Sine Function at Rational Argument, Finite Product of Gamma Functions and Infinite Product Representation

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"It is the spirit that quickeneth; the flesh profiteth nothing: the words that I speak unto you, they are spirit, and they are life." - John 6:63.

Abstract. I corrected the Theorem 21 of previous paper, obtaining an identity for sine function at rational argument involving finite sum of the gamma functions; hence, the representation of infinite product arose.

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1. Introduction

In present paper, I corrected the Theorem 21 in [1, p. 9], obtaining the following identity

$$\frac{q}{p\pi}\sin\left(\frac{p\pi}{q}\right)=\prod_{s=1}^{q}\frac{\Gamma^2\left(\frac{s}{q}\right)}{\Gamma\left(\frac{s}{q}-\frac{p}{q^2}\right)\Gamma\left(\frac{s}{q}+\frac{p}{q^2}\right)},$$

which enabled me to prove the following infinite product

$$\frac{q}{p\pi}\sin\left(\frac{p\pi}{q}\right)=\prod_{\ell=0}^{\infty}\prod_{s=1}^{q}\left(1-\frac{p^2}{(q^2\ell+qs)^2}\right);$$

more specifically, I get

$$\begin{aligned}\frac{2}{\pi}&=\prod_{\ell=0}^{\infty}\left(1-\frac{1}{(4\ell+2)^2}\right)\left(1-\frac{1}{(4\ell+4)^2}\right)\\&=\frac{3}{4}\cdot\frac{15}{16}\cdot\frac{35}{36}\cdot\frac{63}{64}\cdot\frac{99}{100}\cdot\frac{143}{144}\cdot\ldots,\end{aligned}$$

$$\begin{aligned}\frac{3\sqrt{3}}{2\pi}&=\prod_{\ell=0}^{\infty}\left(1-\frac{1}{(9\ell+3)^2}\right)\left(1-\frac{1}{(9\ell+6)^2}\right)\left(1-\frac{1}{(9\ell+9)^2}\right)\\&=\frac{8}{9}\cdot\frac{35}{36}\cdot\frac{80}{81}\cdot\frac{143}{144}\cdot\frac{224}{225}\cdot\frac{323}{324}\cdot\ldots,\end{aligned}$$

$$\frac{2\sqrt{2}}{\pi}=\prod_{\ell=0}^{\infty}\left(1-\frac{1}{(16\ell+4)^2}\right)\left(1-\frac{1}{(16\ell+8)^2}\right)\left(1-\frac{1}{(16\ell+12)^2}\right)\left(1-\frac{1}{(16\ell+16)^2}\right)$$
$$=\frac{15}{16}\cdot\frac{63}{64}\cdot\frac{143}{144}\cdot\frac{255}{256}\cdot\frac{399}{400}\cdot\frac{575}{576}\cdot\frac{783}{784}\cdot\frac{1023}{1024}\cdot\cdots$$

and so on.

2. PRELIMINARY

I use the following classical formula, [2, Section 12.13; 3], which is a Corollary of the Weierstrass infinite product representation for the gamma function:

Corollary 2.1. *If k is a positive integer and $a_1+a_2+\ldots+a_k=b_1+b_2+\ldots+b_k$, where the a_j and b_j are complex numbers and no b_j is zero or a negative integer, then*

$$\prod_{\ell=0}^{\infty}\frac{(\ell+a_1)\cdot\ldots\cdot(\ell+a_k)}{(\ell+b_1)\cdot\ldots\cdot(\ell+b_k)}=\frac{\Gamma(b_1)\cdot\ldots\cdot\Gamma(b_k)}{\Gamma(a_1)\cdot\ldots\cdot\Gamma(a_k)}. \tag{2.1}$$

Proof. See [2, Section 12.13]. □

3. THE MAIN THEOREM

3.1. The sine function at rational argument and the finite product of gamma functions.

Theorem 3.1. *If p and q are positive integers and $p\leqslant q$, then*

$$\frac{q}{p\pi}\sin\left(\frac{p\pi}{q}\right)=\prod_{s=1}^{q}\frac{\Gamma^2\left(\frac{s}{q}\right)}{\Gamma\left(\frac{s}{q}-\frac{p}{q^2}\right)\Gamma\left(\frac{s}{q}+\frac{p}{q^2}\right)}, \tag{3.1}$$

where $\Gamma(z)$ denotes the gamma function and $\sin(z)$ denotes the sine function.

Proof. . Consider the Euler's infinite product representation for sine function [4, p. 321]

$$\frac{\sin(\pi z)}{\pi z}=\prod_{j=1}^{\infty}\left(1-\frac{z^2}{j^2}\right). \tag{3.2}$$

Let $z=p/q$ in (3.2), with $p\in\mathbb{Z}$ and $q\in\mathbb{Z}^+$, and encounter

$$\begin{aligned}\frac{q\sin\left(\frac{p\pi}{q}\right)}{p\pi}&=\prod_{j=1}^{\infty}\left(1-\frac{p^2}{j^2q^2}\right)\\&=\prod_{k=0}^{\infty}\left(1-\frac{p^2}{(k+1)^2q^2}\right).\end{aligned} \tag{3.3}$$

Now, notice that for any $a \in \mathbb{Z}$ and $b \in \mathbb{N}$; there exists unique $c, d \in \mathbb{Z}$, such that $a = bc + d$ and $0 \leqslant d < b$ (division law in $\mathbb{Z}$, see [7, Lemma 7, p. 4]). Hither, this means that any $(k \in \mathbb{N}_0, q \in \mathbb{N})$ uniquely determine the integer r and s, such that $k = qr + s$, where $r = 0, 1, 2, \ldots$ and $s = 1, 2, 3, \ldots, q-1$. Thereupon, it follows (by uniform convergence) that

$$\begin{aligned}
\frac{q \sin\left(\frac{p\pi}{q}\right)}{p\pi} &= \prod_{r=0}^{\infty} \prod_{s=0}^{q-1} \left(1 - \frac{p^2}{(qr+s+1)^2 q^2}\right) \\
&= \prod_{s=0}^{q-1} \prod_{r=0}^{\infty} \left(1 - \frac{p^2}{(qr+s+1)^2 q^2}\right) \\
&= \prod_{s=0}^{q-1} \frac{q^2 \Gamma^2\left(1 + \frac{s+1}{q}\right)}{(s+1)^2 \Gamma\left(\frac{q(s+1)-p}{q^2}\right) \Gamma\left(\frac{q(s+1)+p}{q^2}\right)} \\
&= \prod_{s=0}^{q-1} \frac{q^2}{(s+1)^2} \cdot \prod_{s=0}^{q-1} \frac{\Gamma^2\left(1 + \frac{s+1}{q}\right)}{\Gamma\left(\frac{q(s+1)-p}{q^2}\right) \Gamma\left(\frac{q(s+1)+p}{q^2}\right)} \\
\Rightarrow \frac{q \sin\left(\frac{p\pi}{q}\right)}{p\pi} &= \frac{q^{2q}}{\Gamma^2(q+1)} \cdot \prod_{s=0}^{q-1} \frac{\Gamma^2\left(1 + \frac{s+1}{q}\right)}{\Gamma\left(\frac{q(s+1)-p}{q^2}\right) \Gamma\left(\frac{q(s+1)+p}{q^2}\right)} \\
&= \frac{q^{2q}}{\Gamma^2(q+1)} \cdot \prod_{s=1}^{q} \frac{\Gamma^2\left(1 + \frac{s}{q}\right)}{\Gamma\left(\frac{s}{q} - \frac{p}{q^2}\right) \Gamma\left(\frac{s}{q} + \frac{p}{q^2}\right)},
\end{aligned}$$

using the identity $\Gamma(1+z) = z\Gamma(z)$ in previous equation, I get

$$\begin{aligned}
\frac{q \sin\left(\frac{p\pi}{q}\right)}{p\pi} &= \frac{q^{2q}}{\Gamma^2(q+1)} \cdot \prod_{s=1}^{q} \left(\frac{s}{q}\right)^2 \frac{\Gamma^2\left(\frac{s}{q}\right)}{\Gamma\left(\frac{s}{q} - \frac{p}{q^2}\right) \Gamma\left(\frac{s}{q} + \frac{p}{q^2}\right)} \\
&= \frac{q^{2q}}{\Gamma^2(q+1)} \cdot \prod_{s=1}^{q} \left(\frac{s}{q}\right)^2 \cdot \prod_{s=1}^{q} \frac{\Gamma^2\left(\frac{s}{q}\right)}{\Gamma\left(\frac{s}{q} - \frac{p}{q^2}\right) \Gamma\left(\frac{s}{q} + \frac{p}{q^2}\right)} \\
&= \frac{q^{2q}}{\Gamma^2(q+1)} \cdot \frac{\Gamma^2(q+1)}{q^{2q}} \cdot \prod_{s=1}^{q} \frac{\Gamma^2\left(\frac{s}{q}\right)}{\Gamma\left(\frac{s}{q} - \frac{p}{q^2}\right) \Gamma\left(\frac{s}{q} + \frac{p}{q^2}\right)} \\
&= \prod_{s=1}^{q} \frac{\Gamma^2\left(\frac{s}{q}\right)}{\Gamma\left(\frac{s}{q} - \frac{p}{q^2}\right) \Gamma\left(\frac{s}{q} + \frac{p}{q^2}\right)} \\
\Rightarrow \frac{q}{p\pi} \sin\left(\frac{p\pi}{q}\right) &= \prod_{s=1}^{q} \frac{\Gamma^2\left(\frac{s}{q}\right)}{\Gamma\left(\frac{s}{q} - \frac{p}{q^2}\right) \Gamma\left(\frac{s}{q} + \frac{p}{q^2}\right)},
\end{aligned}$$

which is the desired result. □

Example 3.2. Set $p = 1$ and $q = 2$ in Theorem 3.1

$$\frac{2}{\pi} = 4 \frac{\Gamma^2\left(\frac{1}{2}\right) \Gamma^2\left(\frac{2}{2}\right)}{\Gamma^2\left(\frac{1}{4}\right) \Gamma^2\left(\frac{3}{4}\right)},$$

using simplification, I obtain

$$\frac{1}{2\pi}=\frac{\Gamma^2\left(\frac{1}{2}\right)\Gamma^2\left(\frac{2}{2}\right)}{\Gamma^2\left(\frac{1}{4}\right)\Gamma^2\left(\frac{3}{4}\right)}.$$

Example 3.3. Set $p=1$ and $q=3$ in Theorem 3.1

$$\frac{3\sqrt{3}}{2\pi}=9\frac{\Gamma^2\left(\frac{1}{3}\right)\Gamma^2\left(\frac{2}{3}\right)\Gamma^2\left(\frac{3}{3}\right)}{\Gamma\left(\frac{1}{9}\right)\Gamma\left(\frac{2}{9}\right)\Gamma\left(\frac{4}{9}\right)\Gamma\left(\frac{5}{9}\right)\Gamma\left(\frac{7}{9}\right)\Gamma\left(\frac{8}{9}\right)},$$

using simplification, I obtain

$$\frac{\sqrt{3}}{6\pi}=\frac{\Gamma^2\left(\frac{1}{3}\right)\Gamma^2\left(\frac{2}{3}\right)\Gamma^2\left(\frac{3}{3}\right)}{\Gamma\left(\frac{1}{9}\right)\Gamma\left(\frac{2}{9}\right)\Gamma\left(\frac{4}{9}\right)\Gamma\left(\frac{5}{9}\right)\Gamma\left(\frac{7}{9}\right)\Gamma\left(\frac{8}{9}\right)}.$$

Example 3.4. Set $p=1$ and $q=4$ in Theorem 3.1

$$\frac{4}{\pi\sqrt{2}}=16\frac{\Gamma^2\left(\frac{1}{4}\right)\Gamma^2\left(\frac{2}{4}\right)\Gamma^2\left(\frac{3}{4}\right)\Gamma^2\left(\frac{4}{4}\right)}{\Gamma\left(\frac{1}{16}\right)\Gamma\left(\frac{3}{16}\right)\Gamma\left(\frac{5}{16}\right)\Gamma\left(\frac{7}{16}\right)\Gamma\left(\frac{9}{16}\right)\Gamma\left(\frac{11}{16}\right)\Gamma\left(\frac{13}{16}\right)\Gamma\left(\frac{15}{16}\right)},$$

using simplification, I obtain

$$\frac{\sqrt{2}}{8\pi}=\frac{\Gamma^2\left(\frac{1}{4}\right)\Gamma^2\left(\frac{2}{4}\right)\Gamma^2\left(\frac{3}{4}\right)\Gamma^2\left(\frac{4}{4}\right)}{\Gamma\left(\frac{1}{16}\right)\Gamma\left(\frac{3}{16}\right)\Gamma\left(\frac{5}{16}\right)\Gamma\left(\frac{7}{16}\right)\Gamma\left(\frac{9}{16}\right)\Gamma\left(\frac{11}{16}\right)\Gamma\left(\frac{13}{16}\right)\Gamma\left(\frac{15}{16}\right)}.$$

Example 3.5. Set $p=1$ and $q=5$ in Theorem 3.1

$$\frac{5}{\pi}\sqrt{\frac{5}{8}-\frac{\sqrt{5}}{8}}$$
$$=25\frac{\Gamma^2\left(\frac{1}{5}\right)\Gamma^2\left(\frac{2}{5}\right)\Gamma^2\left(\frac{3}{5}\right)\Gamma^2\left(\frac{4}{5}\right)\Gamma^2\left(\frac{5}{5}\right)}{\Gamma\left(\frac{1}{25}\right)\Gamma\left(\frac{4}{25}\right)\Gamma\left(\frac{6}{25}\right)\Gamma\left(\frac{9}{25}\right)\Gamma\left(\frac{11}{25}\right)\Gamma\left(\frac{14}{25}\right)\Gamma\left(\frac{16}{25}\right)\Gamma\left(\frac{19}{25}\right)\Gamma\left(\frac{21}{25}\right)\Gamma\left(\frac{24}{25}\right)}$$

using simplification, I obtain

$$\frac{1}{5\pi}\sqrt{\frac{5}{8}-\frac{\sqrt{5}}{8}}=\frac{\Gamma^2\left(\frac{1}{5}\right)\Gamma^2\left(\frac{2}{5}\right)\Gamma^2\left(\frac{3}{5}\right)\Gamma^2\left(\frac{4}{5}\right)\Gamma^2\left(\frac{5}{5}\right)}{\Gamma\left(\frac{1}{25}\right)\Gamma\left(\frac{4}{25}\right)\Gamma\left(\frac{6}{25}\right)\Gamma\left(\frac{9}{25}\right)\Gamma\left(\frac{11}{25}\right)\Gamma\left(\frac{14}{25}\right)\Gamma\left(\frac{16}{25}\right)\Gamma\left(\frac{19}{25}\right)\Gamma\left(\frac{21}{25}\right)\Gamma\left(\frac{24}{25}\right)}.$$

3.2. New infinite product representation for the sine function at rational argument.

Corollary 3.6. *If p and q are positive integers and $p\leqslant q$, then*

$$\frac{q}{p\pi}\sin\left(\frac{p\pi}{q}\right)=\prod_{\ell=0}^{\infty}\prod_{s=1}^{q}\left(1-\frac{p^2}{(q^2\ell+qs)^2}\right), \tag{3.4}$$

where $\sin(z)$ *denotes the sine function.*

Proof. Note that

$$\frac{s}{q}+\frac{s}{q}=\frac{s}{q}-\frac{p}{q^2}+\frac{s}{q}+\frac{p}{q^2},$$

satisfy the condition $a_1+a_2=b_1+b_2$; $k=2$ is a positive integer; the a_j and b_j are complex numbers and no b_j is zero or a negative integer. From Corollary 2.1 and the right hand side of the (3.1), it follows that

$$\begin{aligned}\frac{\Gamma^2\left(\frac{s}{q}\right)}{\Gamma\left(\frac{s}{q}-\frac{p}{q^2}\right)\Gamma\left(\frac{s}{q}+\frac{p}{q^2}\right)}&=\prod_{\ell=0}^{\infty}\frac{\left(\ell+\frac{s}{q}-\frac{p}{q^2}\right)\left(\ell+\frac{s}{q}+\frac{p}{q^2}\right)}{\left(\ell+\frac{s}{q}\right)\left(\ell+\frac{s}{q}\right)}\\&=\prod_{\ell=0}^{\infty}\left(1-\frac{p^2}{(q^2\ell+qs)^2}\right).\end{aligned}\tag{3.5}$$

From Theorem 3.1 and (3.5), I conclude that

$$\frac{q}{p\pi}\sin\left(\frac{p\pi}{q}\right)=\prod_{\ell=0}^{\infty}\prod_{s=1}^{q}\left(1-\frac{p^2}{(q^2\ell+qs)^2}\right),$$

which is the desired result. □

Example 3.7. Set $p=1$ and $q=2$ in Corollary 3.6

$$\begin{aligned}\frac{2}{\pi}&=\prod_{\ell=0}^{\infty}\left(1-\frac{1}{(4\ell+2)^2}\right)\left(1-\frac{1}{(4\ell+4)^2}\right)\\&=\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{4^2}\right)\left(1-\frac{1}{6^2}\right)\left(1-\frac{1}{8^2}\right)\left(1-\frac{1}{10^2}\right)\left(1-\frac{1}{12^2}\right)\cdots\\&=\frac{3}{4}\cdot\frac{15}{16}\cdot\frac{35}{36}\cdot\frac{63}{64}\cdot\frac{99}{100}\cdot\frac{143}{144}\cdot\cdots\end{aligned}$$

Example 3.8. Set $p=1$ and $q=3$ in Corollary 3.6

$$\begin{aligned}\frac{3\sqrt{3}}{2\pi}&=\prod_{\ell=0}^{\infty}\left(1-\frac{1}{(9\ell+3)^2}\right)\left(1-\frac{1}{(9\ell+6)^2}\right)\left(1-\frac{1}{(9\ell+9)^2}\right)\\&=\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{6^2}\right)\left(1-\frac{1}{9^2}\right)\left(1-\frac{1}{12^2}\right)\left(1-\frac{1}{15^2}\right)\left(1-\frac{1}{18^2}\right)\cdots\\&=\frac{8}{9}\cdot\frac{35}{36}\cdot\frac{80}{81}\cdot\frac{143}{144}\cdot\frac{224}{225}\cdot\frac{323}{324}\cdot\cdots\end{aligned}$$

Example 3.9. Set $p=1$ and $q=4$ in Corollary 3.6

$$\begin{aligned}\frac{2\sqrt{2}}{\pi}&=\prod_{\ell=0}^{\infty}\left(1-\frac{1}{(16\ell+4)^2}\right)\left(1-\frac{1}{(16\ell+8)^2}\right)\left(1-\frac{1}{(16\ell+12)^2}\right)\left(1-\frac{1}{(16\ell+16)^2}\right)\\&=\left(1-\frac{1}{4^2}\right)\left(1-\frac{1}{8^2}\right)\left(1-\frac{1}{12^2}\right)\left(1-\frac{1}{16^2}\right)\\&\quad\left(1-\frac{1}{20^2}\right)\left(1-\frac{1}{24^2}\right)\left(1-\frac{1}{28^2}\right)\left(1-\frac{1}{32^2}\right)\cdots\\&=\frac{15}{16}\cdot\frac{63}{64}\cdot\frac{143}{144}\cdot\frac{255}{256}\cdot\frac{399}{400}\cdot\frac{575}{576}\cdot\frac{783}{784}\cdot\frac{1023}{1024}\cdot\cdots\end{aligned}$$

Example 3.10. Set $p=3$ and $q=4$ in Corollary 3.6

$$\frac{2\sqrt{2}}{3\pi}=\prod_{\ell=0}^{\infty}\left(1-\frac{9}{(16\ell+4)^2}\right)\left(1-\frac{9}{(16\ell+8)^2}\right)\left(1-\frac{9}{(16\ell+12)^2}\right)\left(1-\frac{9}{(16\ell+16)^2}\right)$$
$$=\left(1-\frac{9}{4^2}\right)\left(1-\frac{9}{8^2}\right)\left(1-\frac{9}{12^2}\right)\left(1-\frac{9}{16^2}\right)$$
$$\left(1-\frac{9}{20^2}\right)\left(1-\frac{9}{24^2}\right)\left(1-\frac{9}{28^2}\right)\left(1-\frac{9}{32^2}\right)\ldots$$
$$=\frac{7}{16}\cdot\frac{55}{64}\cdot\frac{135}{144}\cdot\frac{247}{256}\cdot\frac{391}{400}\cdot\frac{567}{576}\cdot\frac{775}{784}\cdot\frac{1015}{1024}\cdot\ldots$$

Example 3.11. Set $p=1$ and $q=5$ in Corollary 3.6

$$\frac{5}{\pi}\sqrt{\frac{5}{8}-\frac{\sqrt{5}}{8}}=\prod_{\ell=0}^{\infty}\left(1-\frac{1}{(25\ell+5)^2}\right)\left(1-\frac{1}{(25\ell+10)^2}\right)$$
$$\left(1-\frac{1}{(25\ell+15)^2}\right)\left(1-\frac{1}{(25\ell+20)^2}\right)\left(1-\frac{1}{(25\ell+25)^2}\right)$$
$$=\left(1-\frac{1}{5^2}\right)\left(1-\frac{1}{10^2}\right)\left(1-\frac{1}{15^2}\right)\left(1-\frac{1}{20^2}\right)\left(1-\frac{1}{25^2}\right)$$
$$\left(1-\frac{1}{30^2}\right)\left(1-\frac{1}{35^2}\right)\left(1-\frac{1}{40^2}\right)\left(1-\frac{1}{45^2}\right)\left(1-\frac{1}{50^2}\right)\ldots$$
$$=\frac{24}{25}\cdot\frac{99}{100}\cdot\frac{224}{225}\cdot\frac{399}{400}\cdot\frac{624}{625}\cdot\frac{899}{900}\cdot\frac{1224}{1225}\cdot\frac{1599}{1600}\cdot\frac{2024}{2025}\cdot\frac{2499}{2500}\cdot\ldots$$

REFERENCES

[1] Guedes, Edigles, *Infinite Product Representations for Binomial Coefficient, Pochhammer's Symbol, Newton's Binomial and Exponential Function*, June 27, 2016, **viXra:1611.0049**.

[2] Whittaker, E. T. and Watson, G. N., *A Course of Modern Analysis*, Cambridge University Press, Cambridge, 1978.

[3] Sondow,Jonathan and Yi, Huang, *New Wallis- and Catalan-Type Infinite Products for π, e and* $\sqrt{2+\sqrt{2}}$, arXiv:1005.2712.

[4] Spiegel, Murray R., Lipschutz, Seymour and Schiller, John J., *Complex Variables with an Introduction to Conformal Mapping and its Applications*, Schaum's Outline Series, Second Edition, McGrawHill, New York, 2009.