Cosine Function at Rational Argument and Infinite Product Representation

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"It is the spirit that quickeneth; the flesh profiteth nothing: the words that I speak unto you, they are spirit, and they are life." - John 6:63.

ABSTRACT. I used an identity for cosine function at rational argument involving finite product of the gamma functions; hence, the representation of infinite product arose.

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1. INTRODUCTION

In present paper, I used the following identity [1, p. 10, Theorem 22]

$$\cos\left(\frac{p\pi}{q}\right)=\prod_{s=1}^{q}\frac{\Gamma^2\left(\frac{2s-1}{2q}\right)}{\Gamma\left(\frac{2s-1}{2q}+\frac{p}{q^2}\right)\Gamma\left(\frac{2s-1}{2q}-\frac{p}{q^2}\right)},$$

which enabled me to prove the infinite product representation for cosine function at rational argument

$$\cos\left(\frac{p\pi}{q}\right)=\prod_{\ell=0}^{\infty}\prod_{s=1}^{q}\left(1-\frac{4p^2}{(2\ell q^2+2sq-q)^2}\right);$$

more specifically, I get

$$\begin{aligned}\frac{1}{2}=&\prod_{\ell=0}^{\infty}\left(1-\frac{4}{(18\ell+3)^2}\right)\left(1-\frac{4}{(18\ell+9)^2}\right)\left(1-\frac{4}{(18\ell+15)^2}\right)\\ =&\frac{5}{9}\cdot\frac{77}{81}\cdot\frac{221}{225}\cdot\frac{437}{441}\cdot\frac{725}{729}\cdot\frac{1085}{1089}\cdot\ldots,\end{aligned}$$

$$\begin{aligned}\frac{\sqrt{2}}{2}=&\prod_{\ell=0}^{\infty}\left(1-\frac{4}{(32\ell+4)^2}\right)\left(1-\frac{4}{(32\ell+12)^2}\right)\left(1-\frac{4}{(32\ell+20)^2}\right)\left(1-\frac{4}{(32\ell+28)^2}\right)\\ =&\frac{12}{16}\cdot\frac{140}{144}\cdot\frac{396}{400}\cdot\frac{780}{784}\cdot\frac{1292}{1296}\cdot\frac{1932}{1936}\cdot\frac{2700}{2704}\cdot\frac{3596}{3600}\cdot\ldots,\end{aligned}$$

$$\begin{aligned}\frac{1+\sqrt{5}}{4}=&\prod_{\ell=0}^{\infty}\left(1-\frac{4}{(50\ell+5)^2}\right)\left(1-\frac{4}{(50\ell+15)^2}\right)\\ &\left(1-\frac{4}{(50\ell+25)^2}\right)\left(1-\frac{4}{(50\ell+35)^2}\right)\left(1-\frac{4}{(50\ell+45)^2}\right)\\ =&\frac{21}{25}\cdot\frac{221}{225}\cdot\frac{621}{625}\cdot\frac{1221}{1225}\cdot\frac{2021}{2025}\cdot\frac{3021}{3025}\cdot\frac{4221}{4225}\cdot\frac{5621}{5625}\cdot\frac{7221}{7225}\cdot\frac{9021}{9025}\cdot\ldots\end{aligned}$$

and so on.

2. PRELIMINARIES

I use the following classical formula, [2, Section 12.13; 3], which is a Corollary of the Weierstrass infinite product representation for the gamma function:

Corollary 2.1. *If k is a positive integer and $a_1+a_2+\ldots+a_k=b_1+b_2+\ldots+b_k$, where the a_j and b_j are complex numbers and no b_j is zero or a negative integer, then*

$$\prod_{\ell=0}^{\infty}\frac{(\ell+a_1)\cdot\ldots\cdot(\ell+a_k)}{(\ell+b_1)\cdot\ldots\cdot(\ell+b_k)}=\frac{\Gamma(b_1)\cdot\ldots\cdot\Gamma(b_k)}{\Gamma(a_1)\cdot\ldots\cdot\Gamma(a_k)}. \tag{2.1}$$

Proof. See [2, Section 12.13]. □

On the other hand, I will need the Theorem below:

Theorem 2.2. *If p and q are positive integers, $p\leqslant q$ and $p/q\neq 1/2$, then*

$$\cos\left(\frac{p\pi}{q}\right)=\prod_{s=1}^{q}\frac{\Gamma^2\left(\frac{2s-1}{2q}\right)}{\Gamma\left(\frac{2s-1}{2q}+\frac{p}{q^2}\right)\Gamma\left(\frac{2s-1}{2q}-\frac{p}{q^2}\right)}, \tag{2.2}$$

where $\Gamma(z)$ denotes the gamma function and $\cos(z)$ denotes the cosine function.

Proof. See [1, p. 10 and 11, Theorem 22]. □

3. THE MAIN COROLLARY

3.1. New infinite product representation for the cosine function at rational argument.

Corollary 3.1. *If p and q are positive integers, $p\leqslant q$ and $p/q\neq 1/2$, then*

$$\cos\left(\frac{p\pi}{q}\right)=\prod_{\ell=0}^{\infty}\prod_{s=1}^{q}\left(1-\frac{4p^2}{(2\ell q^2+2sq-q)^2}\right), \tag{3.1}$$

where $\sin(z)$ denotes the sine function.

Proof. Note that

$$\frac{2s-1}{2q}+\frac{2s-1}{2q}=\frac{2s-1}{2q}+\frac{p}{q^2}+\frac{2s-1}{2q}-\frac{p}{q^2},$$

satisfy the condition $a_1+a_2=b_1+b_2$; $k=2$ is a positive integer; the a_j and b_j are complex numbers and no b_j is zero or a negative integer. From Corollary 2.1 and the right hand side of the (2.2), it follows that

$$\begin{aligned}\frac{\Gamma^2\left(\frac{2s-1}{2q}\right)}{\Gamma\left(\frac{2s-1}{2q}+\frac{p}{q^2}\right)\Gamma\left(\frac{2s-1}{2q}-\frac{p}{q^2}\right)}&=\prod_{\ell=0}^{\infty}\frac{\left(\ell+\frac{2s-1}{2q}-\frac{p}{q^2}\right)\left(\ell+\frac{2s-1}{2q}+\frac{p}{q^2}\right)}{\left(\ell+\frac{2s-1}{2q}\right)\left(\ell+\frac{2s-1}{2q}\right)}\\&=\prod_{\ell=0}^{\infty}\left(1-\frac{4p^2}{(2\ell q^2+2sq-q)^2}\right).\end{aligned} \tag{3.2}$$

From Theorem 2.2 and (3.2), I conclude that

$$\cos\left(\frac{p\pi}{q}\right)=\prod_{\ell=0}^{\infty}\prod_{s=1}^{q}\left(1-\frac{4p^2}{(2\ell q^2+2sq-q)^2}\right),$$

which is the desired result. □

Example 3.2. Set $p=1$ and $q=3$ in Corollary 3.1

$$\begin{aligned}\frac{1}{2}&=\prod_{\ell=0}^{\infty}\left(1-\frac{4}{(18\ell+3)^2}\right)\left(1-\frac{4}{(18\ell+9)^2}\right)\left(1-\frac{4}{(18\ell+15)^2}\right)\\&=\left(1-\frac{4}{3^2}\right)\left(1-\frac{4}{9^2}\right)\left(1-\frac{4}{15^2}\right)\left(1-\frac{4}{21^2}\right)\left(1-\frac{4}{27^2}\right)\left(1-\frac{4}{33^2}\right)\cdots\\&=\frac{5}{9}\cdot\frac{77}{81}\cdot\frac{221}{225}\cdot\frac{437}{441}\cdot\frac{725}{729}\cdot\frac{1085}{1089}\cdot\cdots\end{aligned}$$

Example 3.3. Set $p=1$ and $q=4$ in Corollary 3.1

$$\begin{aligned}\frac{\sqrt{2}}{2}&=\prod_{\ell=0}^{\infty}\left(1-\frac{4}{(32\ell+4)^2}\right)\left(1-\frac{4}{(32\ell+12)^2}\right)\left(1-\frac{4}{(32\ell+20)^2}\right)\left(1-\frac{4}{(32\ell+28)^2}\right)\\&=\left(1-\frac{4}{4^2}\right)\left(1-\frac{4}{12^2}\right)\left(1-\frac{4}{20^2}\right)\left(1-\frac{4}{28^2}\right)\\&\quad\left(1-\frac{4}{36^2}\right)\left(1-\frac{4}{44^2}\right)\left(1-\frac{4}{52^2}\right)\left(1-\frac{4}{60^2}\right)\cdots\\&=\frac{12}{16}\cdot\frac{140}{144}\cdot\frac{396}{400}\cdot\frac{780}{784}\cdot\frac{1292}{1296}\cdot\frac{1932}{1936}\cdot\frac{2700}{2704}\cdot\frac{3596}{3600}\cdot\cdots\end{aligned}$$

Example 3.4. Set $p=1$ and $q=5$ in Corollary 3.1

$$\begin{aligned}\frac{1+\sqrt{5}}{4}&=\prod_{\ell=0}^{\infty}\left(1-\frac{4}{(50\ell+5)^2}\right)\left(1-\frac{4}{(50\ell+15)^2}\right)\\&\quad\left(1-\frac{4}{(50\ell+25)^2}\right)\left(1-\frac{4}{(50\ell+35)^2}\right)\left(1-\frac{4}{(50\ell+45)^2}\right)\\&=\left(1-\frac{4}{5^2}\right)\left(1-\frac{4}{15^2}\right)\left(1-\frac{4}{25^2}\right)\left(1-\frac{4}{35^2}\right)\left(1-\frac{4}{45^2}\right)\\&\quad\left(1-\frac{4}{55^2}\right)\left(1-\frac{4}{65^2}\right)\left(1-\frac{4}{75^2}\right)\left(1-\frac{4}{85^2}\right)\left(1-\frac{4}{95^2}\right)\cdots\\&=\frac{21}{25}\cdot\frac{221}{225}\cdot\frac{621}{625}\cdot\frac{1221}{1225}\cdot\frac{2021}{2025}\cdot\frac{3021}{3025}\cdot\frac{4221}{4225}\cdot\frac{5621}{5625}\cdot\frac{7221}{7225}\cdot\frac{9021}{9025}\cdot\cdots\end{aligned}$$

References

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