Since $P(z) = \ arcosh[G(z)] = \ln\left[G(z) + \sqrt{G^2(z) - 1}\right]$, $P(z)$ is purely imaginary if $\left|G(z) + \sqrt{G(z) - 1}\right| = 1$.

The Taylor series of $\ln\left[G(z) + \sqrt{G^2(z) - 1}\right]$ is

$$\ln\left[G(z) + \sqrt{G^2(z) - 1}\right] = i \cdot \left[\frac{\pi}{2} - \sum_{n=0}^{\infty} \frac{1}{(2n+1) \cdot 4^n} \cdot \frac{(2n)!}{(n!)^2} G^{2n+1}(z)\right] \tag{17}$$

From (17), to be $P(z)$ purely imaginary, $G(z)$ must be real. Clearly, if $P(z)$ is purely imaginary, $\cosh[P(z)]$ is real, and therefore $G(z)$ is real. Since $\mathrm{F(z)} = u(x,y) + iv(x,y)$, $v(x,y)$ must be zero to be $P(z)$ purely imaginary.

From the equations $G(z) = F(z)/\, a_0 = \frac{1}{a_0}\, u(x,y) + i\frac{1}{a_0} v(x,y)$ and $v(x,y) = 0$, we have

$$\left|G(z) + \sqrt{G^2(z) - 1}\right| = \left|\frac{u(x,y)}{a_0} + \sqrt{\frac{u^2(x,y)}{a_0^2} - 1}\right| = 1 \tag{18}$$

and find out $u(x,y)$ according to (18). We will consider three cases, i.e. $u(x,y) < -a_0$, $u(x,y) > a_0$ and $|u(x,y)| < a_0$. Clearly, $\left|u(x,y)/a_0 + \sqrt{u^2(x,y)/a_0^2 - 1}\right| > 1$ when $u(x,y) < -a_0$ and $u(x,y) > a_0$. In the case of $|u(x,y)| < a_0$, since $u^2(x,y)/a_0^2 < 1$, we have

$$\left|u(x,y)/a_0 + \sqrt{u^2(x,y)/a_0^2 - 1}\right| = \left|u(x,y)/a_0 + i\sqrt{1 - u^2(x,y)/a_0^2}\right| = 1.$$

Hence the necessary condition to be $|F(z)|$ minimum is $v(x,y) = 0$ and $|u(x,y)| < a_0$. However, they are only the necessary condition, not sufficient since $P(z)$ is a series of z and therefore there can exist some points where $v(x,y) = 0$ and $|u(x,y)| < a_0$. Thus, we need to add another condition, that is, $\frac{\partial}{\partial x} u(x,y) = 0$. To sum up, $|F(z)|$ is minimum (global or local) if and only if $v(x,y) = 0$, $|u(x,y)| < a_0$ and $\frac{\partial}{\partial x} u(x,y) = 0$ [2].

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[2] If both $u(x,y)$ and $v(x,y)$ are zero, $|F(z)|$ is not differentiable but since $|F(z)|$ is zero, it is global minimum anyway.

Now, we will consider $F(z)$ on the imaginary axis which is $F(iy)$. From (9), we have $F(iy) = u(0,y)$ and therefore $v(x,y) = 0$. Moreover, since $|F(z)|$ is symmetric by iy-axis, $\frac{\partial}{\partial x}F(x+iy)\Big|_{x=0} = 0$ and also $|F(iy)| \le a_0$, therefore $F(iy)$ is at least local minimum for all y on the horizontal line.

Since $|F(x+iy)|$ has local minima at $x = 0$, $|F(x+iy)|^2$ has also local minima at $x = 0$. $|F(x+iy)|^2$ is differentiable by x even if $F(x+iy) = 0$ unlike $|F(x+iy)|$. Since $|F(x+iy)|^2$ has local minima at $x = 0$, its second derivate at $x = 0$ must be positive, i.e. $D_x^2|F(x+iy)|^2|_{x=0} > 0$ where D_x^2 denotes $\frac{\partial^2}{\partial x^2}$ and since $|F(x+iy)|^2 = u^2(x,y) + v^2(x,y)$, we have

$$D_x^2|F(x+iy)|^2|_{x=0} = 2\left[u(0,y)\cdot D_x^2u(0,y) + \left(D_xu(0,y)\right)^2 + v(0,y)\cdot D_x^2v(0,y) + \left(D_xv(0,y)\right)^2\right] > 0 \tag{19}$$

and $v(0,y) = 0$, $D_xu(0,y) = 0$, thus we have

$$u(0,y)\cdot D_x^2u(0,y) + \left(D_xv(0,y)\right)^2 > 0 \tag{20}$$

and since $D_xv(0,y) = -D_yu(0,y)$, $D_x^2u(0,y) = -D_y^2u(0,y)$ and $\mathrm{F(iy)} = \mathrm{F(y)} = u(0,y)$ [3], we have

$$\left(D_yF(y)\right)^2 - F(y)\cdot D_y^2F(y) > 0 \tag{21}$$

The inequality (21) implies that $F(y)$ is log-concave, meaning $F^2(y)\cdot D_y^2\left[ln\left(F(y)\right)\right] < 0$. Since $F(y)$ is log-concave,

$$D_y\left[ln\left(F(y)\right)\right] = \frac{D_yF(y)}{F(y)} \tag{22}$$

and

$$D_y^2\left[ln\left(F(y)\right)\right] = \frac{d}{dy}\left[\frac{D_yF(y)}{F(y)}\right] = \frac{F(y)\cdot D_y^2F(y) - \left(D_yF(y)\right)^2}{F^2(y)} < 0 \tag{23}$$

Therefore, if $F(y)$ has zeros on iy-axis, $\frac{D_yF(y)}{F(y)}$ is monotonically decreasing and $F(y)$ has a unique extremum between the two contiguous zeros.

4. The co-positive definiteness of $|F(x+iy)|^2$

We have shown that $|F(x+iy)|$ has minima at $x = 0$ but it is unknow that the minima are local or global on the horizontal line where y is fixed. We will show that $|F(x+iy)|^2$ is co-positive semi-definite for x.

Theorem 4:

Let $\Psi(\theta) = \int_a^b \varphi(\theta,t)\mathrm{d}t$, then $\Psi(\theta)$ is (co-)positive semi-definite for θ if and only if $\varphi(\theta,t)$ is (co-)positive semi-definite for θ for all t.

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[3] Since $F(iy)$ is real and all the derivatives of $F(iy)$ are also real, afterward, we omit i from $F(iy)$ unless needed. Hence $F(y)$ refers to $F(iy)$.

Proof

The proof is straightforward. If $\Psi(\theta)$ is positive semi-definite, then $\sum_{n=1}^{N}\sum_{k}^{N} c_n c_k^* \Psi(\theta_n - \theta_k) \geq 0$ for any N ($N \geq 1$), c_n and θ_n, and since $\sum_{n=1}^{N}\sum_{k}^{N} c_n c_k^* \Psi(\theta_n - \theta_k) = \int_a^b \sum_{n=1}^{N}\sum_{k}^{N} c_n c_k^* \varphi(\theta_n - \theta_k, t)\, \mathrm{dt}$, to be $\sum_{n=1}^{N}\sum_{k}^{N} c_n c_k^* \Psi(\theta_n - \theta_k) \geq 0$, $\sum_{n=1}^{N}\sum_{k}^{N} c_n c_k^* \varphi(\theta_n - \theta_k, t)$ must be non-negative for all t. Hence $\varphi(\theta, t)$ must be positive semi-definite for θ for all t. The proof of co-positive definiteness is same. Note that it does not hold for the double integral.

From (1) and $z = x + iy$, we have

$$F(x+iy) = \int_{-\infty}^{\infty} f(t) \cdot e^{-xt} \cdot e^{-iyt} dt \tag{24}$$

which is the Fourier transform of $f(t) \cdot e^{-xt}$. Hence by the Parseval' theorem, we have

$$\int_{-\infty}^{\infty} (f(t) \cdot e^{-xt})^2 dt = \int_{-\infty}^{\infty} f^2(t) \cdot e^{-2xt} dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(x+iy)|^2 dy \tag{25}$$

Since $\int_{-\infty}^{\infty} f^2(t) \cdot e^{-2xt} dt$ is clearly co-positive semi-definite for x, $\int_{-\infty}^{\infty} |F(x+iy)|^2 dy$ must be also co-positive semi-definite for x, and therefore, $|F(x+iy)|^2$ is co-positive semi-definite for x for all y by the theorem 4.

However, $|F(x+iy)|$ and $|F(x+iy)|^2$ are only co-positive semi-definite if $F(x)$ is co-positive semi-definite. Hence the necessary and sufficient condition to be $|F(x+iy)|^2$ co-positive semi-definite is $f(t)$ is non-negative.

Assuming $f(t)$ is an even function, $F(x+iy) = \sum_{n=0}^{\infty} a_{2n} \cdot (x+iy)^{2n}$ from (4) and we have

$$|F(x+iy)|^2 = F(x+iy) \cdot F^*(x+iy) = \left(\sum_{n=0}^{\infty} a_{2n} \cdot (x+iy)^{2n}\right) \cdot \left(\sum_{n=0}^{\infty} a_{2n} \cdot (x-iy)^{2n}\right) \tag{26}$$

and since $|F(x+iy)|^2$ has only even powers of x, the power series expansion of $|F(x+iy)|^2$ is

$$|F(x+iy)|^2 = \sum_{n=0}^{\infty} A_{2n} \cdot x^{2n} \tag{27}$$

where A_{2n} depends on y and since $D_x^{2n}|F(x+iy)|^2|_{x=0} \geq 0$, A_{2n} is non-negative for all n. Since $|\mathrm{F(x+iy)}|^2 = \mathrm{F(x+iy)} \cdot \mathrm{F(x-iy)}$, we have

$$|F(x+iy)|^2 = \left(\int_{-\infty}^{\infty} f(t) \cdot e^{-xt} \cdot e^{-iyt} dt\right) \cdot \left(\int_{-\infty}^{\infty} f(t) \cdot e^{-xt} \cdot e^{iyt} dt\right) \tag{28}$$

and from (28)

$$|F(x+iy)|^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t_1) \cdot f(t_2) \cdot e^{-x(t_1+t_2)} \cdot e^{-iy(t_1-t_2)} dt_1 dt_2 \tag{29}$$

By letting $\mathrm{t} = t_1 + t_2$ and $\tau = t_1$, we have

$$|F(x+iy)|^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\tau) \cdot f(t-\tau) \cdot e^{-xt} \cdot e^{-iy\tau} \cdot e^{iy(t-\tau)} d\tau dt \tag{30}$$

which can be written

$$|F(x+iy)|^2 = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(\tau) \cdot f(t-\tau) \cdot e^{-iy\tau} \cdot e^{iy(t-\tau)} d\tau \right] e^{-xt}\, dt \tag{31}$$

Letting $\tau \mapsto -\tau$, and assuming $f(t)$ is even, we have

$$|F(x+iy)|^2 = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(\tau) \cdot f(t+\tau) \cdot e^{iy\tau} \cdot e^{iy(t+\tau)} d\tau \right] e^{-xt}\, dt \tag{32}$$

or simply,

$$|F(x+iy)|^2 = \int_{-\infty}^{\infty} r_y(t) \cdot e^{-xt}\, dt \tag{33}$$

where

$$r_y(t) = \int_{-\infty}^{\infty} f(\tau) \cdot f(t-\tau) \cdot e^{-iy\tau} \cdot e^{iy(t-\tau)} d\tau = \int_{-\infty}^{\infty} f(\tau) \cdot f(t+\tau) \cdot e^{iy\tau} \cdot e^{iy(t+\tau)} d\tau \tag{34}$$

The conjugate of $r_y(t)$

$$r_y^*(t) = \int_{-\infty}^{\infty} f(\tau) \cdot f(t+\tau) \cdot e^{-iy\tau} \cdot e^{-iy(t+\tau)} d\tau \tag{35}$$

and by letting $\tau \mapsto \tau - t$, and assuming $f(t)$ is even, we have

$$r_y^*(t) = \int_{-\infty}^{\infty} f(\tau - t) \cdot f(\tau) \cdot e^{-iy(\tau-t)} \cdot e^{-iy\tau} d\tau = \int_{-\infty}^{\infty} f(\tau) \cdot f(t-\tau) \cdot e^{-iy\tau} \cdot e^{iy(t-\tau)} \cdot d\tau$$

which is the equation (34), therefore, $r_y(t)$ is real. Moreover, $r_y(t)$ is an even function which can be easily proved. The function $r_y(t)$ is real and even but does not hold the positivity, namely, It can be negative.

The coefficients A_{2n} in (27) can be described with the function $r_y(t)$, that is,

$$A_{2n} = \frac{1}{(2n)!} \int_{-\infty}^{\infty} t^{2n} \cdot r_y(t)\, dt \tag{36}$$

Since A_0 is $|F(iy)|^2$, A_0 can be zero, but can A_2 be also zero? By twice differentiating $|F(x+iy)|^2$ and letting $x = 0$, we get $2A_2$, that is

$$2A_2 = D_x^2 |F(x+iy)|^2|_{x=0} \tag{37}$$

We have derived $D_x^2 |F(x+iy)|^2|_{x=0}$ in (19) and the result was that $\left(D_y F(y)\right)^2 - F(y) \cdot D_y^2 F(y) > 0$ in (21). If A_2 is zero, then it implies that $\left(D_y F(y)\right)^2 - F(y) \cdot D_y^2 F(y) = 0$, i.e.

$$\left(\frac{d}{dx} F(y)\right)^2 - F(y) \cdot \frac{d^2}{dx^2} F(y) = 0 \tag{38}$$

The differential equation (38) is easily solvable by letting $P(y) = \frac{d}{dy} F(y)$ and we get $F(y) = c_1 e^{c_2 y}$ where the constants c_1 and c_2 are to be determined by the initial conditions. Since F(y) is an even function, $\frac{d}{dy}F(y)\Big|_{y=0} = 0$ and $F(0) = a_0$ in (4). Thus, we get $c_1 = a_0$ and $c_2 = 0$, and therefore $F(y) = a_0$, which means $F(y)$ is a constant. Since $F(y) = \int_{-\infty}^{\infty} f(t) \cdot e^{-iyt} dt = a_0$, $f(t) = a_0 \cdot \delta(t)$ where $\delta(t)$ denotes the Dirac delta function. If $f(t) = a_0 \cdot \delta(t)$, $F(z)$ is constant in the whole z-plane. However, since $F(z)$ is different from a constant, A_2 cannot be zero and it should be strictly positive. As a result, $A_2 > 0$ and $A_{2n} \geq 0$ $(\mathrm{n} \neq 1)$, and hence, if y is fixed, $|F(x+iy)|^2$ has a unique global minimum at $x = 0$ and monotonically increasing when $x > 0$. Therefore, $|F(x+iy)|^2$ can have zeros only at $x = 0$ if it has any.

Since $x = r \cdot cos(\theta)$, $|F(x+iy)|^2$ is co-positive semi-definite for $r \cdot cos(\theta)$. However, how is it when θ is fixed or r is fixed. Firstly, we consider when θ is fixed assuming $cos(\theta)$ is non-zero. Since θ is fixed, $cos(\theta)$ is constant. Letting $c = cos(\theta)$, where c is a non-zero constant, and hence, $x = c \cdot r$. Putting it to (27), we have

$$|F(r+iy)|^2 = \sum_{n=0}^{\infty} B_{2n} \cdot r^{2n} \tag{39}$$

where $B_{2n} = A_{2n} \cdot c^{2n}$.

Since c is non-zero, $c^{2n} > 0$, and since $A_2 > 0$, B_2 is strictly positive and other coefficients B_{2n} are non-negative. Thus, when θ is fixed, $|F(r+iy)|^2$ behaves like $|F(x+iy)|^2$, meaning, it has a unique minimum at $r = 0$ and monotonically increasing when $r > 0$.

Secondly, we will consider the case when r is fixed and θ varies. It is the behavior of $|F(x+iy)|^2$ on a circle centered at the origin. By changing the variable in the eq. (27), namely, $x = r \cdot cos(\theta)$, we have

$$|F(r \cdot cos(\theta) + iy)|^2 = \sum_{n=0}^{\infty} C_{2n} \cdot cos^{2n}(\theta) \tag{40}$$

where $C_{2n} = A_{2n} \cdot r^{2n}$.

Assuming that r is non-zero, then C_2 is strictly positive and other coefficients C_{2n} are non-negative. By differentiating eq. (40) by θ, we have

$$-sin(\theta) \sum_{n=1}^{\infty} 2n \cdot C_{2n} \cdot cos^{2n-1}(\theta) = -sin(2\theta) \cdot \sum_{n=1}^{\infty} n \cdot C_{2n} \cdot cos^{2n-2}(\theta)$$

Since C_2 is strictly positive and $n \cdot C_{2n} \cdot cos^{2n-2}(\theta)$ is non-negative when $n \geq 2$, the slop is only depending on $-\sin(2\theta)$. In the interval of $0 < \theta < \pi/2$, $-\sin(2\theta)$ is negative and hence, the slop is negative, therefore, $|F(\theta+iy)|^2$ is monotonically decreasing. In the interval of $\pi/2 < \theta < \pi$, $-\sin(2\theta)$ and the slop are positive and thus, $|F(\theta+iy)|^2$ is monotonically increasing. Note that the slop is zero when $\theta = 0$ and $\theta = \pi/2$.

By the Paley-Wiener theorem, to be $F(z)$ entire, $f(t)$ must be decreasing rapidly and $F(z)$ does not have any poles on x-axis as mentioned. Moreover, if $F(z)$ has zeros only on the iy-axis, that is, $F(z)$ has only real zeros, $F(z)$ must be entire.

We conclude with the theorem.

Theorem 5:

If a function $f(t)$ is non-negative and even and its two-sided Laplace transform is denoted by $F(z)$, then $|F(\mathrm{x}+\mathrm{iy})|$ and $|F(x+iy)|^2$ have a unique minimum at $x=0$ on the horizontal line when y is fixed and monotonically increasing in the region of convergence, and therefore, $|F(x+iy)|$ and $|F(x+iy)|^2$, and thus $F(x+iy)$ can have zeros only at $x=0$, and since $F(iy)$ is real when y is real, $F(x+iy)$ have only real zeros. Moreover, $F(iz)$ has zeros only on the x-axis and consequently, only real zeros, if $f(t)$ is a rapidly decreasing non-negative even function.

5. The Laguerre inequalities and generalized Laguerre inequalities

We have shown that the two-sided Laplace transform of an even and non-negative functions have only real zeros. We will show that the Laguerre inequalities and generalized Laguerre inequalities are true for the two-sided Laplace transform of an even and non-negative functions.

Theorem 6: The convexity

1) A function $f(x)$ is convex if and only if $f''(x) \geq 0$ for all x.
2) The sufficient but not necessary condition to be a function $f(x)$ strictly convex is $f''(x) > 0$ for all x.

Theorem 7: The concavity

1) A function $f(x)$ is concave if and only if $f''(x) \leq 0$ for all x.
2) The sufficient but not necessary condition to be a function $f(x)$ strictly concave is $f''(x) < 0$ for all x.

Theorem 8: The log- convexity and log-concavity

A function $f(x)$ is log-convex if $ln[f(x)]$ is convex. Similarly, a function $f(x)$ is log-concave if $ln[f(x)]$ is concave.

1) A function $f(x)$ is log-convex, if

$$f(\lambda x_1 + \mu x_2) \leq [f(x_1)]^{\lambda} \cdot [f(x_2)]^{\mu}$$

where $\lambda, \mu > 0$ and $\lambda + \mu = 1$.

2) A function $f(x)$ is log-concave, if

$$f(\lambda x_1 + \mu x_2) \geq [f(x_1)]^{\lambda} \cdot [f(x_2)]^{\mu}$$

where $\lambda, \mu > 0$ and $\lambda + \mu = 1$.

3) If $f(x)$ and $g(x)$ are both log-convex, then $f(x) \cdot g(x)$ is also log-convex. Similarly, if $\mathrm{f(x)}$ and $g(x)$ are both log-concave, then $f(x) \cdot g(x)$ is also log-concave.

The necessary but not sufficient condition to have $F(y)$ only real zeros is that $F(y)$ and all the derivatives of $F(y)$ are log-concave, hence we have the theorem.

Theorem 9: The Laguerre inequalities

$F(y)$ belongs to the Laguerre-Pólya class if

$$\left[F^{(n)}(y)\right]^2 - F^{(n-1)}(y) \cdot F^{(n+1)}(y) \geq 0 \tag{41}$$

where $n = 1, 2, 3, \ldots$ and for all $y \in \mathbb{R}$.

Note that the theorem 9 is the necessary conditions, not sufficient.

Theorem 10:

Let be $F(x) = \int_{-\infty}^{\infty} f(t) \cdot e^{-xt} dt$ where $f(t)$ is non-negative and x is real, then $F(x)$ is log-convex. If $f(t)$ is non-negative, then $t^{2n} \cdot f(t)$ is also non-negative. Consequently, $(2n)^{th}$ derivative of $F(x)$, that is, $F^{(2n)}(\mathrm{x}) = \int_{-\infty}^{\infty} t^{2n} \cdot f(t) \cdot e^{-xt} dt$ is also log-convex for $n = 0, 1, 2, \ldots$ It can be easily proved using theorem 8 and the Hölder inequality. Since $F^{(2n)}(\mathrm{x})$ is log-convex, we have

$$F^{(2n)}(\mathrm{x}) \cdot F^{(2n+2)}(\mathrm{x}) - \left[F^{(2n+1)}(x)\right]^2 \geq 0 \tag{42}$$

Note that $(2n+1)^{th}$ derivative of $F(x)$, that is, $F^{(2n+1)}(\mathrm{x}) = \int_{-\infty}^{\infty} t^{2n+1} \cdot f(t) \cdot e^{-xt} dt$ is generally not log-convex.

Theorem 11:

Let be $F(x) = \int_{-\infty}^{\infty} f(t) \cdot e^{-xt} dt$ where $f(t)$ is a continuous non-negative function and x is real, then we have the inequalities:

$$\left[F^{(n+m)}(x)\right]^2 \leq F^{(2n)}(x) \cdot F^{(2m)}(x) \tag{43}$$

<u>Proof</u>

Since $F^{(n+m)}(x) = \int_{-\infty}^{\infty} t^{n+m} \cdot f(t) \cdot e^{-xt} dt$, we have

$$\left[F^{(n+m)}(x)\right]^2 = \left(\int_{-\infty}^{\infty} t^{n+m} \cdot f(t) \cdot e^{-xt} dt\right)^2 = \left(\int_{-\infty}^{\infty} t^n \cdot \sqrt{f(t)} \cdot e^{-\frac{1}{2}xt} \cdot t^m \cdot \sqrt{f(t)} \cdot e^{-\frac{1}{2}xt} dt\right)^2$$

By the Cauchy-Schwarz inequality, we have

$$\left(\int_{-\infty}^{\infty} t^n \cdot \sqrt{f(t)} \cdot e^{-\frac{1}{2}xt} \cdot t^m \cdot \sqrt{f(t)} \cdot e^{-\frac{1}{2}xt} dt\right)^2 \leq \left(\int_{-\infty}^{\infty} t^{2n} \cdot f(t) \cdot e^{-xt} dt\right) \cdot \left(\int_{-\infty}^{\infty} t^{2m} \cdot f(t) \cdot e^{-xt} dt\right)$$

Hence, $\left[F^{(n+m)}(x)\right]^2 \leq F^{(2n)}(x) \cdot F^{(2m)}(x)$.

Now, we consider the theorem (9) in the case of $n = 0$. Since $F(x)$ is log-convex,

$$F(x) \cdot F''(x) - [F'(x)]^2 \geq 0 \tag{44}$$

and if $f(t)$ is non-negative and even, then we have

$$F(x) = \int_{-\infty}^{\infty} f(t) \cdot cosh(xt)\, dt$$
$$F'(x) = \int_{-\infty}^{\infty} t \cdot f(t) \cdot sinh(xt)\, dt$$

$$F''(x) = \int_{-\infty}^{\infty} t^2 \cdot f(t) \cdot cosh(xt)\, dt$$

From eq. (9), we have

$$F(iy) = \int_{-\infty}^{\infty} f(t) \cdot cos(yt)\, dt$$

$$F'(iy) = -\int_{-\infty}^{\infty} t \cdot f(t) \cdot sin(yt)\, dt$$

$$F''(iy) = -\int_{-\infty}^{\infty} t^2 \cdot f(t) \cdot cos(yt)\, dt$$

By letting $x \mapsto iy$, we have

$$F(x)|_{x=iy} = \int_{-\infty}^{\infty} f(t) \cdot \cos(yt)\, dt = F(iy)$$

$$F'(x)|_{x=iy} = i\int_{-\infty}^{\infty} t \cdot f(t) \cdot \sin(yt)\, dt = -i \cdot F'(iy)$$

$$F''(x)|_{x=iy} = \int_{-\infty}^{\infty} t^2 \cdot f(t) \cdot \cos(yt)\, dt = -F''(iy)$$

Therefore, by letting $x \mapsto iy$, we get the inequality (43) with the function of $F(iy)$, that is,

$$(F(x) \cdot F''(x) - [F'(x)]^2)_{x=iy} = -F(iy) \cdot F''(iy) - [i \cdot F'(iy)]^2 \geq 0$$

From this inequality, we have

$$[F'(iy)]^2 - F(iy) \cdot F''(iy) \geq 0 \tag{45}$$

More intuitively, the inequality (44) is

$$F(x) \cdot \frac{d^2}{dx^2} F(x) - \left[\frac{d}{dx} F(x)\right]^2 \geq 0$$

and by changing the variable $\mathrm{x} \mapsto \mathrm{iy}$ we have

$$F(iy) \cdot \frac{d^2}{d(iy)^2} F(iy) - \left[\frac{d}{d(iy)} F(iy)\right]^2 \geq 0 \tag{46}$$

and the inequality (46) is

$$\frac{1}{(i)^2} F(iy) \cdot \frac{d^2}{dy^2} F(iy) - \frac{1}{(i)^2}\left[\frac{d}{dy} F(iy)\right]^2 = -F(iy) \cdot \frac{d^2}{dy^2} F(iy) + \left[\frac{d}{dy} F(iy)\right]^2 \geq 0$$

and since $F(iy)$ and all the derivatives of $F(iy)$ are real, we have

$$[F'(iy)]^2 - F(iy) \cdot F''(iy) \geq 0$$

which is the same result in (45).

Generally, we get the theorem:

Theorem 12:

If $F(x)$ does not have any pole on the x-axis, that is, $F(z)$ is entire, and $F(iy)$ is real when y real, then

$$F^{(n)}(x) \cdot F^{(m)}(x)\Big|_{x=iy} = \frac{1}{(i)^{n+m}} F^{(n)}(iy) \cdot F^{(m)}(iy) \tag{47}$$

where n and m are non-negative integer.

Proof

$$F^{(n)}(x) \cdot F^{(m)}(x) = \frac{d^n}{dx^n}\mathrm{F(x)} \cdot \frac{d^m}{dx^m}\mathrm{F(x)}$$

By changing the variable x to iy, we have

$$\frac{d^n}{dx^n}F(x) \cdot \frac{d^m}{dx^m}F(x)\bigg|_{x=iy} = \frac{d^n}{d(iy)^n}F(iy) \cdot \frac{d^m}{d(iy)^m}F(iy) = \frac{1}{(i)^n}\frac{d^n}{dy^n}F(iy) \cdot \frac{1}{(i)^m}\frac{d^m}{dy^m}F(iy)$$

and therefore, eq. (47) is valid.

In general, if

$$\sum_{k=1}^{N} F^{(n_k)}(x) \cdot F^{(m_k)}(x) \geq 0$$

then

$$\sum_{k=1}^{N} F^{(n_k)}(x) \cdot F^{(m_k)}(x)\Bigg|_{x=iy} = \sum_{k=1}^{N} \frac{1}{(i)^{n_k+m_k}} F^{(n_k)}(y) \cdot F^{(m_k)}(y) \geq 0 \tag{48}$$

The inequality does not change since we have only changed the variable.

The theorem 12 is very useful, since we can prove some inequalities on the x-axis more easily than on the iy-axis.

Theorem 13:

We get the inequality below by applying (47) to (42) [4]

$$\left[F^{(2n+1)}(y)\right]^2 - F^{(2n)}(\mathrm{y}) \cdot F^{(2n+2)}(\mathrm{y}) \geq 0 \tag{49}$$

which are the even terms of the Laguerre inequalities.

Theorem 14:

We get the inequality below by applying eq. (47) to (43)

[4] Since $F(iy)$ and all the derivatives of $F(iy)$ are real when y is real, for the sake of convenience, i is omitted afterward, i.e. $F(y)$ means $F(iy)$.

$$\begin{aligned} \left[F^{(n+m)}(y)\right]^2 - F^{(2n)}(y)\cdot F^{(2m)}(y) \geq 0 \qquad & when\ (n+m)\ is\ odd \\ \left[F^{(n+m)}(y)\right]^2 - F^{(2n)}(y)\cdot F^{(2m)}(y) \leq 0 \qquad & when\ (n+m)\ is\ even \end{aligned} \tag{50}$$

We shall show the odd terms of the Laguerre inequalities, but before that, we prove Hölder inequality for the double integral.

Theorem 15: Hölder inequality for the double integral

$$\left|\int_a^b \int_c^d f(x,y)\cdot g^*(x,y)dxdy\right| \leq \left(\int_a^b \int_c^d |f(x,y)|^p dxdy\right)^{\frac{1}{p}} \cdot \left(\int_a^b \int_c^d |g(x,y)|^q dxdy\right)^{\frac{1}{q}} \tag{51}$$

where $\mathrm{p},\mathrm{q} > 1$ and $\frac{1}{p}+\frac{1}{q}=1$.

<u>Proof</u>

We define $\mathrm{U(x,y)}$ and $\mathrm{V(x,y)}$ such that

$$\mathrm{U(x,y)} = \frac{|f(x,y)|}{\left(\int_a^b \int_c^d |f(x,y)|^p dxdy\right)^{\frac{1}{p}}}$$

and

$$\mathrm{V(x,y)} = \frac{|g(x,y)|}{\left(\int_a^b \int_c^d |g(x,y)|^q dxdy\right)^{\frac{1}{q}}}$$

Since $\mathrm{U(x,y)} \geq 0$ and $\mathrm{V(x,y)} \geq 0$, by Young's inequality for products, we have

$$\mathrm{U(x,y)}\cdot\mathrm{V(x,y)} \leq \frac{U^p(x,y)}{p} + \frac{V^q(x,y)}{q}$$

which is

$$\frac{|f(x,y)\cdot g(x,y)|}{\left(\int_a^b \int_c^d |f(x,y)|^p dxdy\right)^{\frac{1}{p}} \cdot \left(\int_a^b \int_c^d |g(x,y)|^q dxdy\right)^{\frac{1}{q}}} \leq \frac{1}{p}\frac{|f(x,y)|^p}{\int_a^b \int_c^d |f(x,y)|^p dxdy} + \frac{1}{q}\frac{|g(x,y)|^q}{\int_a^b \int_c^d |g(x,y)|^q dxdy}$$

Since both sides of the inequality are non-negative, the inequality does not change by integrating. By integrating twice, we have

$$\frac{\int_a^b \int_c^d |f(x,y)\cdot g(x,y)|dxdy}{\left(\int_a^b \int_c^d |f(x,y)|^p dxdy\right)^{\frac{1}{p}} \cdot \left(\int_a^b \int_c^d |g(x,y)|^q dxdy\right)^{\frac{1}{q}}} \leq \frac{1}{p}\frac{\int_a^b \int_c^d |f(x,y)|^p dxdy}{\int_a^b \int_c^d |f(x,y)|^p dxdy} + \frac{1}{q}\frac{\int_a^b \int_c^d |g(x,y)|^q dxdy}{\int_a^b \int_c^d |g(x,y)|^q dxdy}$$

Since $\frac{1}{p}+\frac{1}{q}=1$ we have

$$\left|\int_a^b \int_c^d f(x,y)\cdot g^*(x,y)dxdy\right| \leq \left(\int_a^b \int_c^d |f(x,y)|^p dxdy\right)^{\frac{1}{p}} \cdot \left(\int_a^b \int_c^d |g(x,y)|^q dxdy\right)^{\frac{1}{q}}$$

Since $F(x) = \int_{-\infty}^{\infty} f(t)\cdot\cosh(xt)\,dt$, by differentiating $(2n+1)$ times, we have

$$F^{(2n+1)}(x) = \int_{-\infty}^{\infty} t^{2n+1} \cdot f(t) \cdot \sinh(xt)\, dt \qquad (52)$$

and we can write (52)

$$F^{(2n+1)}(x) = \int_{-\infty}^{\infty} t^{2n+1} \cdot f(t) \cdot \sinh(xt)\, dt = x \int_{-\infty}^{\infty} t^{2n+2} \cdot f(t) \cdot \frac{\sinh(xt)}{xt}\, dt$$

and since

$$\int_0^1 \cosh(xt\tau)\, d\tau = \frac{\sinh(xt)}{xt}$$

We have

$$F^{(2n+1)}(x) = x \int_0^1 \int_{-\infty}^{\infty} t^{2n+2} \cdot f(t) \cdot \cosh(xt\tau)\, dt d\tau = x \int_0^1 \int_{-\infty}^{\infty} t^{2n+2} \cdot f(t) \cdot e^{-xt\tau} dt d\tau$$

We define $P(x)$ such that

$$P(x) = \int_0^1 \int_{-\infty}^{\infty} t^{2n+2} \cdot f(t) \cdot \cosh(xt\tau)\, dt d\tau = \int_0^1 \int_{-\infty}^{\infty} t^{2n+2} \cdot f(t) \cdot e^{-xt\tau} dt d\tau$$

The purpose of changing $\cosh(xt\tau)$ to $e^{-xt\tau}$ is to prove the log-convexity of $P(x)$ easily. Since $t^{2n+2} \cdot f(t)$ is even, the change is valid.

Theorem 16:

The function $P(x)$, which is defined as

$$P(x) = \int_0^1 \int_{-\infty}^{\infty} t^{2n+2} \cdot f(t) \cdot e^{-xt\tau} dt d\tau$$

is log-convex.

<u>Proof</u>

$$\int_0^1 \int_{-\infty}^{\infty} t^{2n+2} \cdot f(t) \cdot e^{-(\lambda x_1 + \mu x_2) \cdot t\tau} dt d\tau = \int_0^1 \int_{-\infty}^{\infty} t^{2n+2} \cdot f(t) \cdot e^{-\lambda x_1 t\tau} \cdot e^{-\mu x_2 t\tau}\, dt d\tau$$

where $\lambda + \mu = 1$.

Since $f(t) \geq 0$, we have

$$\int_0^1 \int_{-\infty}^{\infty} t^{2n+2} \cdot f(t) \cdot e^{-\lambda x_1 t\tau} \cdot e^{-\mu x_2 t\tau}\, dt d\tau = \int_0^1 \int_{-\infty}^{\infty} (t^{2n+2} \cdot f(t) \cdot e^{-x_1 t\tau})^{\lambda} \cdot (t^{2n+2} \cdot f(t) \cdot e^{-x_2 t\tau})^{\mu}\, dt d\tau$$

By the Hölder inequality for the double integral (theorem 15), we have

$$\int_0^1 \int_{-\infty}^{\infty} (t^{2n+2} \cdot f(t) \cdot e^{-x_1 t\tau})^{\lambda} \cdot (t^{2n+2} \cdot f(t) \cdot e^{-x_2 t\tau})^{\mu}\, dt d\tau$$
$$\leq \left(\int_0^1 \int_{-\infty}^{\infty} t^{2n+2} \cdot f(t) \cdot e^{-x_1 t\tau}\, dt d\tau \right)^{\lambda} \cdot \left(\int_0^1 \int_{-\infty}^{\infty} t^{2n+2} \cdot f(t) \cdot e^{-x_2 t\tau}\, dt d\tau \right)^{\mu}$$

which is

$$\int_0^1 \int_{-\infty}^{\infty} t^{2n+2} \cdot f(t) \cdot e^{-(\lambda x_1+\mu x_2)\cdot t\tau} dt d\tau$$
$$\leq \left(\int_0^1 \int_{-\infty}^{\infty} t^{2n+2} \cdot f(t) \cdot e^{-x_1 t\tau}\, dt d\tau\right)^{\lambda} \cdot \left(\int_0^1 \int_{-\infty}^{\infty} t^{2n+2} \cdot f(t) \cdot e^{-x_2 t\tau}\, dt d\tau\right)^{\mu}$$

This inequality yields

$$P(\lambda x_1 + \mu x_2) \leq [P(x_1)]^{\lambda} \cdot [P(x_2)]^{\mu}$$

and by the theorem 9, $P(x)$ is log-convex.

Since $\mathrm{P(x)}$ is log-convex, we have

$$P(x) \cdot P''(x) - [P'(x)]^2 \geq 0 \tag{52}$$

From (9), we have $F(y) = \int_{-\infty}^{\infty} f(t) \cdot \cos(yt)\, dt$ and its $(2n+1)^{th}$ derivative is

$$F^{(2n+1)}(y) = (-1)^{n+1} \int_{-\infty}^{\infty} t^{2n+1} \cdot f(t) \cdot \sin(yt)\, dt$$

where $n = 0, 1, 2, \ldots$

Since

$$\int_0^1 \cos(xt\tau)\, d\tau = \frac{\sin(xt)}{xt}$$

we have

$$F^{(2n+1)}(y) = (-1)^{n+1} \cdot y \int_0^1 \int_{-\infty}^{\infty} t^{2n+2} \cdot f(t) \cdot \cos(yt\tau)\, dt d\tau$$

and since $P(x)|_{x=iy} = P(y) = \int_0^1 \int_{-\infty}^{\infty} t^{2n+2} \cdot f(t) \cdot \cos(yt\tau)\, dt d\tau$, we have

$$F^{(2n+1)}(y) = (-1)^{n+1} \cdot y \cdot P(y)$$

and from (52), we get

$$[P'(y)]^2 - P(y) \cdot P''(y) \geq 0$$

thus, $P(y)$ is log-concave.

Letting $g(y) = (-1)^{n+1} \cdot y$, we have

$$[g'(y)]^2 - g(x) \cdot g''(x) = 1 > 0$$

and therefore, $(-1)^{n+1} \cdot y$ is log-concave. Since both $(-1)^{n+1} \cdot y$ and $P(y)$ are log-concave, $F^{(2n+1)}(y)$ is log-concave. Thus, $F^{(n)}(y)$ is log-concave for all non-negative integer n. Consequently, the Laguerre inequalities are valid for all n.

Theorem 16: The generalized Laguerre inequalities

We define $L_n(y)$ as follows:

$$L_n(y) = (-1)^n \cdot \frac{1}{n!} \sum_{k=0}^{2n} (-1)^k \cdot \binom{2n}{k} \cdot F^{(k)}(y) \cdot F^{(2n-k)}(y) \tag{53}$$

where $n = 0, 1, 2, \dots$

$F(\mathrm{y})$ has only real zeros if and only if $L_n(y) \geq 0$ for all n.

We return to eq. (33), which is

$$|F(x+iy)|^2 = \int_{-\infty}^{\infty} r_y(t) \cdot e^{-xt}\, dt \tag{54}$$

where

$$r_y(t) = \int_{-\infty}^{\infty} f(\tau) \cdot f(t+\tau) \cdot e^{iy\tau} \cdot e^{iy(t+\tau)} d\tau \tag{55}$$

and reform (55), so that

$$r_y(t) = \int_{-\infty}^{\infty} f(\tau) \cdot e^{iy\tau} \cdot f(t+\tau) \cdot e^{iy(t+\tau)} d\tau \tag{56}$$

and by letting $g(\tau) = f(\tau) \cdot e^{-iy\tau}$, $r_y(t)$ is the cross-correlation function of $g(\tau)$ and $g^*(\tau)$ where $g^*(\tau) = f(\tau) \cdot e^{iy\tau}$. Let $F(\omega)$ be the Fourier transform of $f(\tau)$, then the Fourier transform of $g(\tau)$ is $F(\omega - y)$ and the Fourier transform of $g^*(\tau)$ is $F(\omega + y)$. By the cross-correlation theorem, we have

$$r_y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega - y) \cdot F(\omega + y) \cdot e^{it\omega} d\omega$$

and since $F(y)$ is even, we have

$$r_y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(y - \omega) \cdot F(y + \omega) \cdot e^{it\omega} d\omega \tag{57}$$

which is similar to the Wigner-Ville distribution function. By changing variable $x = i\theta$, from (54), we have

$$|F(i\theta + iy)|^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(y - \omega) \cdot F(y + \omega) \cdot e^{i\omega t} \cdot e^{-i\theta t}\, d\omega\, dt \tag{58}$$

and

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(y - \omega) \cdot F(y + \omega) \cdot e^{i\omega t} \cdot e^{-i\theta t}\, d\omega\, dt = \int_{-\infty}^{\infty} F(y - \omega) \cdot F(y - \omega) \cdot \left[\int_{-\infty}^{\infty} e^{i\omega t} \cdot e^{-i\theta t} dt \right] d\omega$$

and

$$\int_{-\infty}^{\infty} e^{i\omega t} \cdot e^{-i\theta t} dt = 2\pi \cdot \delta(\theta - \omega)$$

thus, we have

$$|F(i\theta + iy)|^2 = \int_{-\infty}^{\infty} F(y - \omega) \cdot F(y + \omega) \cdot \delta(\omega - \theta)\, d\omega$$

and by omitting i for the convenience, we have,

$$|F(\theta + y)|^2 = F(y - \theta) \cdot F(y + \theta) \tag{59}$$

which is the characteristic equation of $|F(x + iy)|^2$ where $\mathrm{x} = i\theta$, hence, from (54)

$$|F(\theta + y)|^2 = \int_{-\infty}^{\infty} r_y(t) \cdot e^{-i\theta t}\, dt \tag{60}$$

The n^{th} moment of $|F(x+iy)|^2$, which is denoted as M_n, is defined as follows

$$M_n(y) = \int_{-\infty}^{\infty} t^n \cdot r_y(t)\, dt \tag{61}$$

or

$$M_n(y) = (-1)^n \cdot D_x^n |F(x+iy)|^2|_{x=0} \tag{62}$$

Another method to get $M_n(y)$ is differentiating (60), that is,

$$M_n(y) = \frac{1}{(-i)^n} \cdot D_\theta^n |F(\theta + y)|^2|_{\theta=0}$$

or by (59), $M_n(y)$ is $\frac{1}{(-i)^n} \cdot D_\theta^n [F(\theta - y) \cdot F(\theta + y)]_{\theta=0}$ which can be computed using the Leibniz rule, that is,

$$M_n(y) = \frac{1}{(-i)^n} \cdot D_\theta^n [F(y-\theta) \cdot F(y+\theta)]_{\theta=0} = \frac{1}{(-i)^n} \cdot \sum_{k=0}^{n} (-1)^k \cdot \binom{n}{k} \cdot F^{(k)}(y) \cdot F^{(n-k)}(y) \tag{63}$$

However, since $r_y(t)$ is an even function, $M_n(y)$ vanishes when n is odd and we need to compute only for even n, hence,

$$M_{2n}(y) = D_\theta^{2n} [F(y-\theta) \cdot F(y+\theta)]_{\theta=0} = (-1)^n \sum_{k=0}^{2n} (-1)^k \cdot \binom{2n}{k} \cdot F^{(k)}(y) \cdot F^{(2n-k)}(y) \tag{64}$$

and we have

$$|F(x+iy)|^2 = \sum_{n=0}^{\infty} \frac{1}{(2n)!} \cdot M_{2n}(y) \cdot x^{2n} = \sum_{n=0}^{\infty} L_n(y) \cdot x^{2n} \tag{65}$$

where $L_n(y)$ is defined in (53).

If $L_n(y) \geq 0$ for all n, $|F(x+iy)|^2$ has a unique minimum at $x = 0$ when y is fixed and hence, $F(y)$ has only real zeros.

Now, we will prove the generalized Laguerre inequalities.

In fact, $F(y+\theta) = F(iy + i\theta) = F[i(y+\theta)]$, i.e. this function lies on the iy-axis. $F(y-\theta)$ is the same. We will map $F(y-\theta) \cdot F(y+\theta)$ on x-axis, i.e. $F(x-\theta) \cdot F(x+\theta)$, and we have

$$F(x-\theta) = \int_{-\infty}^{\infty} f(t) \cdot e^{-xt} \cdot e^{\theta t} dt$$

$$\sum_{n=1}^{N}\sum_{k}^{N} c_n c_k^* F(x-(\theta_n+\theta_k)) = \int_{-\infty}^{\infty} f(t)\cdot e^{-xt}\cdot \sum_{n=1}^{N}\sum_{k}^{N} c_n c_k^*\, e^{(\theta_n+\theta_k)t}\,dt$$

$$= \int_{-\infty}^{\infty} f(t)\cdot e^{-xt}\cdot \left|\sum_{n=1}^{N} c_n e^{\theta_n t}\right|^2 dt \geq 0$$

Therefore, $F(x-\theta)$ is co-positive semi-definite. In the same way, we can prove that $F(x+\theta)$ is co-positive semi-definite. Since both $F(x-\theta)$ and $F(x+\theta)$ are co-positive semi-definite, $F(x-\theta)\cdot F(x+\theta)$ is co-positive semi-definite for θ.

Since $F(x-\theta)\cdot F(x+\theta)$ is co-positive semi-definite for θ, $D_\theta^{2n}[F(x-\theta)\cdot F(x+\theta)]_{\theta=0} \geq 0$ and we have

$$M_{2n}(x) = D_\theta^{2n}[F(x-\theta)\cdot F(x+\theta)]_{\theta=0} = \sum_{k=0}^{2n}(-1)^k\cdot\binom{2n}{k}\cdot F^{(k)}(x)\cdot F^{(2n-k)}(x) \geq 0 \qquad (66)$$

By letting $x = iy$, we have

$$M_{2n}(iy) = D_\theta^{2n}[F(y-\theta)\cdot F(y+\theta)]_{\theta=0} = \sum_{k=0}^{2n}(-1)^k\cdot\binom{2n}{k}\cdot\frac{1}{(i)^k}F^{(k)}(y)\cdot\frac{1}{(i)^{2n-k}}F^{(2n-k)}(y) \geq 0$$

which implies

$$M_{2n}(y) = (-1)^n\cdot\sum_{k=0}^{2n}(-1)^k\cdot\binom{2n}{k}\cdot F^{(k)}(y)\cdot F^{(2n-k)}(y) \geq 0$$

and hence $L_n(y) \geq 0$ for all n.

The function

$$F(iy) = \int_{-\infty}^{\infty} f(t)\cdot\cos(yt)\,dt = 2\int_0^{\infty} f(t)\cdot\cos(yt)\,dt$$

belongs to the Laguerre-Pólya class and has only real zeros if $f(t)$ is a non-negative even function and rapidly decreasing so that $\mathrm{F}(\mathrm{z})$ is entire where $\mathrm{F}(\mathrm{z})$ is defined as

$$F(z) = \int_{-\infty}^{\infty} f(t)\cdot\cosh(zt)\,dt$$

We can also define $\mathrm{F}(\mathrm{i}z)$, that is,

$$F(iz) = 2\int_0^{\infty} f(t)\cdot\cos(zt)\,dt$$

which is the famous form in the literature because of the Riemann's big-xi function $\Xi(\mathrm{z})$, then $\mathrm{F}(\mathrm{x})$ defined as

$$F(x) = 2\int_0^{\infty} f(t)\cdot\cos(xt)\,dt$$

belongs to the Laguerre-Pólya class and has only real zeros. Basically, the two definitions are same.

From eq. (4), by letting $\mathrm{z} = iy$, we have

$$F(iy) = F(y) = \sum_{n=0}^{\infty} (-1)^n \cdot a_{2n} \cdot y^{2n} = a_0 - a_2 y^2 + a_4 y^4 - a_6 y^6 + \cdots \tag{67}$$

where $a_{2n} \geq 0$ and $F(iy)$ has only real zeros.

Since the Laguerre inequalities holds for any y, by letting $y = 0$, we have

$$\left[F^{(n)}(0)\right]^2 \geq F^{(n-1)}(0) \cdot F^{(n-1)}(0) \tag{68}$$

which is always valid if the power expansion of a function has the form of (67). We consider it in two cases, that is, when n is odd and n is even.

i. n is odd: $F^{(n)}(0) = 0$
$F^{(n-1)}(0)$ and $F^{(n-1)}(0)$ have the opposite sign and hence, $F^{(n-1)}(0) \cdot F^{(n-1)}(0) \leq 0$.
Therefore, the inequalities (68) are valid.

ii. n is even: $\left[F^{(n)}(0)\right]^2 \geq 0$
Both $F^{(n-1)}(0)$ and $F^{(n-1)}(0)$ are zero, thus the inequalities (68) are also valid.

Therefore, all functions whose power series expansions have the form of (66) hold the Laguerre inequalities at $y = 0$. However, not all functions whose power series expansions have the form of (67) have only real zeros, therefore, the Laguerre inequalities are the necessary condition, but not sufficient to have only real zeros.

By letting $u = \sqrt{y}$ in (67), we have

$$F(u) = \sum_{n=0}^{\infty} (-1)^n \cdot a_{2n} \cdot u^n = a_0 - a_2 u + a_4 u^2 - a_6 u^3 + \cdots \tag{69}$$

which also have only real zeros when $u > 0$. By applying (68), we have

$$n \cdot a_{2n}^2 \geq (n+1) \cdot a_{2n-2} \cdot a_{2n+2} \tag{70}$$

The equalities hold if and only if $F(z) = e^{\alpha z^2}$ where $\alpha > 0$, which is the two-sided Laplace transform of $f(t) = \frac{1}{2\sqrt{\pi\alpha}} e^{-t^2/4\alpha}$. Therefore, if $f(t)$ is Gaussian, its two-sided Laplace transform $F(z)$ does not have any zero. Hence, if $f(t) = e^{-\varphi(t)}$ or sum of $e^{-\varphi(t)}$ and its two-sided Laplace transform is $F(z)$, then $F(z)$ has only real zeros if the order of $\varphi(t)$ is greater than two. Further, if the order of $\varphi(t)$ is less than two, the two-sided Laplace transform of $e^{-\varphi(t)}$ does not have only real zeros.

6. The Riemann hypothesis

The Riemann zeta function $\zeta(s)$ is defined

$$\zeta(\mathrm{s}) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$$

where $\mathrm{s} = \sigma + \mathrm{i}\omega$.

It is known that the zeros of $\zeta(s)$ are located only on the strip $0 < \sigma < 1$. Riemann conjectured that all the zeros of $\zeta(s)$ are located on the line $\sigma = \frac{1}{2}$, so-called "Riemann hypothesis".

Using the Riemann's functional equation, an entire and symmetric function can be obtained which is called the xi function $\xi(s)$ where

$$\xi(s) = \frac{1}{2}\pi^{-\frac{s}{2}}s(s-1)\Gamma\left(\frac{s}{2}\right)\zeta(s) \tag{71}$$

and

$$\xi(s) = \xi(1-s)$$

hence $\xi(s)$ is symmetric at $\sigma = \frac{1}{2}$ and the zeros of $\xi(s)$ are located at the same position of $\zeta(s)$, that is, on the strip $0 < \sigma < 1$. If the Riemann hypothesis is true, all the zeros of $\xi(s)$ are located on the line $\sigma = \frac{1}{2}$.

It is well-known that

$$\xi(s) = \int_{-\infty}^{\infty} \varphi(t) \cdot e^{-\left(s-\frac{1}{2}\right)t} dt \tag{72}$$

where

$$\varphi(t) = 2\pi \sum_{n=1}^{\infty} n^2 \cdot e^{-\pi n^2 e^{2t}} \cdot \left(2\pi n^2 e^{\frac{9}{2}t} - 3e^{\frac{5}{2}t}\right) \tag{73}$$

and it can be shown that $\varphi(t) > 0$ for all t and an even function.

By letting $z = s - \frac{1}{2}$ where $z = x + iy$, and $\varphi(t)$ is even, we have

$$\Phi(z) = \int_{-\infty}^{\infty} \varphi(t) \cdot e^{-zt} dt = \int_{-\infty}^{\infty} \varphi(t) \cdot \cosh(zt)\, dt \tag{74}$$

and since $\Phi(z)$ is a shifted function by $\frac{1}{2}$ of $\xi(s)$, $\Phi(z)$ is entire and the zeros of $\Phi(z)$ should be located on the strip $-\frac{1}{2} < x < \frac{1}{2}$. From (66), we have

$$\Phi(z) = \frac{1}{2}\pi^{-\frac{1}{4}} \cdot \pi^{-\frac{z}{2}} \cdot \left(z^2 - \frac{1}{4}\right) \cdot \Gamma\left(\frac{z}{2} + \frac{1}{4}\right) \cdot \zeta\left(z + \frac{1}{2}\right) \tag{75}$$

which is Riemann's original definition of xi-function.

We consider the function $\varphi(t)$ defined in (72). It is positive and even. Moreover, it is decreasing very rapidly (otherwise, $\xi(s)$ and $\Phi(z)$ cannot be entire). Therefore, $\Phi(iy)$ belongs to the Laguerre-Pólya class and has only real zeros. It means that all the zeros of $\Phi(z)$ are located at $x = 0$, and hence, all the zeros of $\xi(s)$ and $\zeta(s)$ are located at $\sigma = \frac{1}{2}$. Thus, the Riemann hypothesis is true.

From eq. (73), we have

$$\Phi(iz) = \int_{-\infty}^{\infty} \varphi(t) \cdot \cos(zt)\, dt = 2\int_{0}^{\infty} \varphi(t) \cdot \cos(zt)\, dt$$

and by letting $t \mapsto 2t$, we have

$$\Phi(iz) = 4\int_0^\infty \varphi(2t)\cdot\cos(2zt)\,dt \tag{76}$$

We define $\emptyset(t)$ as

$$\emptyset(t) = \pi\sum_{n=1}^{\infty} n^2\cdot e^{-\pi n^2 e^{2t}}\cdot(2\pi n^2 e^{9t} - 3e^{5t})$$

then $\emptyset(t) = \frac{1}{2}\,\varphi(2t)$ and eq. (71) will be

$$\Phi(iz) = 8\int_0^\infty \emptyset(t)\cdot\cos(2zt)\,dt$$

and by defining $\Xi(z) = \frac{1}{8}\,\Phi(iz/2)$, we have

$$\Xi(z) = \int_0^\infty \emptyset(t)\cdot\cos(zt)\,dt \tag{77}$$

or simply,

$$\Xi(z) = 2\,\Phi(iz) \tag{78}$$

This function is called the big-xi or upper-case xi function and used to prove the Riemann hypothesis and to find the location of zeros in most literatures.

Since, $\emptyset(t)$ is a positive even function and decreasing rapidly, $\Xi(x)$ belongs to the Laguerre-Pólya class and has only real zeros, which leads that the Riemann hypothesis is true.